Linear Convergence in Bundle Progressive Hedging and its link to Proximal Decomposition

XVIIth Conference on Stochastic Programming

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The multizonal optimization problem

Based on P. Mahey, J. Koko, and A. Lenoir, Decomposition methods for a spatial model for long-term energy pricing problem, in Math Meth Oper Res (2017) 85:137–153

$$\min_{(q,f)} \sum_{z\in Z} G_z(\boldsymbol{q}_z) + \sum_{e\in E} L_e(\boldsymbol{f}_e)$$
$$B\boldsymbol{q}_t + A\boldsymbol{f}_t = \boldsymbol{d}_t, \quad t = 1, \ldots, T-1$$
$$x_{z,t+1} = x_{zt} - u_{zt} + i_{zt} \quad \forall z, t$$
$$q_{z,t} \in \mathcal{P}_{z,t},\ f_{e,t} \in F_{e,t} \quad \forall z, t, e$$

where :

usage of storage

production

storage

$$\mathcal{P}_{z,t} = \left\{0 \le p_{z,t} \le P^{max}_{z,t},\ 0 \le u_{z,t} \le U^{max}_{z,t},\ X^{min}_{z,t} \le x_{z,t} \le X^{max}_{z,t}\right\}$$
$$F_{e,t} = \left\{0 \le f_{e,t} \le \kappa_{et}\right\}.$$

flow

Dissociating in and outgoing flows allows to write

$$\min_{q,f,\phi} \sum_{z} Q_z(\boldsymbol{f}_z, \boldsymbol{\phi}_z) \quad (\boldsymbol{f}, \boldsymbol{\phi}) \in \mathcal{A} \coloneqq \{(f,\phi)\ ,\ \phi_{ez,t} = f_{ez',t}, \quad \text{for} \quad e = (z', z) \in z'^{+} \cap z^{-}, \quad \forall z, z' \in Z, \forall t \ \}$$

More generally : multistage stochastic problems

Based on F. Atenas's PhD thesis: Proximal decomposition methods for optimization problems with structure, State University of Campinas, SP, Brazil – September 2023.

$$\begin{cases} \min_{x} & f(x) = \sum_{s=1}^{S} p_s f_s(x_s) \\ \text{s.t.} & x_s \in C_s, \quad \text{for all } s = 1, \dots, S, \\ & x \in A \end{cases}$$

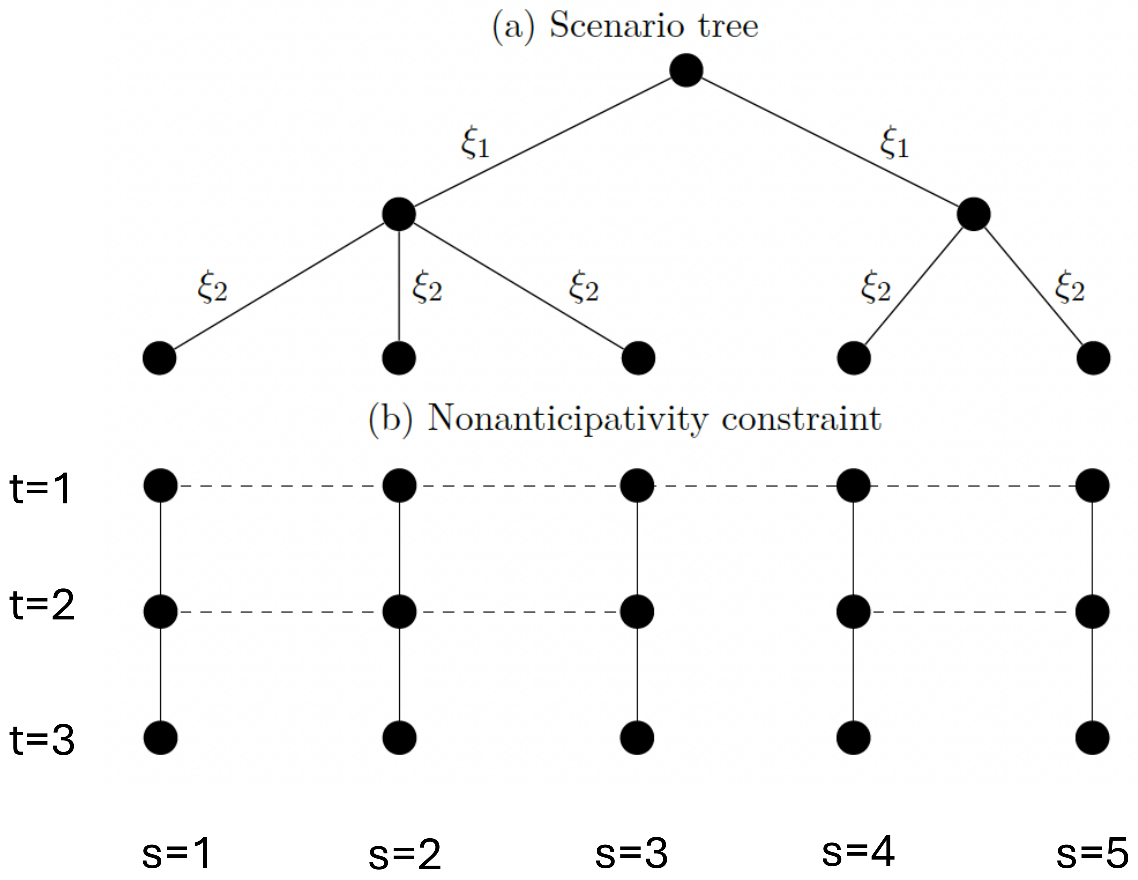
convex compact

subspace (multistage), easy projection

$$P_A\,[x]_t = \mathbb{E}_{[t-1]}[x_t].$$

$$\begin{cases} \min_{w} & h(w) \\ \text{s.t.} & w \in A^{\perp} \end{cases} \quad \text{with} \quad \langle u, v \rangle_S = \sum_{s=1}^{S} p_s u_s \cdot v_s\,.$$

$$h(w) = \sum_{s=1}^{S} p_s h_s(w_s)\,, \ \text{where } h_s(w_s) = \max_{x_s \in C_s} (-\mathcal{L}_s)(x_s, w_s)\,.$$



Progressive Hedging

Introduced in Rockafellar, R. T., and Wets, R. J-B, "Scenarios and policy aggregation in optimization under uncertainty." Mathematics of Operations Research 16 (1991), 119–1473

fixed !

Proximal step : For all s,

$$x_s^{k+1/2} = \text{argmin}_{x_s \in \mathbb{R}^n} f_s(x_s) + x_s \cdot w_s^k + \frac{t}{2}\|x_s - x_s^k\|^2$$

and

$$w^{k+1/2} = w^k + t(x^{k+1/2} - x^k).$$

Projection step: $x_{k+1} = P_{A,S}(x^{k+1/2})$, $w_{k+1} = P_{A^\perp,S}(w^{k+1/2})$ and $k \leftarrow k+1$.

blind ?

Proximal Decomposition

Introduced in Mahey P, Oualibouch S, Pham DT (1995) Proximal decomposition on the graph of a maximal monotone operator. SIAM J Optim 5:454–466

Proximal step : For all s,

$$x_s^{k+1/2} = \text{argmin}_{x_s \in \mathbb{R}^n} f_s(x_s) - x_s \cdot w_s^k + \frac{t}{2}\|x_s - x_s^k\|^2$$

and

$$w^{k+1/2} = w^k + t(x^{k+1/2} - x^k).$$

Projection step: $x_{k+1} = P_{A,\cancel{S}}(x^{k+1/2})$, $w_{k+1} = P_{A^\perp,\cancel{S}}(w^{k+1/2})$ and $k \leftarrow k+1$.

➡ When A is a subspace and distribution is uniform, PD and PH are the same !

Bundle methods

Based on *Correa, R., Lemaréchal, C. Convergence of some algorithms for convex minimization. Mathematical Programming 62, 261–275 (1993)*

Goal : Minimize a convex and finite-valued function f over the whole of X

Having only a Black Box : given $x \in X$, the value $f(x)$ and some $g = g(x) \in \partial f(x)$

Step 1 : Choose a convex function $\varphi^k : \mathcal{H} \longrightarrow \mathbb{R}$, and compute

$$y^{k+1} = \text{argmin}_y \varphi^k(y) + \frac{1}{2t_k}||y - x^k||^2.$$

Step 2 : Compute $f(y^{k+1})$. If a good decrease if obtained, namely if

$$f(x^k) - f(y^{k+1}) \geq m[f(x^k) - \varphi^k(y^{k+1})], \quad m \in (0,1)$$

then set $x^{k+1} = y^{k+1}$. Otherwise, $x^{k+1} = x^k$.

Bundle Progressive Hedging

Based on F. Atenas's PhD thesis: Proximal decomposition methods for optimization problems with structure, State University of Campinas, SP, Brazil – September 2023.

Algorithm 3: Bundle Progressive Hedging

Initialization : $k = 0, (x_0, w_0) \in A \times A^{\perp}, 0 < t_{min} \leq t_0 \leq t_{max}, m, l \in]0,1[$, and $TOL > 0$.

Proximal step : For all s,

(3.9)
$$w_s^{k+1/2} = \text{argmin}_{w_s \in \mathbb{R}^n} h_s(w_s) + w_s \cdot x_s^k + \frac{2}{t_k}|w_s - w_s^k|^2$$

and

(3.10)
$$x^{k+1/2} = x^k + \frac{1}{t_k}(w^{k+1/2} - w^k).$$

Stopping test : Compute the model descent :

$$\delta_{mod}^k = h(w^k) - h(w^{k+1/2}) - \langle w^{k+1/2}, x^k \rangle_S.$$

If $\delta_{mod}^k < TOL$: return (x^k, w^k).

Projection step: $x_{k+1} = P_{A,S}(x^{k+1/2})$, $u_{k+1} = P_{A^{\perp},S}(w^{k+1/2})$.

Descent test : Compute the real descent :

$$\delta_{ex}^k = h(w^k) - h(u^{k+1}).$$

If $\delta_{ex}^k \geq m \cdot \delta_{mod}^k$: we have a serious step, $w^{k+1} = u^{k+1}$, and $t_{k+1} = \max(t_{min}, \min(t_{max}, t_k/l))$.

Else : we have a null step, $w^{k+1} = w^k$, and $t_{k+1} = \max(t_{min}, \min(t_{max}, l \cdot t_k))$.

$k \leftarrow k + 1$

(3.9) and (3.10) can also be written

$$x_s^{k+1/2} = argmin_{x_s} f_s(x_s) + w_s^k x_s + \frac{t_k}{2}||x_s - x_s^k||^2$$

$$w^{k+1/2} = w^k + t_k(x^{k+1/2} - x^k).$$

Same can be done for Proximal Decomposition

Step size can be changed !

Cost: One more dual functional evaluation to be made

Convergence analysis of BPH :previous results

Based on F. Atenas's PhD thesis: Proximal decomposition methods for optimization problems with structure, State University of Campinas, SP, Brazil – September 2023.

If finite termination (and $TOL = 0$) : last primal and dual iterates are optimal

set of dual solutions

If infinitely many serious steps + error bound : $d(w, S) \leq \alpha ||x||$ for $x \in \partial h(w) \cap B(0, \epsilon)$

- dual functional values at serious steps converge towards optimum monotonically, linearly
- dual serious (intermediate) iterates converge towards optimum linearly
- primal serious (intermediate) iterates converge subsequentially towards optimum

If infinite tail of null steps : (with stabilizing step size)

- last serious dual iterate W is optimal (and $u^k, w^{k+\frac{1}{2}}$ converge towards it)
- primal (intermediate) iterates converge subsequentially towards optimum … can we say more?

Convergence analysis of BPH : strongly convex function

Projected gradient (PG) algorithm : $y^{k+1} = P_A(y^k - \alpha_k \nabla f(y^k))$

Last serious dual iterate, optimal

Here primal iterates are given by PG on the Moreau envelope of $\phi(x) = f(x) + \langle W, x \rangle$

set of primal solutions

Projection error bound (PEB): $d(x, \bar{S}) \leq l||x - P_A(x - \frac{1}{t}\nabla f(x))||$

Theorem[1]: a function is strongly convex iff its Moreau envelope is strongly convex

Proposition[2]: PEB holds for strongly convex function

Theorem[3]: if PEB holds , then iterates of PG converge R-linearly

1 : *Theorem 2.1 from Zhi-Quan Luo and Paul Tseng. Error bounds and convergence analysis of feasible descent methods: a general approach. Annals of Operations Research, 46(1):157–178, 1993.*

2 : *Proposition 2.19 from C. Planiden and X. Wang. Strongly convex functions, moreau envelopes, and the generic nature of convex functions with strong minimizers. SIAM Journal on Optimization, 26(2):1341–1364, 2016.*

3 : *Theorem 5.3 from Zhi-Quan Luo and Paul Tseng. Error bounds and convergence analysis of feasible descent methods: a general approach. Annals of Operations Research, 46(1):157–178, 1993*

Convergence analysis : general case

We show that if PEB holds and the zeroes of f are isolated, then

- primal iterates and intermediate iterates converge towards an optimum
- primal iterates converge Q-linearly

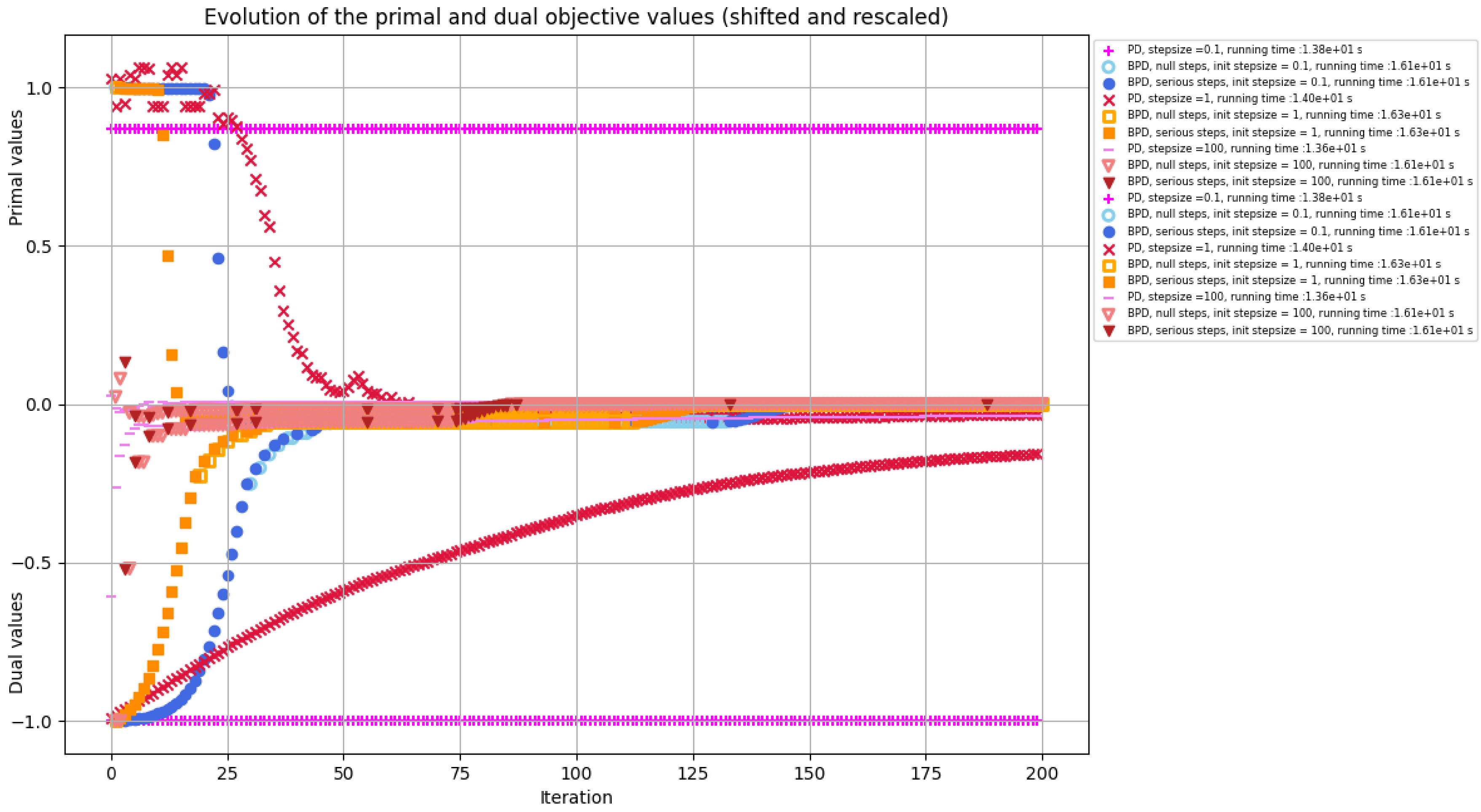
What does it mean ?

We found the dual optimal... but we don't know it yet! (stopping test is a sufficient condition for optimality, not a necessary one)

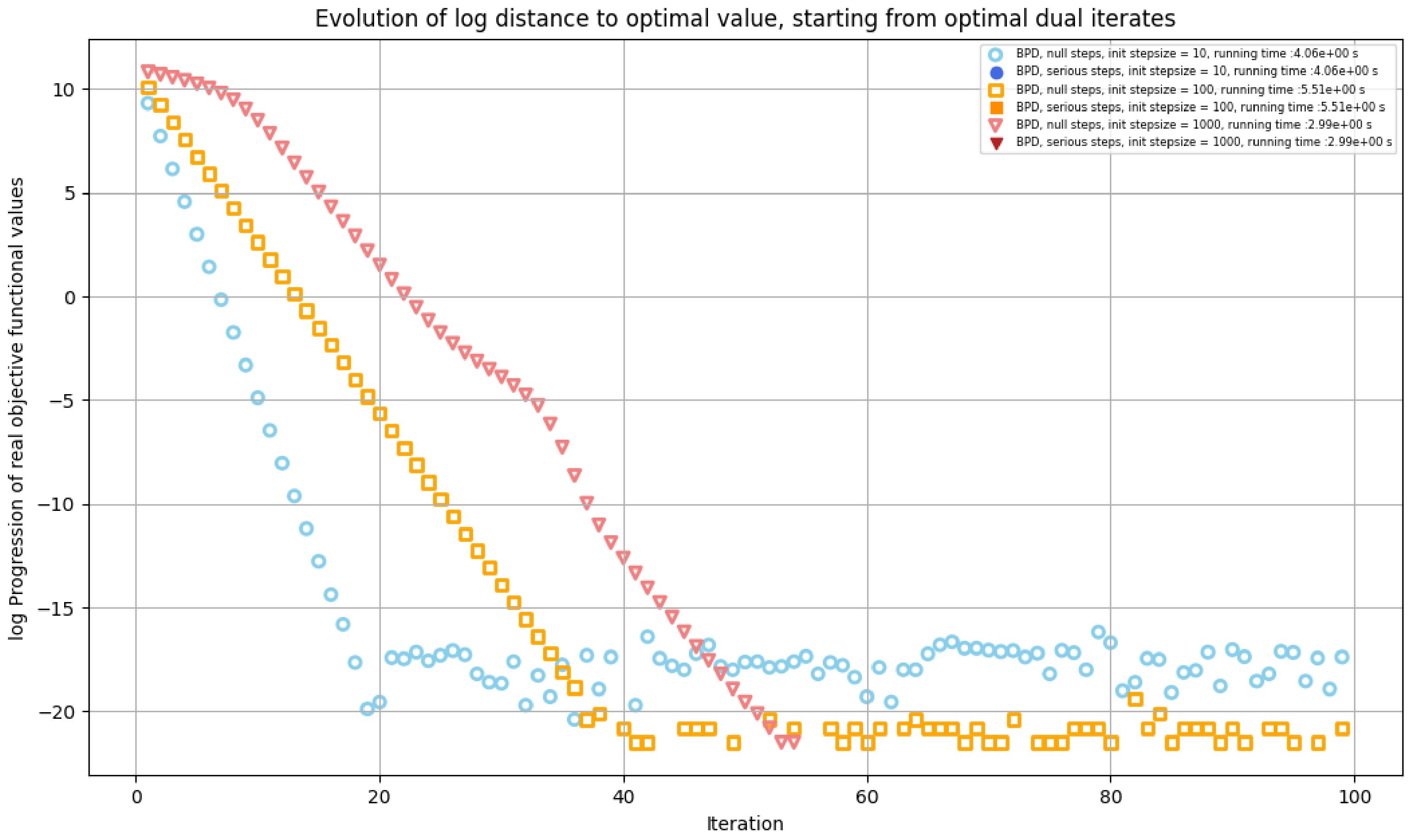
We proved that it takes a "linear time" to realize it !

⚠ This is a convergence rate analysis… not one of arithmetic complexity !

Simulations on the multizonal optimization problem



Simulations on the multizonal optimization problem (2)



Conclusion… what's next?

Extend the convergence results to non subspace constraint sets

Allowing to use a model of the objective function itself

Why? Extension to not so easy subproblems

How? With a *super-serious* descent test ? More soon!

Thank you for your time and attention 😌