Generalized Riemann-Hilbert correspondence: old and new

Yan Soibelman

KANSAS STATE UNIVERSITY and IHES

February 28, 2025

Classical RH-correspondence

Riemann-Hilbert correspondence is typically understood as a correspondence between differential equations and their solutions.
For regular singular differential equations (e.g. $z\frac{df}{dz} = a \cdot f, a \in \mathbb{C}$) the only invariant of the locally constant sheaf of solutions is its monodromy (in this example $f = Cz^a$ and monodromy about $z = 0$ is $e^{2\pi ia}$). Reconstruction of the regular singular equation from its sheaf of solutions is the essence of Hilbert's 21st problem. Kashiwara proved in 80's the general statement about the derived equivalence of the category of regular singular D-modules on a complex manifold X and the category of constructible sheaves on X. The equivalence functor is $\mathcal{M} \mapsto \mathbf{R}Hom_{D_X}(\mathcal{M}, \mathcal{O}_X)$. Under this equivalence the abelian category of regular singular D_X-modules corresponds to the category of perverse sheaves on X.
For irregular singular equations (e.g. $z^2\frac{df}{dz} = a \cdot f, a \in \mathbb{C}$) the monodromy is not the only invariant of the sheaf of solutions (e.g. $f = Ce^{-\frac{a}{z}}$ has trivial monodromy). RH-correspondence in dimension 1 is due to Deligne-Malgrange. For higher-dimensional case see recent papers by Agnolo-Kashiwara.

HFT and GRHC

Besides of differential equations there are other interesting classes of equations: difference, q-difference, elliptic difference equations. It is a challenging problem to formulate a universal RH-correspondence which serves those classes as well. We did that with Maxim Kontsevich in 2015 as a part of our program "Holomorphic Floer Theory" (HFT). Our generalized RH-correspondence (GRHC) relates two different areas of mathematics: deformation quantization and Floer theory. Because of that it can be used for study representations of quantized algebras which are not related to a particular class of equations, e.g. Cherednik or Sklyanin algebras.

From the perspective of GRHC constructible sheaves which are crucial in the case of D-modules is just a computational tool in the Floer theory of cotangent bundles.

Also, I will explain that there are global and local versions of GRHC.

Riemann-Hilbert correspondence vs Homological Mirror Symmetry

Our GRHC slightly resembles Homological Mirror Symmetry (HMS). The "A-side" of the GRHC will be the Fukaya category, similarly to HMS. The "B-side" will be a deformation of the derived category of coherent sheaves in the "non-commutative direction". This is slightly different from HMS. More important difference with HMS: both categories in the GRHC correspond to the same complex symplectic manifold.
Instead of A- and B-sides of HMS in GRHC we are talking about equivalence of de Rham and Betti sides which are related to deformation quantization and Floer theory respectively.

Category of DQ-modules

Given a complex symplectic manifold $(M, \omega^{2,0})$, $dim_{\mathbb{C}} M = 2n$ deformation quantization story (Kontsevich, Kashiwara, Schapira,...) gives rise to the following structures:
1) A sheaf of categories over $\mathbb{C}[[\hbar]]$ which modulo $\hbar$ is equivalent to the sheaf of categories of coherent $\mathcal{O}_M$-modules. The "quantized" sheaf is not necessarily a sheaf of modules over a sheaf of algebras.
Assume: there is a $*$-product on $\mathcal{O}_M[[\hbar]] = \{\sum_{k \geq 0} \hbar^k f_k, f_k \in \mathcal{O}_M\}$. Then we have a sheaf of algebras $\mathcal{O}_{M,\hbar} = (\mathcal{O}_M[[\hbar]], *)$ which mod $\hbar$ is isomorphic to the sheaf of Poisson algebras $\mathcal{O}_M$.
A DQ-module is a finitely generated $\mathcal{O}_{M,\hbar}$-module. More pedantically, one should quotient this category by the subcategory of $\hbar$-torsion modules.

Holonomic DQ-modules: de Rham side of the global RH-correspondence

To define the de Rham side category one should make additional choices. They include e.g. a partial Poisson log compactification $P_{log} \supset M$, such that on the normal crossing divisor $D_{log} = P_{log} - M$ the form $\omega^{2,0}$ has poles of order 1. After that one can define the category $Hol_{glob} := Hol_{glob}(M)$ of holonomic DQ-modules. Roughly, it is the category of global sections of the sheaf of categories of finite rank $\mathcal{O}_{M,\hbar}$-modules which modulo $\hbar$ have Lagrangian support, and the support "behaves nicely" near the D_{log}.
Assume: the category Hol_{glob} is defined in fact over the subring $\mathbb{C}\{\hbar\} \subset \mathbb{C}[[\hbar]]$ of analytic germs at $\hbar = 0$.

Fukaya category: Betti side of the global RH-correspondence

One can also define a family of global Fukaya categories $\mathcal{F}_{glob,\hbar} := \mathcal{F}(M, \frac{1}{\hbar}(Re(\omega^{2,0}) + i\, Im(\omega^{2,0})))$. It can be thought of as a single category $\mathcal{F}_{glob}$, linear over the Novikov ring of series $Nov := \sum_{\{\lambda_i | Re(\lambda_i) \to +\infty\}} c_i e^{-\frac{\lambda_i}{\hbar}}$.

Rigorous definition of $\mathcal{F}_{glob}$ is quite involved. In fact $\mathcal{F}_{glob}$ is an A_∞-category of functors to the category of finite-dimensional complexes of the so-called *partially wrapped Fukaya category*. Some of the objects of the latter are real-analytic Lagrangian submanifolds of the real symplectic manifold $(M, \frac{1}{\hbar}(Re(\omega^{2,0})))$ behaving "nicely" near D_{log} and carrying vector bundles with connections whose curvature is $\frac{1}{\hbar} Im(\omega^{2,0})$ (i.e. $\frac{1}{\hbar} Im(\omega^{2,0})$ is the so-called B-field). The A_∞-structure is defined in terms of pseudo-holomorphic discs with boundaries on given Lagrangians ("instanton corrections" to the graded vector spaces generated by intersection points).
Assume that $\mathcal{F}_{glob}$ is defined over the subring $\mathcal{A} = \lim_{\varepsilon \to 0} \mathcal{O}^{an}(0 < |\hbar| < \varepsilon)$ which is the inductive limit of rings of analytic functions on punctured discs $0 < |\hbar| < \varepsilon$.

Generalized global RH-correspondence

Global Riemann-Hilbert correspondence (conjectural in general) says:

after extension of scalars to $\mathcal{A}$ there is a derived equivalence $Hol_{glob} \simeq \mathcal{F}_{glob}$. The t-structure with the heart consisting of holonomic DQ-modules corresponds to the subcategory of $\mathcal{F}_{glob}$ in which objects are supported on complex Lagrangian analytic subsets.

Short formulation: de Rham and Betti sides of the RH-correspondence are equivalent.

Probably the equivalence comes from a faithful embedding of $\mathcal{F}_{glob}$ to the category of all DQ-modules.

If instead of $\mathcal{A}$ both sides can be defined over $\mathcal{O}^{an}(\mathbb{C}^*)$ we can fix $\hbar$ and obtain the RH-correspondence over $\mathbb{C}$.

Example:rational case

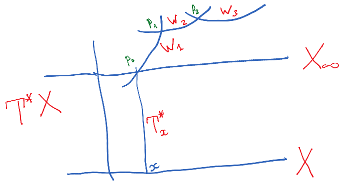
1) Meromorphic connections $(\mathcal{E}, \nabla)$ on a marked smooth complex curve $(X, x_1, ..., x_n)$ with prescribed singular behavior at each x_i. By Hukuhara-Levelt-Turrittin (HLT) theorem formally $(\mathcal{E}, \nabla) = \oplus_{i,\alpha} e^{c_\alpha^{(i)}} \otimes \nabla_\alpha^{(i),RS}$. Here finite collection of singular terms $c_\alpha^{(i)} = \sum_{\lambda \in \mathbb{Q}_{\leq 0}} c_{\alpha,\lambda}^{(i)} (x - x_i)^\lambda$ controls the irregular behavior at x_i, and $\nabla_\alpha^{(i),RS}$ are connections which are regular singular at x_i.

Let $Conn(X, c_\alpha^{(i)})$ be the category of bundles with meromorphic connections on X such that their formal types at the marked points are as above. Then *irregular RH-correspondence* (Deligne-Malgrange, 1982) says that $Conn(X, c_\alpha^{(i)})$ is equivalent to the category of locally-constant sheaves on $X - \{x_1, ..., x_n\}$ endowed with filtrations (described in terms of $c_\alpha^{(i)}$) on the fibers close to the points x_i (filtrations control exponential growth as $x \to x_i$, i.e. the Stokes phenomenon). This category can be interpreted as a subcategory of the category of holonomic DQ-modules after a choice of an appropriate partial log compactification P_{log}.

In this case P_{log} is obtained from the fiberwise compactification $\overline{T^*X} = T^*X \cup X_\infty$ by a finite sequence of blow-ups and then by adding to $T^*X = T^*X$ those divisors $D_\alpha^{(i)} \subset D_{log}$ on which the symplectic form $\omega^{2,0}_{T^*X}$ has pole of order one. Divisors $D_\alpha^{(i)}$ bijectively correspond to the singular terms $c_\alpha^{(i)}$. The RH-correspondence can be restated as an equivalence of $Conn(X, c_\alpha^{(i)})$ with a certain subcategory of the Fukaya category of T^*X generated by conic Lagrangian submanifolds defined in terms of $\{c_\alpha^{(i)}\}_{i,\alpha}$. On the figure below we fix $x = x_i$ and use the notation $W_\alpha := D_\alpha^{(i)}$.

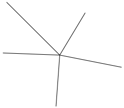
Figure: blow-ups

Thursday, October 29, 2015
9:02 AM



Example:trigonometric case

2) q-difference equations $f(qx) = A(x)f(x), x \in (\mathbb{C}^*)^n$. They are holonomic modules over the quantum torus $A_q(n)$, $0 < |q| < 1$, i.e. a $\mathbb{C}$-algebra with invertible generators $x_i, y_i, 1 \leq i \leq n$ and relations $x_i y_j = q^{\delta_{ij}} y_j x_i, 1 \leq i, j \leq n$. Here $q = e^{\hbar}$.
Then $M = (\mathbb{C}^*)^{2n}, \omega^{2,0} = \sum_{1 \leq i \leq n} \frac{dx_i}{x_i} \wedge \frac{dy_i}{y_i}$, and P_{log} is a toric variety corresponding to a "Lagrangian fan" in $\mathbb{R}^{2n}$. Analogs of singular terms are unions of rational Lagrangian cones in $\mathbb{R}^{2n}$ with vertices at the origin (hence the term "Lagrangian fan") .



For $n = 1$ the $q - RH$-correspondence is due to Ramis-Sauloy-Zhang (2009). Our reformulation claims an equivalence of the category of coherent sheaves on $\mathbb{C}^*/q^{\mathbb{Z}}$ endowed with two anti-Harder-Narasimhan filtrations (slopes of consecutive semistable factors increase and equal to the slopes of the rays) and a certain Fukaya category associated with the corresponding toric surface.

Example: elliptic case

Take $\tau = 2\pi i\hbar$, and E be the corresponding elliptic curve.
3a) (difference equations on elliptic curve) $f(x+u) = A(x)f(x), x \in E$,where $u \in E$ is fixed.
Then $M = E \times \mathbb{C}^*_z, \omega^{2,0} = dx \wedge \frac{dz}{z}$, and $P_{log} = E \times \mathbb{P}^1$.
3b) (Sklyanin algebras). Here we consider holonomic modules over the elliptic algebra corresponding to the quantization of $M = \mathbb{P}^2 - E$ endowed with a simplectic form $\omega^{2,0}$ which has a pole of order 1 on the smooth cubic E. Then $P_{log} = \mathbb{P}^2$.

Local RH-correspondence

For any (possibly singular) complex Lagrangian subvariety $L \subset M$ we can define local versions of the de Rham and Betti sides, $Hol_{L,loc}$ and $\mathcal{F}_{L,loc}$. The category $Hol_{L,loc}$ consists of holonomic DQ-modules supported on L. It is linear over $\mathbb{C}((\hbar))$. The category $\mathcal{F}_{L,loc}$ is defined over $\mathbb{Z}$ (no instanton corrections). There is a local version of the RH-correspondence which claims equivalence $Hol_{L,loc} \simeq \mathcal{F}_{L,loc}$ after extending scalars to $\mathbb{C}((\hbar))$. Informally, we can think about objects of these local categories as those "living in a small Stein neighborhood of L".

From local to global on the Betti side

Local Fukaya category gives rise to a *local system* of A_∞-categories over the circle of directions $\theta = Arg(\hbar)$ (or over $\mathbb{C}^*_\hbar$) via dilation $\omega^{2,0} \mapsto \omega^{2,0}/\hbar$.
There is in general countable set of Stokes rays $\theta = Arg(\int_\gamma \omega^{2,0})$, where $\gamma \in H_2(M, L, \mathbb{Z})$, as well as by the Stokes isomorphisms for each Stokes ray. This package of data is an example of the wall-crossing structure, the notion we introduced in arXiv:1303.3253.
Stokes isomorphisms are equivalences of the fibers of $\mathcal{F}_{L,loc}$ slightly on the left and on the right of a Stokes ray. They are defined in terms of the virtual count of pseudo-holomorphic discs with boundary on L (instanton corrections). Generically the discs are absent, but they appear on Stokes rays.

Sometimes (e.g. if one has exponential bounds on the number of discs) the corresponding wall-crossing structure belongs to a smaller class of analytic wall-crossing structures (see our arXiv:2005.10651).

If this is the case, one can start with the local system $\mathcal{F}_{L,loc}$ then use Stokes isomorphisms and glue a new meromorphic family of categories, i.e. a new category $\mathcal{F}_{L,loc}^{an}$ over the ring $\mathcal{O}^{an}(0 < |\hbar| < \varepsilon)$ for some $\varepsilon > 0$. It is no longer a local system. There is a faithful embedding $\mathcal{F}_{L,loc}^{an} \hookrightarrow \mathcal{F}_{glob}$. The inductive limit over all L conjecturally coincides with $\mathcal{F}_{glob}$ (or one can simply define $\mathcal{F}_{glob}$ as such a limit).

Local de Rham side: naive picture

The local RH-correspondence which I formulated before is not sufficiently general. One reason for that: there are different possibilities for the behavior of the Lagrangian support of holonomic DQ-modules as $\hbar \to 0$.
In the local RH-correspondence which I mentioned before objects of $Hol_{L,loc}$ for a smooth L were direct sums of cyclic modules such that the cyclic vector (quantum wave function) is given in local symplectic coordinates (x, p) by the WKB-expansion
$\psi(x, \hbar) = exp(\frac{S_{-1}(x)}{\hbar} + S_0(x) + \hbar S_1(x) + ...) = exp(\sum_{i \geq -1} \hbar^i S_i(x))$, where
$L = \{p = dS_{-1}(x)\}$. Notice that the series in the exponent gives a deformation of L corresponding to the formal path of closed 1-forms $\alpha_{-1} + \sum_{i \geq 1} \hbar^i \alpha_i$, where $\alpha_i = dS_i$ locally.

In fact there are more types of holonomic DQ-modules associated with a smooth L than those above. There are also holonomic DQ-modules associated with non-smooth L which are not covered by the previous theory.

Motivating example: fractional WKB expansions

Even for a smooth L one can consider a DQ-module supported on L which is a cyclic module with the generator $\psi(x,\hbar) = exp(\sum_{\lambda\in\mathbb{Q}\cap[-1,0)} \hbar^{\lambda} S_{\lambda}(x) + \sum_{i\geq 0} \hbar^i S_i(x))$.

Such solutions can appear as formal flat sections of connections $\nabla = d + \frac{A_{-1}}{\hbar} + \sum_{i\geq 0} \hbar^i A_i$, where $A_i = A_i(x)$ are holomorphic matrix-valued functions and A_{-1} is nilpotent.

Enhanced Lagrangian subvarieties

From the point of view of deformation theory this means that we should consider deformations L' of L which are more general than those corresponding to formal paths of closed 1-forms of the type $dS_{-1} + \sum_{i \geq 1} \hbar^i \alpha_i(x)$, where locally $L = graph(dS_{-1})$. For example we can consider deformations L' of L corresponding to formal paths of closed 1-forms of the type $dS_{-1} + \sum_{\lambda \in \mathbb{Q} \cap (0,1)} \hbar^{\lambda} \alpha_{\lambda}(x) + \sum_{i \geq 1} \hbar^i \alpha_i(x)$. Notice the analogy of the finite sum over $\lambda \in \mathbb{Q} \cap (0,1)$ with singular terms for meromorphic connections in the rational case of GRHC. The difference is that now we are talking about singular behavior w.r.t. $\hbar \to 0$, and not with respect to x. These new " singular terms at $\hbar = 0$" should correspond to certain divisors. I will explain how to construct them on the next slide.

Remark that $graph(dS_{-1} + \sum_{\lambda \in \mathbb{Q} \cap (0,1)} \hbar^{\lambda} \alpha_{\lambda}(x))$ is a family of Lagrangian varieties which survives when we throw away terms of the size $O(\hbar^{\geq 1})$. Thus it can be thought of as an "enhancement" of the support L of the corresponding DQ-module.

Local de Rham side: enhancements in general

Assume first that M is compact, so we can ignore P_{log}. Consider the manifold $P = M \times \mathbb{C}_\hbar$ endowed with the Poisson structure $\pi_P = \hbar(\omega^{2,0})^{-1}$. It is foliated by symplectic leaves $P_\hbar = M \times \{\hbar\} \simeq M, \hbar \neq 0$, while the fiber P_0 although isomorphic to M as a variety consists of 0-dimensional symplectic leaves.
Let us fix an enhancement L' of L (it is a certain deformation of L).
After that we make consecutive blow-ups of P. If L is smooth and $L' = L \times \mathbb{C}_\hbar$ (trivial deformation) the first blow-up is $Bl_{\overline{L'} \cap P_0}(P)$. We continue this procedure taking as centers of blow-ups intersections of exceptional divisors with proper transforms of $\overline{L'}$. Even for smooth L we can start with a more complicated enhancement L' than the trivial family $L \times \mathbb{C}_\hbar$. If L is non-smooth we first resolve singularities of L. We continue to do blow-ups until the proper transform of $\overline{L'}$ intersects only symplectic divisors ("symplectic" means that on the complement with intersections with other divisors the Poisson structure π_P gives rise to a symplectic form). These symplectic divisors parametrize "singular terms at $\hbar = 0$".

For $n = 1$ one can show that smooth parts of symplectic divisors are twisted cotangent bundles to some smooth curves. Moreover in this case the common curve of two intersecting symplectic divisors is a log-curve for each of the two symplectic forms. We expect similar story in general.

In the above example of the fractional WKB expansion the Lagrangian subvariety $graph(\alpha_\lambda)$ intersects a unique symplectic divisor D_λ^{sympl}. The set $\Delta_{L'}$ of symplectic divisors determines the local category $Hol_{\Delta_{L'},loc}$ of holonomic DQ-modules. Roughly, objects of this category are families of holonomic DQ-modules "supported" on L' such that their appropriately defined "limits" as $\hbar \to 0$ are certain D-modules. E.g. they are D-modules associated with twisted cotangent bundles in the above example $n = 1$.

If M is non-compact one adds to $P_0 := M \times \{0\}$ the divisor which is the intersection of the closure of $D_{log} \times \mathbb{C}^*_\hbar$ with P_0. Then one repeats the above construction with blow-ups adding these divisors to the story.
Assume that the categories $Hol_{\Delta_{L'},loc}$ are well-defined over the ring of analytic germs $\mathbb{C}\{\hbar\}$. In our approach L and L' can be singular, so the categories form a filtered system since the union of two singular analytic Lagrangian subsets is a singular analytic Lagrangian subset. Then Hol_{glob} coincides with the inductive limit of $Hol_{\Delta_{L'},loc}$ over all Lagrangian analytic subsets L and all their enhancements L'.

Local Betti side with enhancements

On the Betti side the enhancements affect the wall-crossing structure (i.e. Stokes rays and Stokes isomorphisms). E.g. in the enhancement L' of L associated with the fractional WKB expansion, one can have pseudo-holomorphic discs of the area $O(e^{-\hbar^{\lambda}}), \lambda \in \mathbb{Q} \cap (0,1)$ which are discs with boundaries on $graph(\alpha_\lambda) \cup graph(\alpha_\mu), \lambda, \mu \in \mathbb{Q} \cap (0,1)$. This phenomenon leads to the new "enhanced" wall-crossing structure (without enhancements the areas of pseudo-holomorphic discs will be of the size $O(e^{-\hbar^{-1}})$) .

From local to global in the enhanced setting

After that we should take into account "large discs in M with boundary on L". This is another interesting of the story for which I have no time. It is based on Symplectic Field Theory of Eliashberg-Givental-Hoffer as well as on the geometric approach to deformation theory of A_∞-categories which we developed with Maxim about 2003 (see first volume of our unpublished book on my home page at KSU) .
Then we modify the local systems $\mathcal{F}_{L,loc}$ on S^1 using the Stokes isomorphisms coming from all pseudo-holomorphic discs. If the corresponding wall crossing structure is analytic we obtain the category $\mathcal{F}^{an}_{L,loc}$ over the ring $\mathcal{A} = \mathcal{O}^{an}(0 < |\hbar| < \varepsilon)$. The inductive limit is a category linear over the ring $\mathcal{A}$.
After that we can formulate the global RH-correspondence as an equivalence of categories over $\mathcal{A}$.

Few applications

a) Extension of the non-abelian Hodge theory (NAHT) in dimension one beyond the case of bundles with connections on curves (where it is due to Simpson). In our generalized NAHT periodic monopoles in $\mathbb{R}^3$ play the role of harmonic bundles.
b) In the generalized NAHT one has analogs of several questions typically associated with Simpson's NAHT, e.g. the generalized $P = W$ conjecture.
c) Relation to the representation theory of quantized algebras (e.g. quantum tori, rational Cherednik algebras, Sklyanin algebras etc.).
d) Adding $\mathbb{C}^*$-actions to our story one can relate it to quantized Coulomb branches.

Appendix: non-archimedean analytic geometry and skeleta

Assume for simplicity that M is algebraic. Our global Betti and de Rham categories are inductive limits over the $\mathbb{Q}PL$-sets constructed from a collection of compactifying divisors of P (symplectic divisors serving $\hbar = 0$ and log divisors serving $\hbar \neq 0$). More precisely, each divisor gives rise to a valuation on the field of rational functions on the scheme M_K obtained from M by extension of scalars. Hence it gives a point in the Berkovich analytic space M_K^{an}. Then we have a compact $\mathbb{Q}PL$-subspace ("skeleton") of M_K^{an} which is a union of simplices "spanned" by above divisorial valuations. These skeleta resemble those which appeared in our paper on homological mirror symmetry 25 years ago.

Hope: with each skeleton one can associate de Rham and Betti categories which are equivalent via the GRHC.