# INFINITE CURVATURE, QUANTUM RANDOM WALKS, AND PITMAN'S CELEBRATED 2M-X THEOREM

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This talk is based on a joint work with François Chapon [1].

The classical theorem by Pitman (1975) states that a Brownian motion minus twice its running infimum enjoys the Markov property. It has the same law as the norm of a 3-dimensional Brownian motion.

We start by recalling the long history of this theorem. First and foremost, it is for for this example that the technology of intertwining has been developed by Rogers and Pitman in their seminal 1981 paper. Then, it was further understood that this theorem, in its discrete form, is heavily connected to combinatorial representation theory. Even in the context of solvable models in mathematical physics, this theorem bridges random matrix theory and directed percolation.

After recalling this history, we present a problem which remained and which is to explain this theorem through a relation between the representation theory and the geometry of $SL_2$. Let us explain the problem in more details.

**The problem:** On the one hand, Biane understood that Pitman's theorem is intimately related to the representation theory of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$, in the so-called crystal regime $q \to 0$. On the other hand, Bougerol and Jeulin showed the appearance of exactly the same Pitman transform in the infinite curvature limit $r \to \infty$ of a Brownian motion on the hyperbolic space $\mathbb{H}^3 = SL_2(\mathbb{C})/SU_2$. Our work aims at understanding this phenomenon by giving a unifying point of view.

**The result of** [1]**:** We shall give a unifying point of view using commutative diagrams of spaces and commutative diagrams of processes.

Although we will only give glimpses of the main result in the talk, here is the full abstract of the paper. We exhibit a presentation $\mathcal{U}_q^{\overline{h}}(\mathfrak{sl}_2)$ of the Jimbo-Drinfeld quantum group which isolates the role of curvature r and that of the Planck constant $\overline{h}$. The simple relationship between parameters is $q = e^{-r}$ . The semi-classical limits $\overline{h} \to 0$ are the Poisson-Lie groups dual to $SL_2(\mathbb{C})$ with varying curvatures $r \in \mathbb{R}^+$. We also construct classical and quantum random walks, drawing a full picture which includes Biane's quantum walks and the construction of Bougerol-Jeulin. Taking the curvature

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parameter $r$ to infinity leads indeed to the crystal regime at the level of representation theory ($\overline{h} > 0$) and to the Bougerol-Jeulin construction in the classical world ($\overline{h} = 0$).

All these results are neatly in accordance with a philosophy in representation theory called the Kirillov orbit method.

## References

[1] Chapon, F., Chhaibi, R. Quantum $SL_2$, infinite curvature and Pitman's 2M-X theorem. *Probab. Theory Relat. Fields*, 179, 835–888 (2021). https://doi.org/10.1007/s00440-020-01002-8

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