# Solving 2-stage DRO with mixed-integer ambiguity sets via a hybrid Benders and C&C

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# Outline

# 1. Defender-Attacker-Defender

$$\underbrace{\min_{x\in\mathcal{X}} c^T x}_{\text{Supplier}} + \underbrace{\max_{\mathbb{P}_Z\in\mathcal{P}_Z} \max_{\mathbb{P}_\xi\in\mathcal{P}_\xi} \mathbb{E}_{\mathbb{P}_Z}\mathbb{E}_{\mathbb{P}_\xi}}_{\text{"Nature"}} \underbrace{\min_{y\in\mathcal{Y}(x,Z,\xi)} q^T y}_{\text{Operator } Q(x,Z,\xi)} \tag{1.1}$$

Figure 1: 2-stage DRO with continuous and discrete uncertainties, resp. $\xi$ and $Z$

1. (Supplier) $x$: the defender's investments (integer)
2. (Attacker) $Z$: the attacker's disruption (binary)
3. (Operator) $y$: the operator's operations (continuous)

**Modelling Discrete Adverse Events: stronger modelling power**

Events with negative outcomes can be modelled by discrete random variables $Z$, e.g., physical failure, connectivity loss, unsafe operation, etc.

Some relevant problems: network design, facility location, network hardening, vehicle routing

**Assume relatively complete recourse**

As long as "last resort" operational actions are modelled, the assumption is verified.

### Random variables

Let $Z$ be a discrete random vector modelling $k$ failures with support

$$\mathcal{Z} : \left\{ z \in \{0,1\}^N : \sum z \geq N - k \right\}. \tag{1.2}$$

Let $\xi$ be a continuous random vector with polyhedral support

$$\Xi = \{\xi \in \mathbb{R} : C\xi \leq d\}. \tag{1.3}$$

We are given empirical distributions $\hat{\mathbb{P}}_Z^J$ and $\hat{\mathbb{P}}_\xi^I$ of $Z$ and $\xi$, resp. with $J$ and $I$ samples.

### Ambiguity sets: $\phi$-divergence and $p$-Wasserstein distance

$$\mathcal{P}_Z(\kappa) = \left\{ \mathbb{P} \in \mathcal{M}(\mathcal{Z}) : I_\phi\left(\mathbb{P}, \hat{\mathbb{P}}_Z^I\right) \leq \kappa \right\} \tag{1.4}$$

$$\mathcal{P}_\xi(\varepsilon) = \left\{ \mathbb{P} \in \mathcal{M}(\Xi) : W_p\left(\mathbb{P}, \hat{\mathbb{P}}_\xi^J\right) \leq \varepsilon \right\} \tag{1.5}$$

In our work, we choose $\phi(t) = |t-1|$ (total variation) and $p = 1$.

# 2. Approximating the robust recourse problem

The worst-case distributions $\mathbb{P}_Z^*, \mathbb{P}_\xi^*$ are given by the robust recourse problem

$$\max_{\mathbb{P}_Z \in \mathcal{P}_Z} \max_{\mathbb{P}_\xi \in \mathcal{P}_\xi} \mathbb{E}_{\mathbb{P}_Z} \mathbb{E}_{\mathbb{P}_\xi} Q(x, Z, \xi) \tag{2.1}$$

The above is difficult to solve.

Instead, we can naively approximate by taking the supremum successively

$$\begin{aligned} \max_{\mathbb{P}_Z} \max_{\mathbb{P}_\xi} \mathbb{E}_{\mathbb{P}_Z} \mathbb{E}_{\mathbb{P}_\xi} Q(x, Z, \xi) &\leq \overbrace{\max_{\mathbb{P}_Z} \mathbb{E}_{\mathbb{P}_Z} \overbrace{\max_{\mathbb{P}_\xi} \mathbb{E}_{\mathbb{P}_\xi} Q(x, Z, \xi)}^{\psi(x, Z)}}^{v(x)} \\ &\leq \max_{\mathbb{P}_Z} \mathbb{E}_{\mathbb{P}_Z} \psi(x, Z) \end{aligned} \tag{2.2}$$

Roughly, we consider a finite number of independent subproblems $\psi(x, Z(\omega))$.

# 2.2 Scenario reduction: truncated problem $v(x)$

The number of outcomes $Z(\omega), \forall \omega \in \mathcal{Z}$ is finite, but combinatorial. Let $\Omega = |\mathcal{Z}|$ and $p_j = \mathbb{P}_Z(z_j), j \in [\Omega]$. Then, we write equivalently

$$\max_{\mathbb{P}_Z} \mathbb{E}_{\mathbb{P}_Z} \psi(x, z_j) = \max_{p_j} \sum_{j=1}^{\Omega} p_j \psi(x, z_j) \tag{2.3}$$

**Scenario reduction**

To reduce the combinatorial complexity, we truncate the number of scenarios $\Omega$ to be exactly $J$, the number of available data points.

**Assume that the dataset $\{z_j\}_{j=1}^{J}$ is representative of worst-case scenarios.**

**Scenario reduction $v(x)^J$**

Consider the quantized problem

$$\nu^J(x) := \max_{p_j} \sum_{j=1}^{J} p_j \psi(x, z_j) \tag{2.4}$$

**Conservative Decision problem**

Adding back the 1st stage gives us an upper bound of the original problem.

$$\min_{x \in \mathcal{X}} c^T x + v(x) = \min_{x \in \mathcal{X}} c^T x + \max_{\mathbb{P}_Z} \mathbb{E}_{\mathbb{P}_Z} \psi(x, Z) \tag{2.5}$$

where $\psi(x, z)$ is the robust recourse problem wrt to $\xi$.

$$\psi(x, z) := \max_{\mathbb{P}_\xi \in \mathcal{P}_\xi} \mathbb{E}_{\mathbb{P}_\xi} Q(x, z, \xi) \tag{2.6}$$

# 3. Reformulations

**Reformulation of $\psi(x, z)$ (Mohajerin Esfahani and Kuhn 2018)**

With dataset $\left\{\hat{\xi}_i\right\}_{i=1}^{I}$, we can write the strong duality reformulation as follows

$$
\begin{aligned}
\psi(x, z) :=& \\
& \min_{\theta, s_i, \gamma_{i\omega}} \theta\varepsilon + \frac{1}{I}\sum_{i=1}^{I} s_i \\
& \quad a_\omega^T \hat{\xi}_i + b_\omega + \gamma_{i\omega}^T \left(d - C\hat{\xi}_i\right) \le s_i \quad \forall i \in [\![1, I]\!], \forall \omega \in \mathcal{V}(z) \\
& \quad \left\|C^T \gamma_{i\omega} - a_\omega\right\|_* \le \theta \quad \forall i \in [\![1, I]\!], \forall \omega \in \mathcal{V}(z) \\
& \quad \gamma_{i\omega} \ge 0 \quad \forall i \in [\![1, I]\!], \forall \omega \in \mathcal{V}(z)
\end{aligned}
\tag{3.1}
$$

where $\mathcal{V}(z)$ is the set of all extreme points of the dual feasible space of $Q(x, z, \xi)$, $(a_\omega, b_\omega)$ are the coefficients of the linear function $Q(x, z, \xi) : \xi \mapsto a_\omega^T \xi + b_\omega$ for the vertex $\omega$.

**Formulation of $\upsilon(x)$ (Ben-Tal et al. 2013)**

The Lagrange dual function of $\upsilon(x)$ is given by

$$g(\alpha, \beta) := \beta + \kappa\alpha + \alpha \sum_{j=1}^{J} \hat{p}_j \phi^* \left( \frac{\psi(x, z) - \beta}{\alpha} \right), \alpha \geq 0 \tag{3.2}$$

Given the variation distance $\phi(t) = |t - 1|$, the Lagrange dual problem of $\upsilon(x)$ is

$$\begin{aligned} \upsilon(x) := \inf_{\alpha, \beta} & \ \beta + \kappa\alpha + \sum_{j=1}^{L} \hat{p}_j u_j \\ \text{s.t. } & u_j \geq -\alpha && \forall j \in [\![1, J]\!] \\ & u_j \geq \psi(x, z_j) - \beta && \forall j \in [\![1, J]\!] \\ & \psi(x, z_j) - \beta \leq \alpha && \forall j \in [\![1, J]\!] \\ & \alpha \geq 0, \beta \text{ free} \end{aligned} \tag{3.3}$$

**Robust recourse problem**

Developping $v(x)$ with the dual formulation of $\psi(x, z_j), \forall j \in [\![1, J]\!]$, we have

$$
\begin{aligned}
v(x) := \inf_{\alpha,\beta}\ & \beta + \kappa\alpha + \sum_{j=1}^{L} \hat{p}_j u_j \\
\text{s.t. } & u_j \geq -\alpha && \forall j \in [\![1, J]\!], \forall i \in [\![1, I]\!] \\
& u_j \geq r_j && \forall j \in [\![1, J]\!], \forall i \in [\![1, I]\!] \\
& r_j \leq \alpha && \forall j \in [\![1, J]\!], \forall i \in [\![1, I]\!] \\
& \alpha \geq 0, \beta \text{ free} \\
& r_j := \theta_j \varepsilon + \frac{1}{I}\sum_{i=1}^{I} s_{ji} - \beta \\
& && \text{finite but combinatorial} \\
& a_\omega^T \hat{\xi}_i + b_\omega + \gamma_{ji\omega}{}^T \left(d - C\hat{\xi}_i\right) \leq s_{ji} && \forall j \in [\![1, J]\!], \forall i \in [\![1, I]\!], \forall \omega \in \mathcal{V}(z) \\
& \left\| C^T \gamma_{ji\omega} - a_\omega \right\|_* \leq \theta && \forall j \in [\![1, J]\!], \forall i \in [\![1, I]\!], \forall \omega \in \mathcal{V}(z) \\
& \gamma_{ji\omega} \geq 0 && \forall j \in [\![1, J]\!], \forall i \in [\![1, I]\!], \forall \omega \in \mathcal{V}(z)
\end{aligned}
\tag{3.4}
$$

one variable for each $\omega$

### Decompose with respect to ... what?

Benders:

- $Q(x, z, \xi)$ is a LP given $x \in \mathcal{X}$, otherwise it is bilinear. Then:
  1. Start with $x := x_0$
  2. Solve for a feasible solution $\tilde{x}$
  3. Solve $v(\tilde{x})$
  4. Create an optimality cut

C&CG:

- Given $x \in \mathcal{X}$, $z \in \mathcal{Z}$, $\psi(x, z)$ is combinatorial in the due to enumerating all the BFS of $Q(x, z, \xi)$. For every $\psi(x, z_j, \xi_i)$:
  1. Start with one vertex in a set $V(l, i) := \{\omega_0 \in \mathcal{V}(z)\}$
  2. Solve the current problem and get feasible solutions
  3. Find **"the best"** new vertex $\omega^k$ and add it to $V(l, i) = V(l, i) \cup \{w^k\}$
  4. Add the new variables and constraints associated with $\omega^k$

**(Generalized) Benders decomposition (Geoffrion 1972)**

The master problem $\left(\mathrm{MP}^{Q}\right)$ with $Q$ cuts is defined by

$$\begin{aligned} \min_{x\in\mathcal{X}} \ & c^T x + \nu \\ \text{s.t. } & \nu \geq \mathrm{SP}(x_0) \\ & \nu \geq \pi_q^T\left(x - x_q\right) + \mathrm{SP}\left(x_q\right) \ \forall q \in [\![Q]\!] \end{aligned} \tag{$\mathrm{MP}^Q$}$$

where (SP) is the subproblem $\upsilon(x)$ given by

$$\upsilon(x) := \max_{\mathbb{P}_Z\in\mathcal{P}_Z} \mathbb{E}_{\mathbb{P}_Z}\psi(x, Z) \tag{SP}$$

and $\pi_Q$ are the reduced costs of fixing $x = x_q$ in (SP).

We can initialize $\left(\mathrm{MP}^{Q}\right)$ with an initial feasible solution $x_0$.

**Dual values**

We need the reduced costs $\pi$ of $x$, so **$\upsilon(x)$ must be a convex, continuous** problem. This is why we separate the Benders and C&CG procedures.

**C&CG**

The reduced master problem is

$$\begin{aligned}
&\inf_{\alpha,\beta} \beta + \kappa\alpha + \sum_{j=1}^{L} \hat{p}_j u_j \\
&\text{s.t. } u_j \geq -\alpha, u_j \geq r_j, r_j \leq \alpha && \forall j \in [\![1, J]\!], \forall i \in [\![1, I]\!] \\
&\quad\ \alpha \geq 0, \beta \text{ free} \\
&\quad\ a_\omega^T \hat{\xi}_i + b_\omega + \gamma_{ji\omega}^T \left(d - C\hat{\xi}_i\right) \leq s_{ji} && \forall j \in [\![1, J]\!], \forall i \in [\![1, I]\!], \forall \omega \in V(j,i) \\
&\quad\ \left\| C^T \gamma_{ji\omega} - a_\omega \right\|_* \leq \theta && \forall j \in [\![1, J]\!], \forall i \in [\![1, I]\!], \forall \omega \in V(j,i) \\
&\quad\ \gamma_{ji\omega} \geq 0 && \forall j \in [\![1, J]\!], \forall i \in [\![1, I]\!], \forall \omega \in V(j,i)
\end{aligned} \tag{RMP}$$

where $V(j,i)$ is the set of vertices for a given scenario $\left(z_j, \xi_i\right)$ and represent a "worst-case realization".

**Pricing**

In order to populate $V(j, i)$, we need a new $\omega \in \mathcal{V}(z_j)$ from which we can derive a new constraint

$$a_\omega^T \hat{\xi}_i + b_\omega + \gamma_{ji\omega}^T \left(d - C\hat{\xi}_i\right) \leq s_{ji}. \tag{3.8}$$

$\mathcal{V}(z_j)$ is the dual feasible set of $Q(x, z_j, \xi_i)$, therefore we can simply solve the dual problem.

$$\begin{aligned} \min_{y \in \mathcal{Y}(x, Z, \xi)} \ & q^T y \\ \text{s.t. } & Wy \leq h - Tx \\ & V_Z y = g_\xi - S_\xi x \\ & x \in \mathcal{X} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max_{\lambda, \nu} \ & \lambda^T (Tx - h) + \nu^T \left(S_\xi x - g_\xi\right) \\ \text{s.t. } & W^T \lambda + V_z^T \nu + q = 0 \\ & \lambda \geq 0 \\ & \nu \text{ free} \end{aligned}$$

Then,

$$\mathcal{V}(z_j) := \{\omega := (\lambda, \nu) : W^T \lambda + V_z^T \nu + q = 0, \lambda \geq 0, \nu \text{ free}\} \tag{3.9}$$

**Pricing**

The pricing problem must be derived to "find $\omega$ that has the most violation" with respect to

$$a_\omega^T \hat{\xi}_i + b_\omega + \gamma_{ji\omega}^T \left(d - C\hat{\xi}_i\right) \le s_{ji}. \tag{3.10}$$

By construction, the LHS of (3.10) is the objective of the dual problem for a given $\omega$. Then, we also write

$$h\left(\lambda, \nu; \tilde{\xi}\right) := \max_{\omega \in \mathcal{V}(z_l)} a(\omega)^T \tilde{\xi} + b(\omega) + \gamma_{ji}^{*T} \left(d - C\tilde{\xi}\right) - s^*(ji) \tag{3.11}$$

given optimal dual values $\gamma_{ji}^*$ and $s_{ji}^*$ of the (RMP).

**However, unless we found a new scenario $\tilde{\xi}$, $h(\lambda, \nu)$ will always return an identical $\omega$.**

But, can’t you just enumerate them all??

But, can’t you just enumerate them all??

**No, not really.**

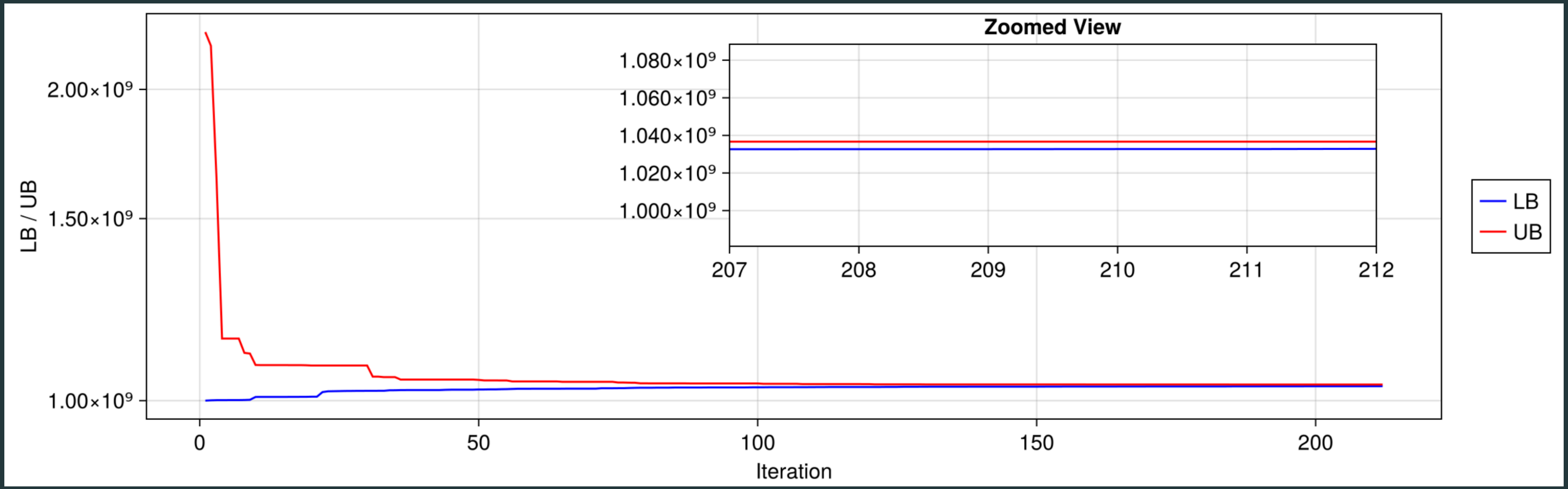


Figure 2: (Non)convergence of C&CG

**Solving the nonconvex nonlinear problem** $\max_{\mathbb{P}_\xi} \mathbb{E}_{\mathbb{P}_\xi} Q(x, z, \xi)$

Given $x \in \mathcal{X}$, $z \in \mathcal{Z}$, and samples $\left\{\hat{\xi}_i\right\}_{i=1}^{I}$, can we directly solve $\nu(x)$?

$$
\begin{aligned}
\min_{\xi_i} \; & \sum_{i}^{M} \tau_i Q(x, z, \xi) \\
\text{s.t.} \; & \sum_{j} \Pi_{ij} = \tau_i & \forall j \in [\![1, M]\!] \\
& \sum_{i} \Pi_{ij} = \frac{1}{I} & \forall i \in [\![1, I]\!] \\
& \sum_{i} \tau_i = 1 \\
& \sum_{ij} \left\| \xi_i - \hat{\xi}_j \right\| \Pi_{ij} \leq \varepsilon \\
& 0 \leq \Pi_{ij} \leq 1 \\
& 0 \leq \tau_i \leq 1 \\
& \xi_i \in \Xi
\end{aligned}
\tag{3.12}
$$

## Summary

1. Unlocking the modelling power of mixed-integer models is primordial for industrial applications.
2. We approximate a discrete-continuous "2-DRO" problem leveraging existing strong duality formulations to derive a finite, convex reformulation.
3. **WIP**: We propose an nested Benders and C&CG decomposition algorithm to solve the problem. (**pricing to be refined!**)

Thank you!

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- take robust *ex ante* mitigation and adaptation decisions under deep uncertainty of future disruptions caused by natural extremes
- plan effective ex post response and recovery, with emphasis on computational efficiency and scalability

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