Robust chance-constrained optimization with discrete distributions via a bundle approach

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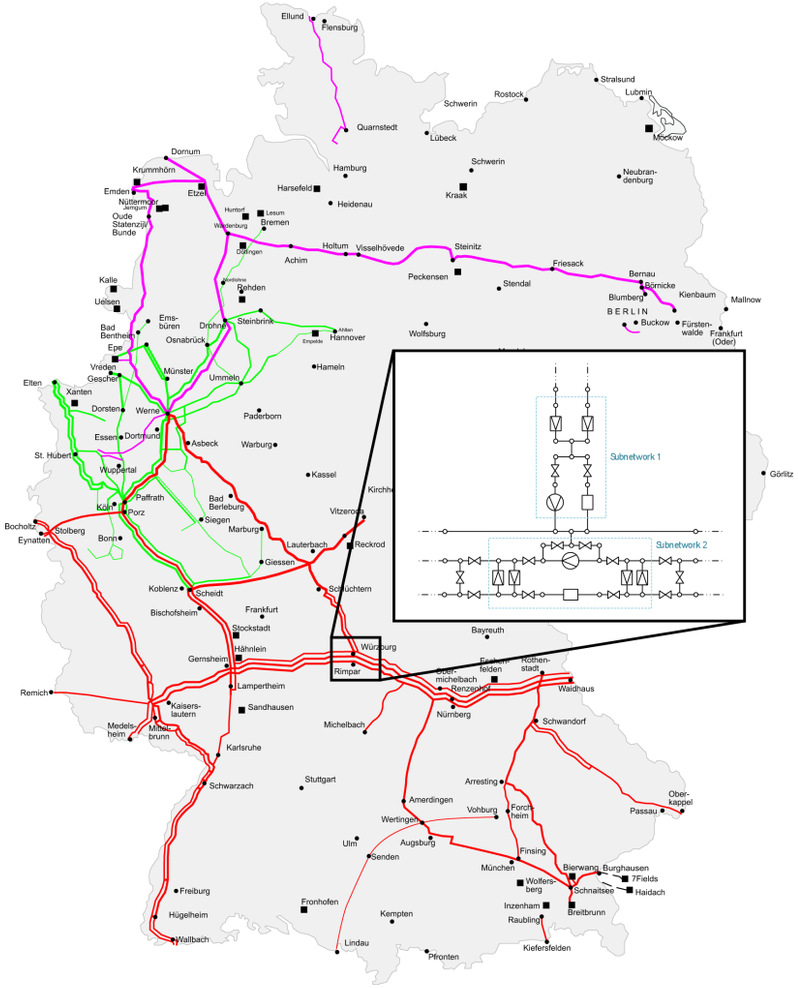
Motivation - Gas network with uncertain demands



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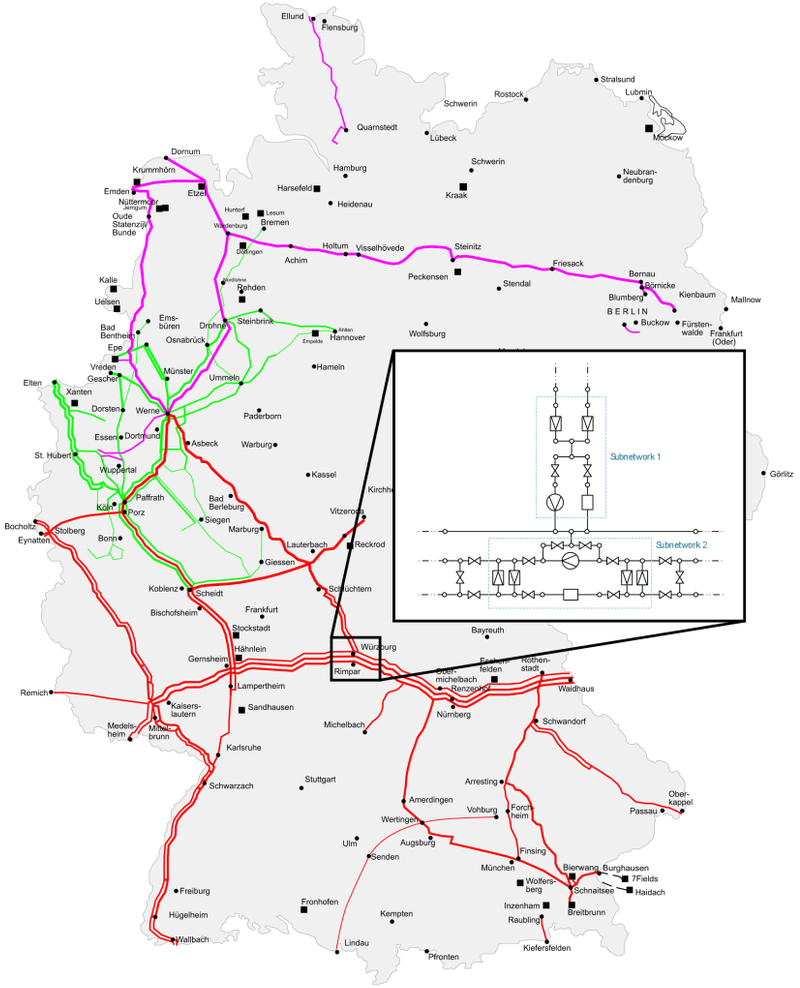

$\min$ costs

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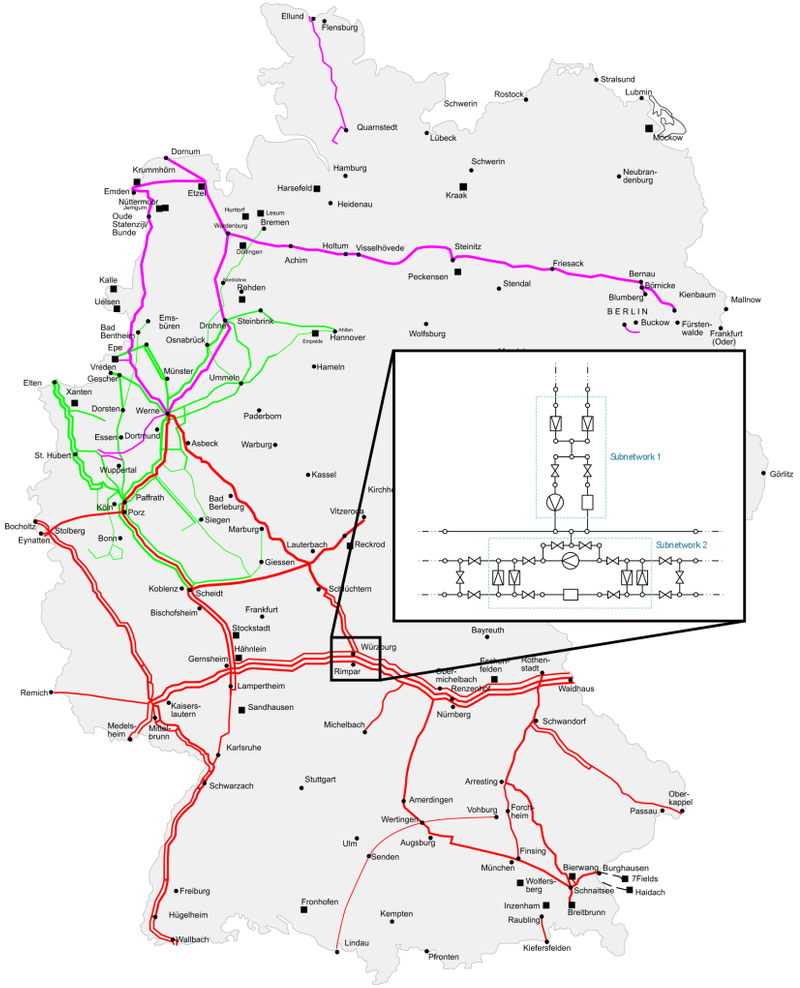

$$\min \quad \text{costs}$$

$$s.t. \quad \mathbb{P}(\text{pressure in bounds}) \geq 0.9 \quad \forall\, \mathbb{P} \in \mathcal{U}$$

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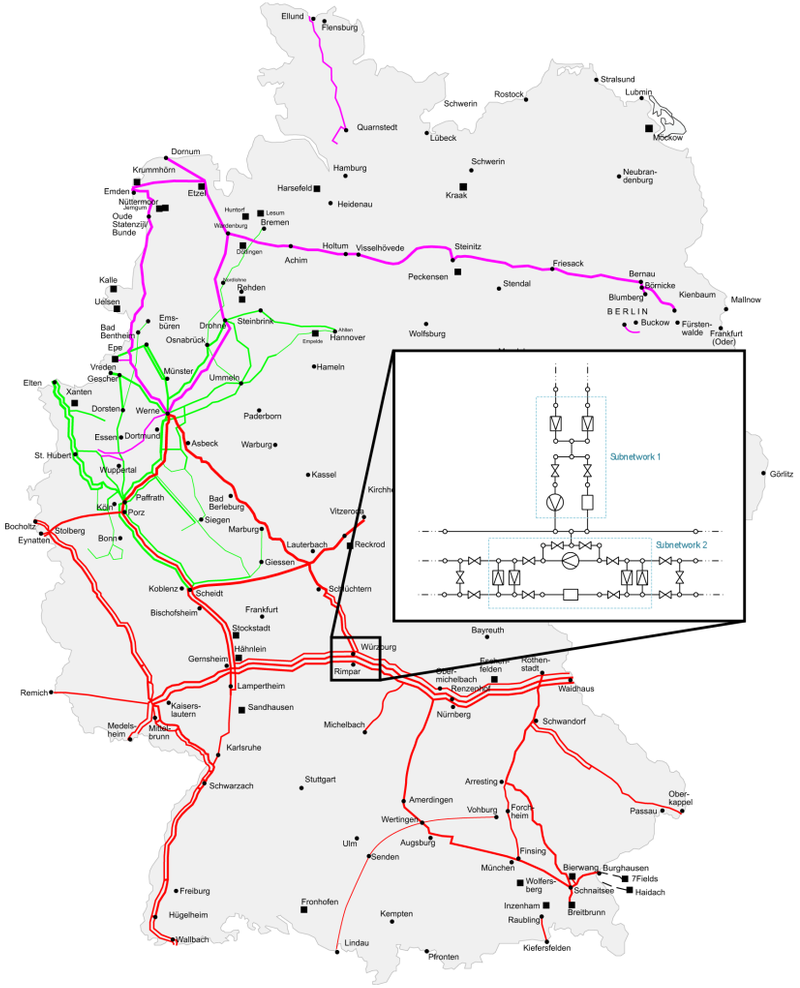

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$$\text{variables in bounds}$$

Robust chance-constrained optimization

Robust chance-constrained optimization

general idea

combination of two uncertainties

- uncertain data in the constraints

  $\rightarrow$ feasibility with a probability p less than 1

- uncertain probability distribution

  $\rightarrow$ robust protection against all distributions in ambiguity set $\mathcal{U}$

Robust chance-constrained optimization

general idea

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  → feasibility with a probability p less than 1
- uncertain probability distribution
  → robust protection against all distributions in ambiguity set $\mathcal{U}$

robust chance-constrained optimization problem

$$\min_{x} \; g_0(x)$$

$$s.t. \;\; \mathbb{P}_{\delta \sim \zeta} \left(g_j(x, \delta) \leq 0 \; \forall \; j \in \mathcal{J} \right) \geq p \qquad \forall \, \zeta \in \mathcal{U}$$

$$x \in \mathcal{X}$$

with $\mathcal{X} := \{x \in \mathbb{R}^n : \; h_i(x) \leq 0 \, \forall \, i \in \mathcal{I}, \; h_k(x) = 0 \, \forall \, k \in \mathcal{K}\}$

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- **surveys**

  Lin et al. (2022), Rahimian and Mehrotra (2022), . . .

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Overview of our solution approach[1]

our setting

[1] B., Liers, Stingl: https://opus4.kobv.de/opus4-trr154/564

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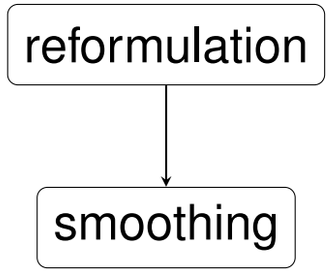

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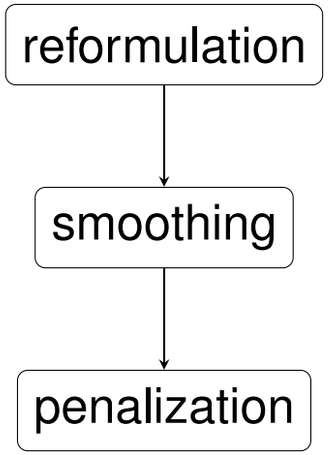

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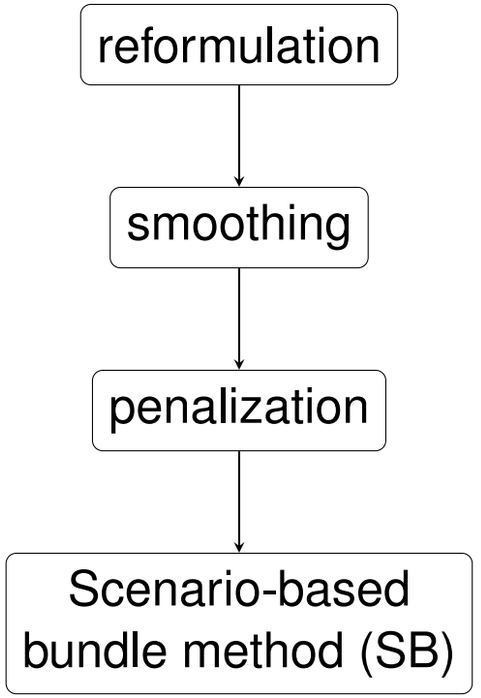

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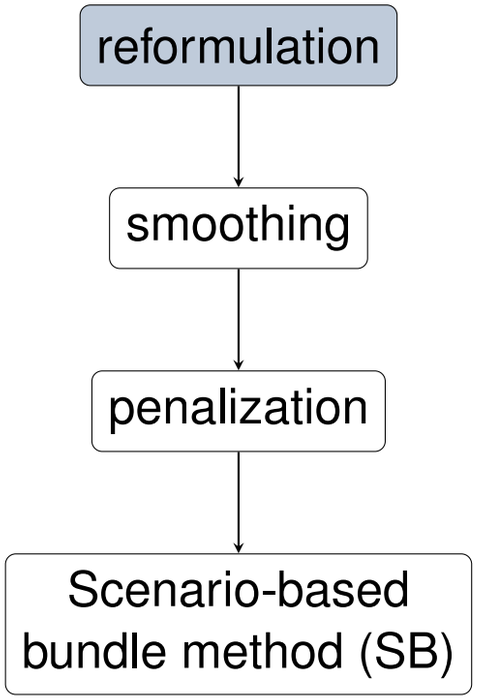
Equivalent reformulation

robust joint chance constraint

$$\mathbb{P}_{\delta\sim\zeta}\left(g_j(x,\delta)\leq 0\;\forall\, j\in\mathcal{J}\right)\geq p\quad\forall\,\zeta\in\mathcal{U}$$



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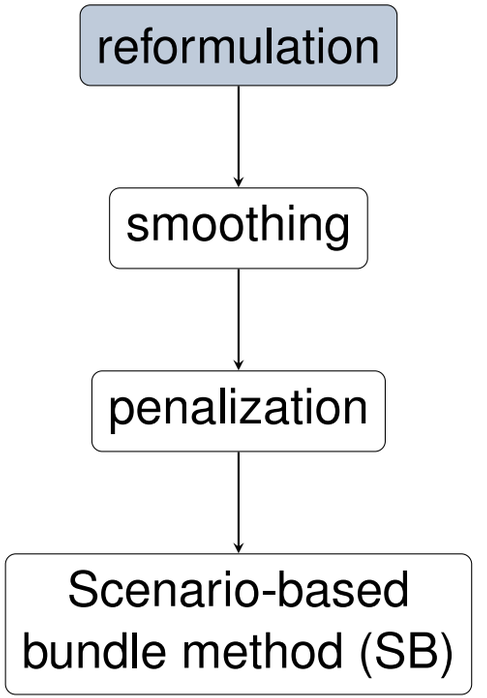
$\rightarrow$ equivalent reformulation to single chance constraint

$$g_j(x,\delta)\leq 0\quad\forall\, j\in\mathcal{J}\iff g(x,\delta):=\max_{j\in\mathcal{J}}\{g_j(x,\delta)\}\leq 0$$



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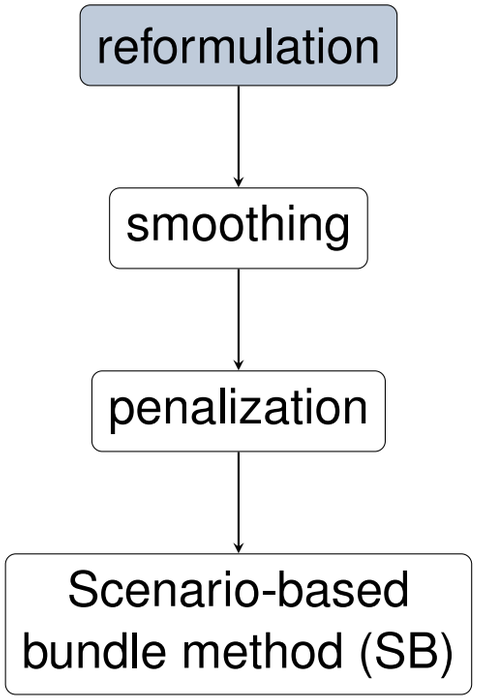
$$\sum_{i=1}^{|\Delta|}\zeta_i\chi_{(-\infty,0]}(g(x,\delta_i))\geq p\quad\forall\,\zeta\in\mathcal{U}$$



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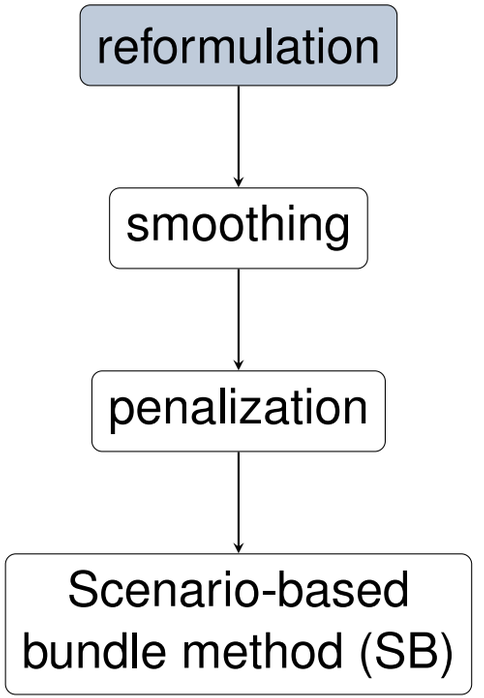
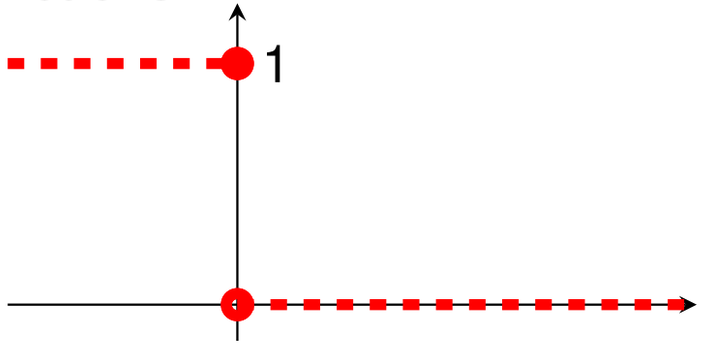


Smoothing approximation

robust chance constraint with indicator functions

$$\min_{\zeta \in \mathcal{U}} \sum_{i=1}^{|\Delta|} \zeta_i \chi_{(-\infty,0]}(g(x,\delta_i)) \geq p$$

$\rightarrow$ problem: indicator function not continuous



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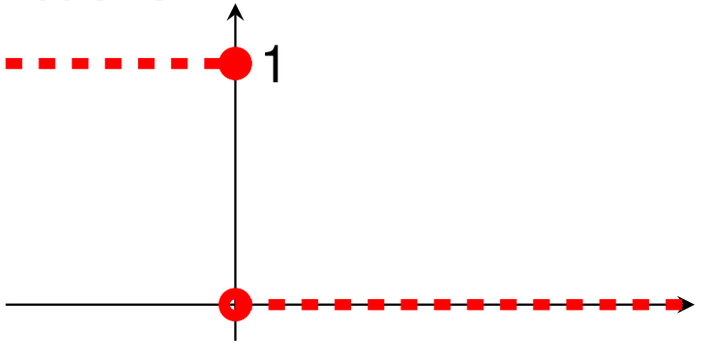

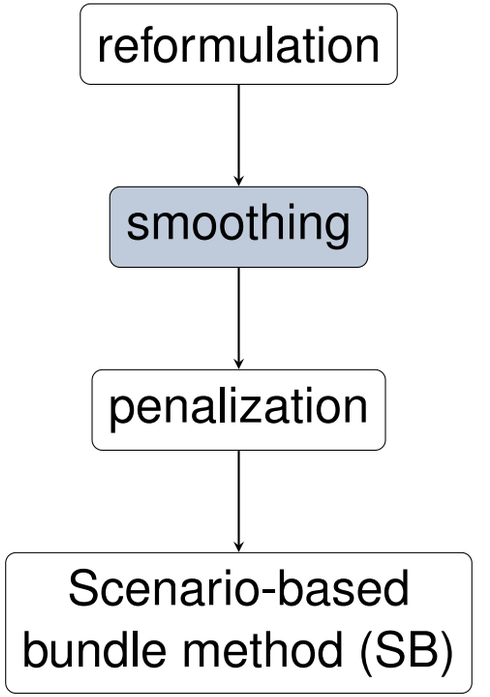

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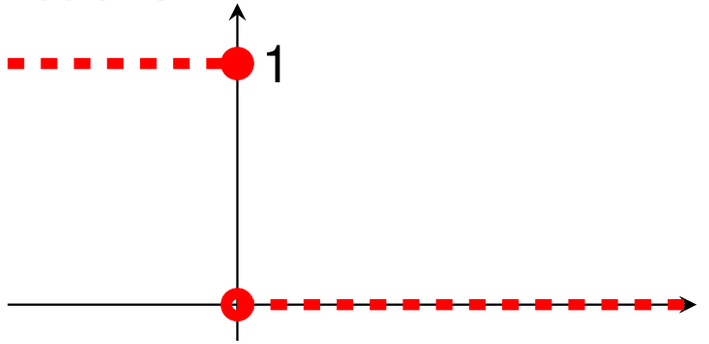
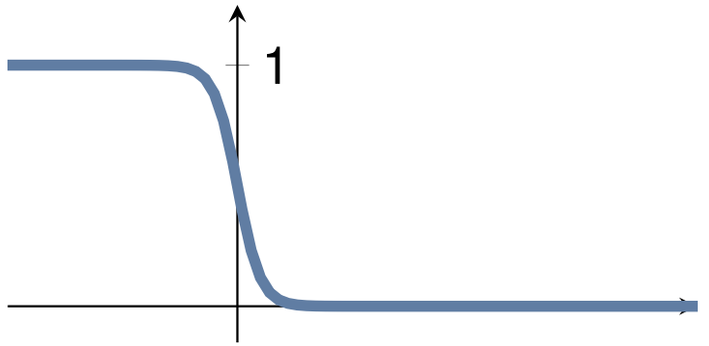
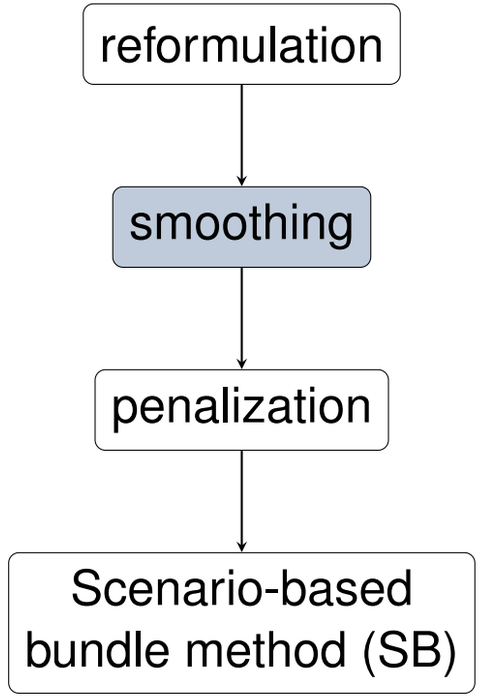


smoothing approximation

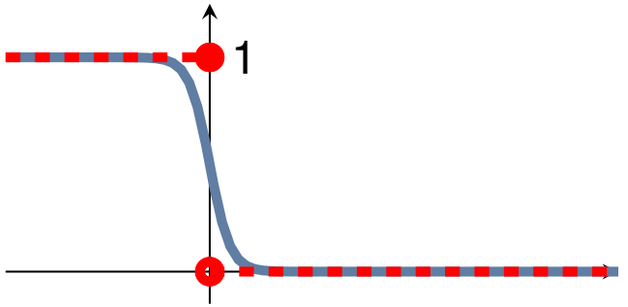
$$\min_{\zeta \in \mathcal{U}} \sum_{i=1}^{|\Delta|} \zeta_i s(g(x,\delta_i),\beta) \geq p$$

$\rightarrow$ approximate indicator function by continuously differentiable function s





Definition of smoothing functions



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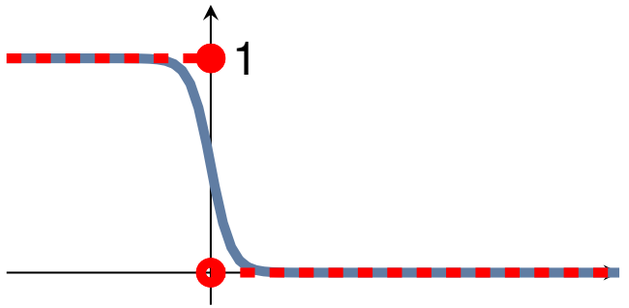
Smoothing function

A function $s : \mathbb{R} \times \mathbb{R}_{>0} \to [0, 1]$ is called a smoothing function for the indicator function $\chi_{(-\infty,0]}$ if

1) $s(\cdot, \beta)$ is continuously differentiable and monotonically decreasing for all $\beta > 0$
2) s is jointly continuous in both arguments
3) $\lim_{\beta\to\infty} s(0, \beta) = a$ with $a \in [0, 1]$
4) $\lim_{\beta\to\infty} s(y, \beta) = 0$ for all $y > 0$
5) $\lim_{\beta\to\infty} s(y, \beta) = 1$ for all $y < 0$.



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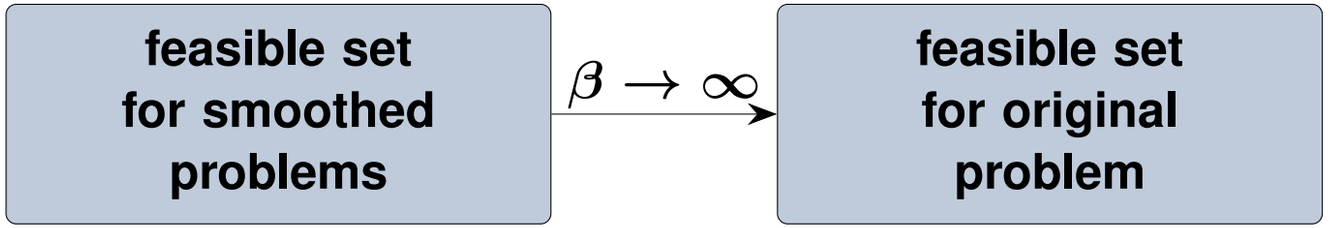

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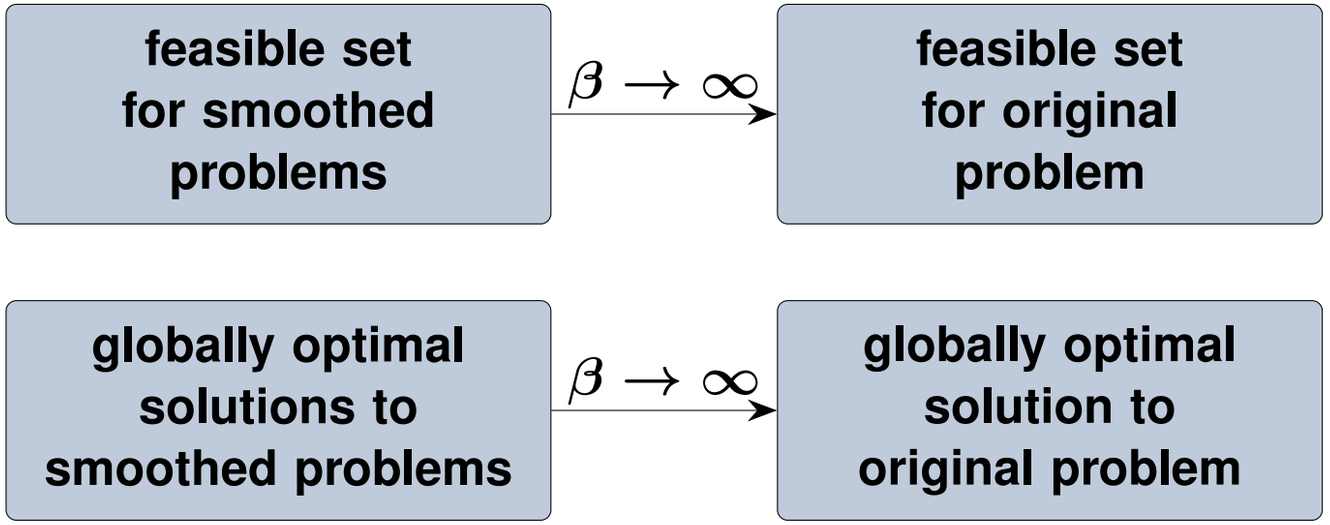

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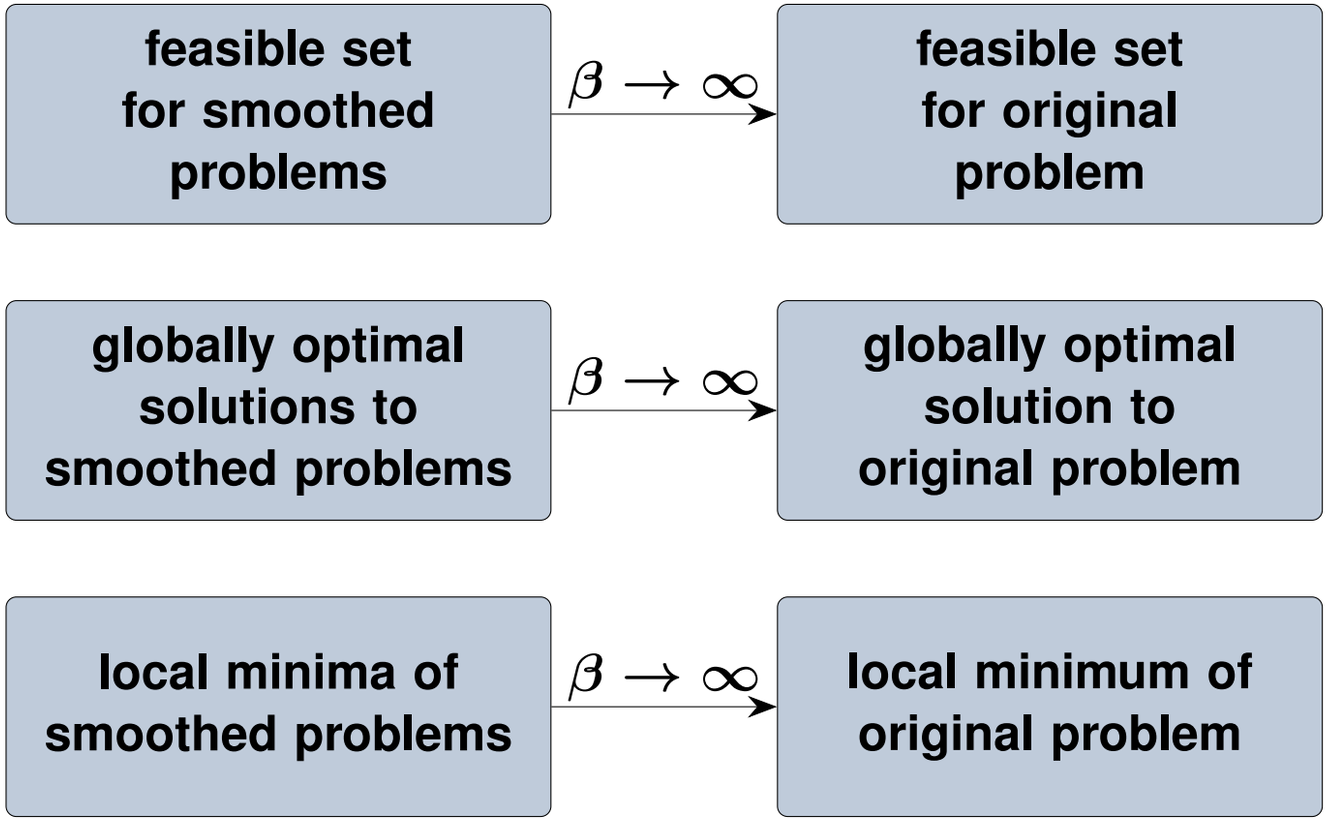

Penalty approximation

smoothed robust chance-constrained problem

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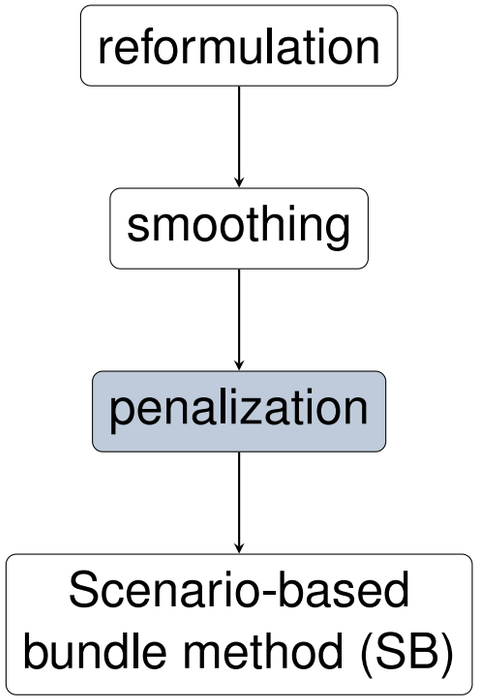

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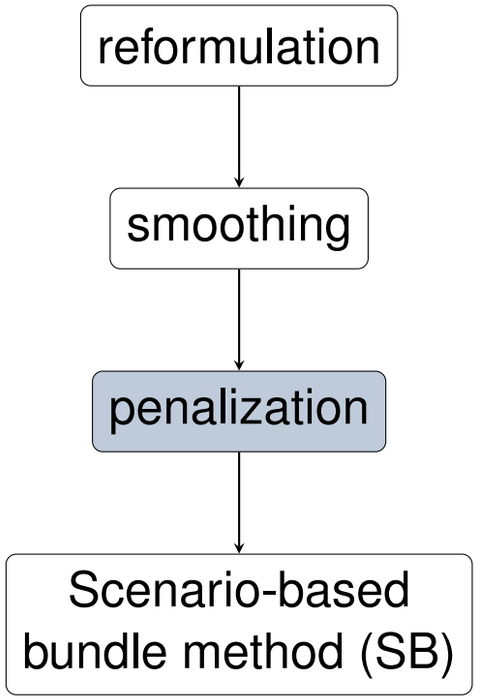

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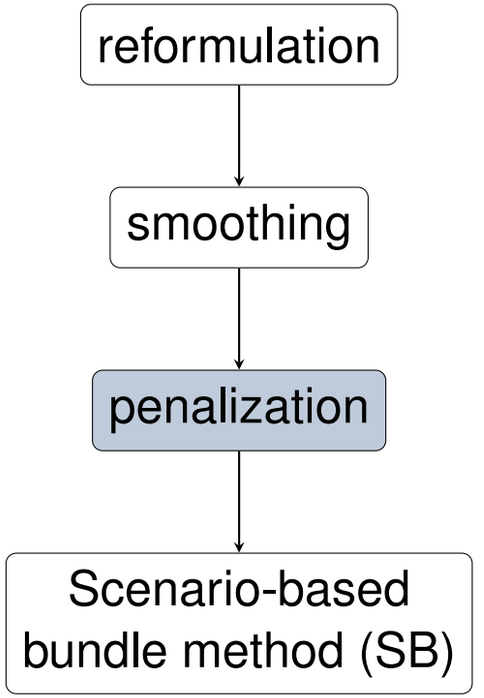

Properties of approximation problem

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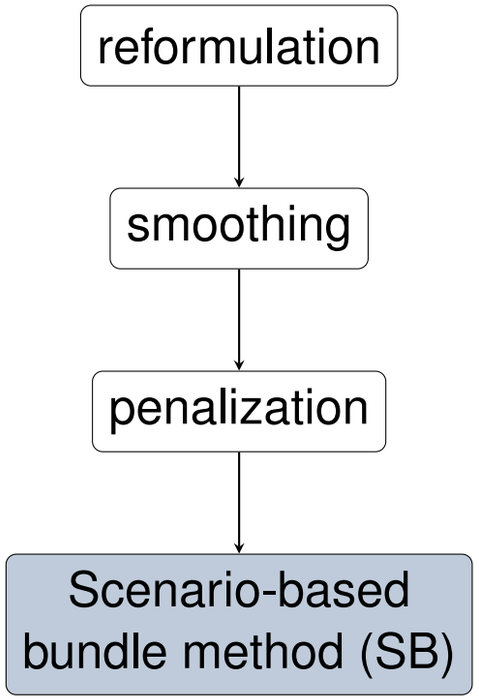

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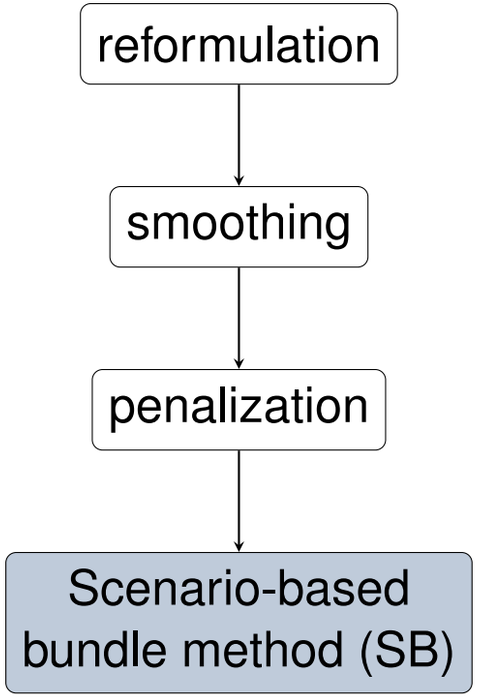
Scenario-based bundle method (SB)



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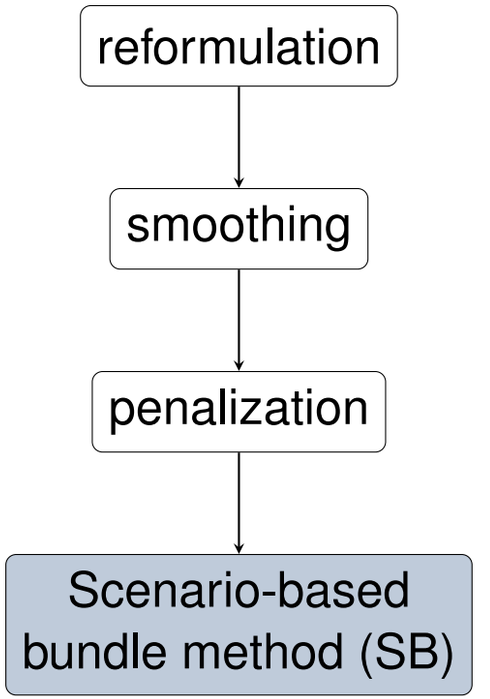

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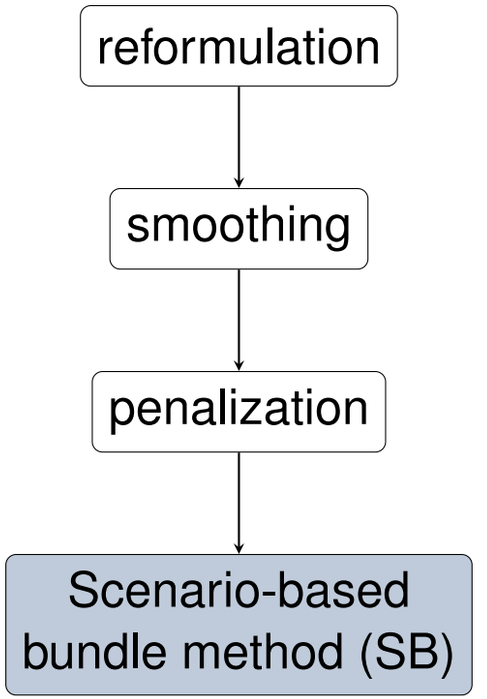

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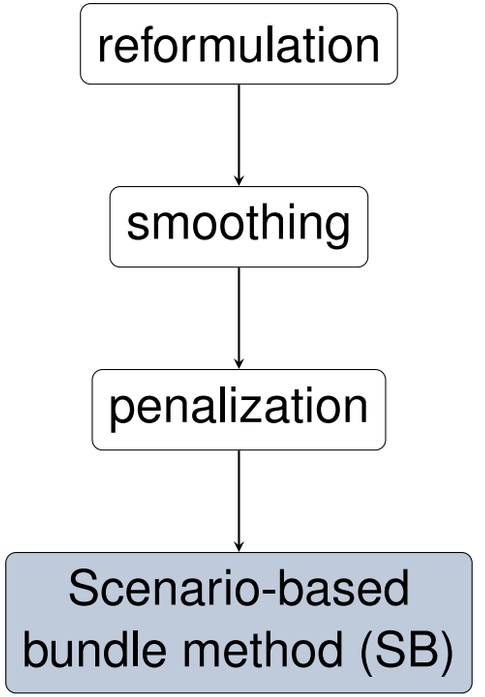
solution: use bundle solution for scenario-based postprocessing to guarantee feasibility



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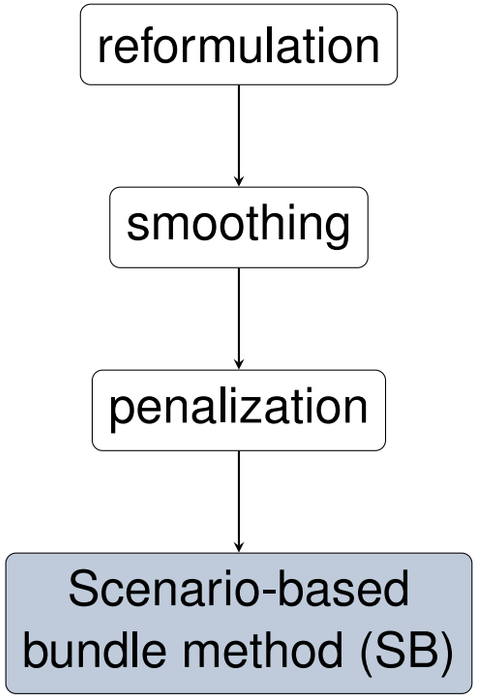

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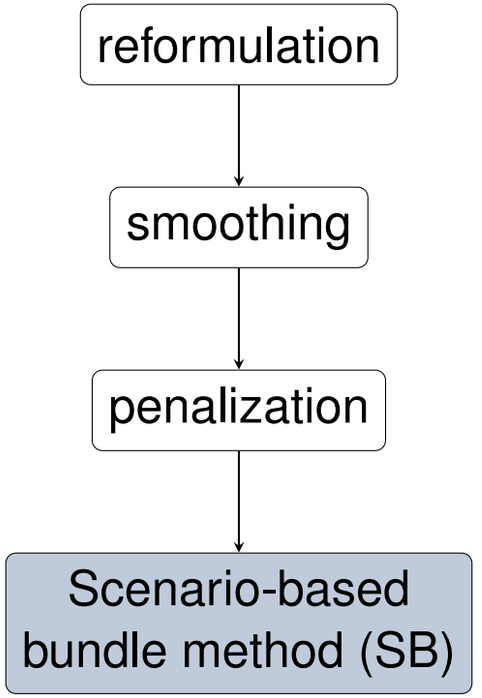

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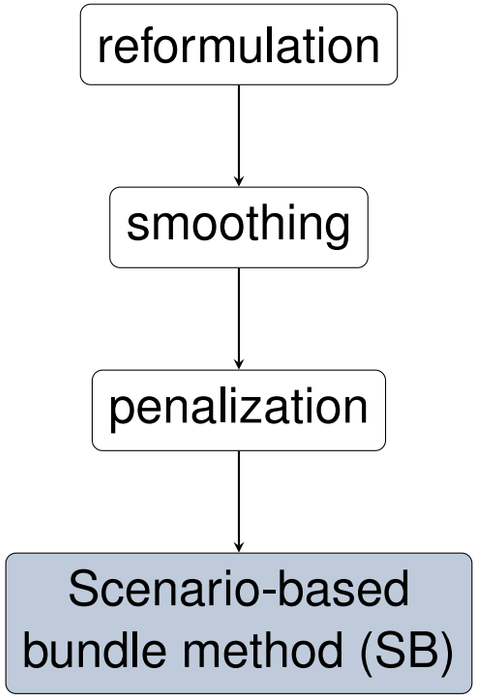

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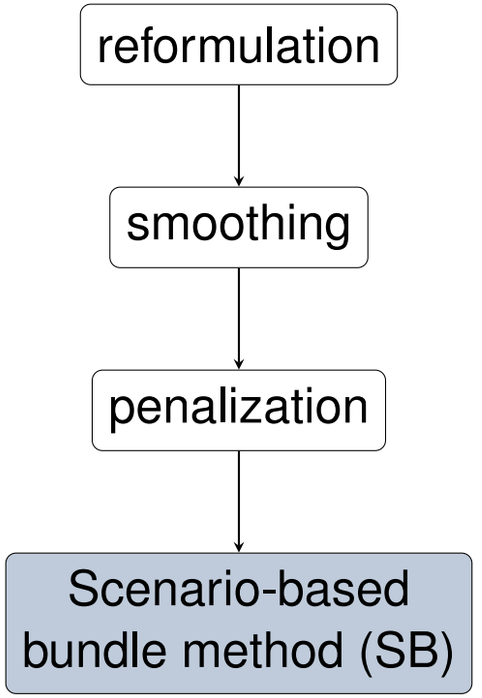

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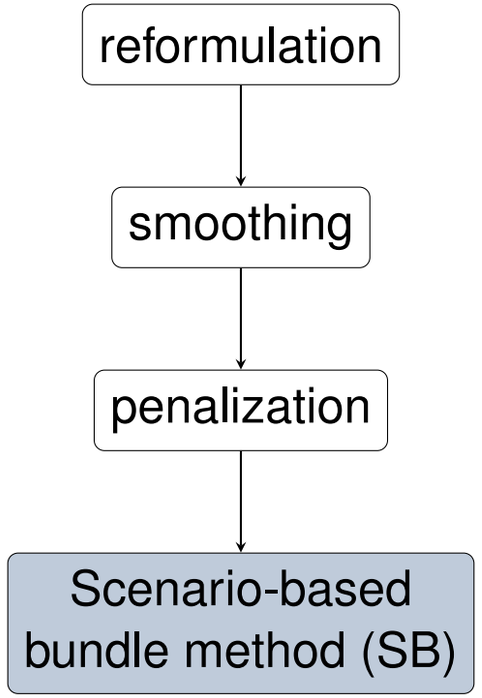

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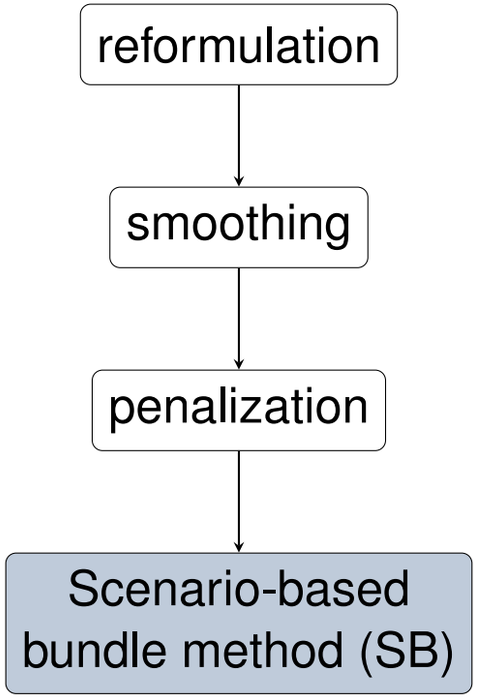

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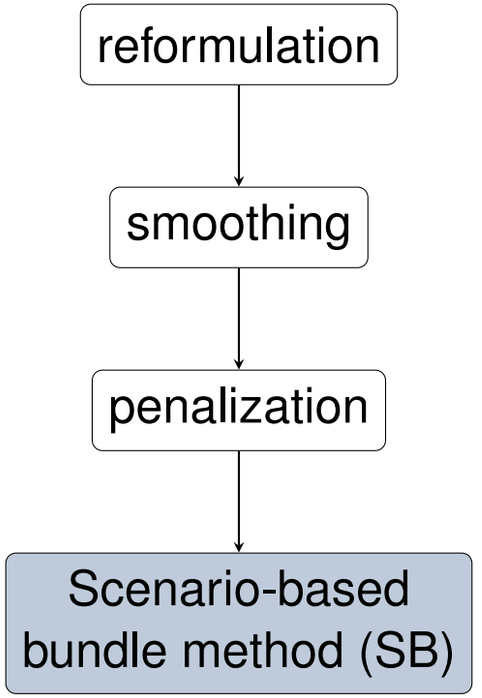
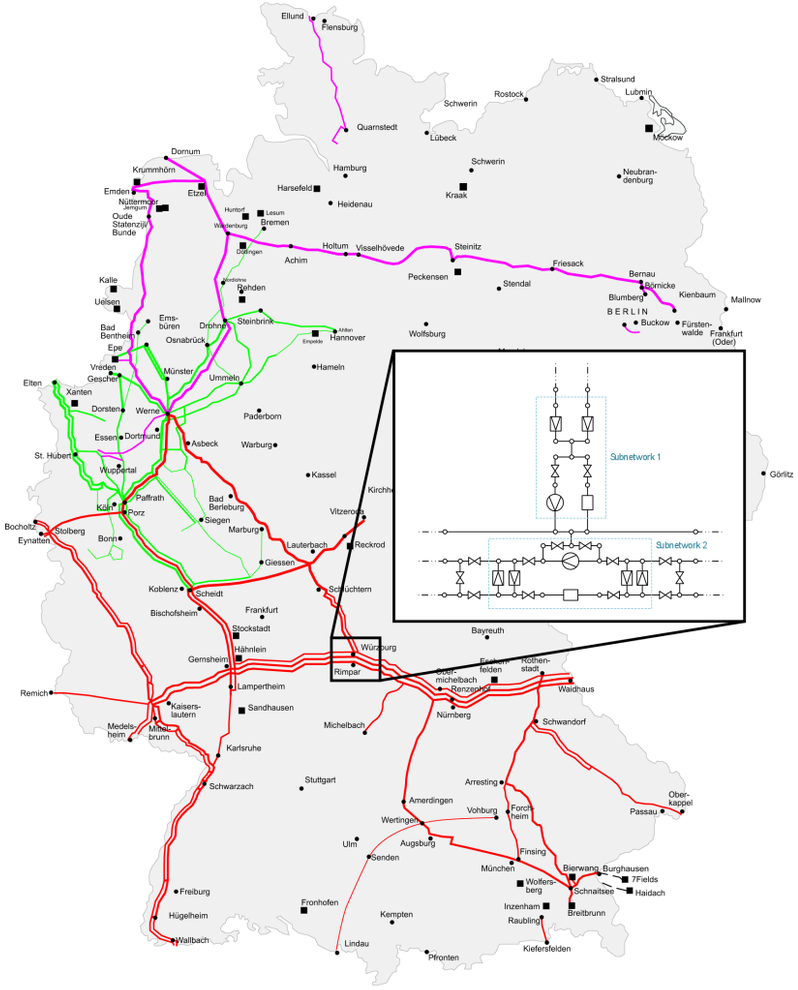


$\rightarrow$ guaranteed to find feasible solution, if underlying robust problem feasible

Robust chance-constrained gas transport problem



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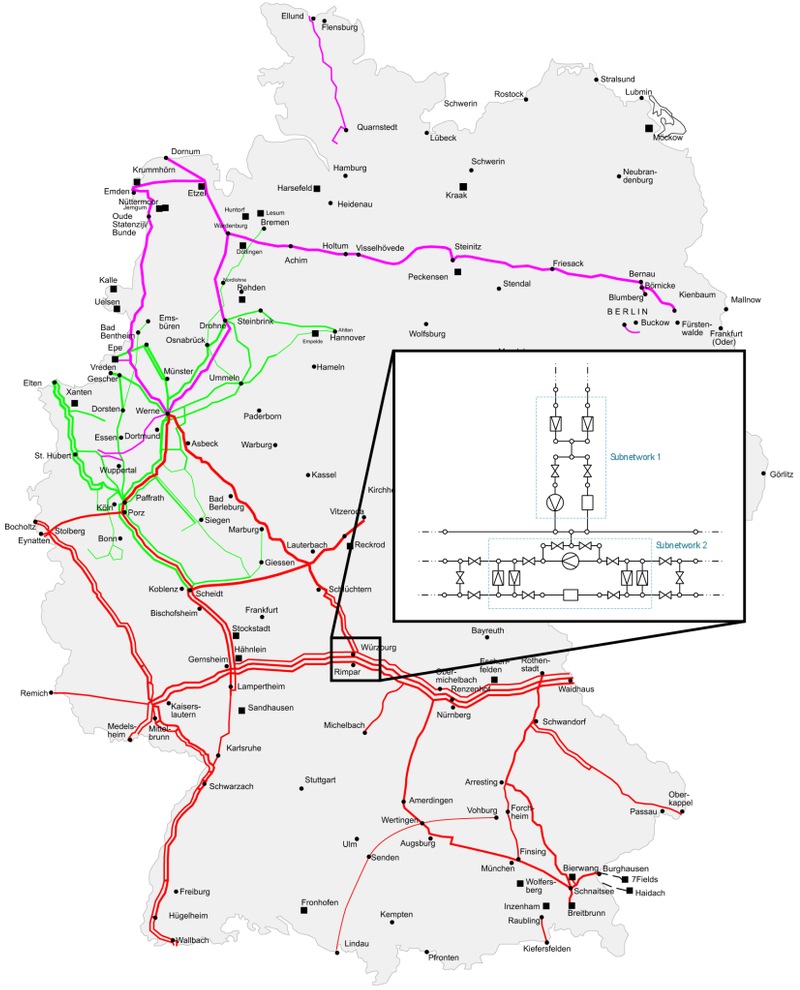

$\min$ costs

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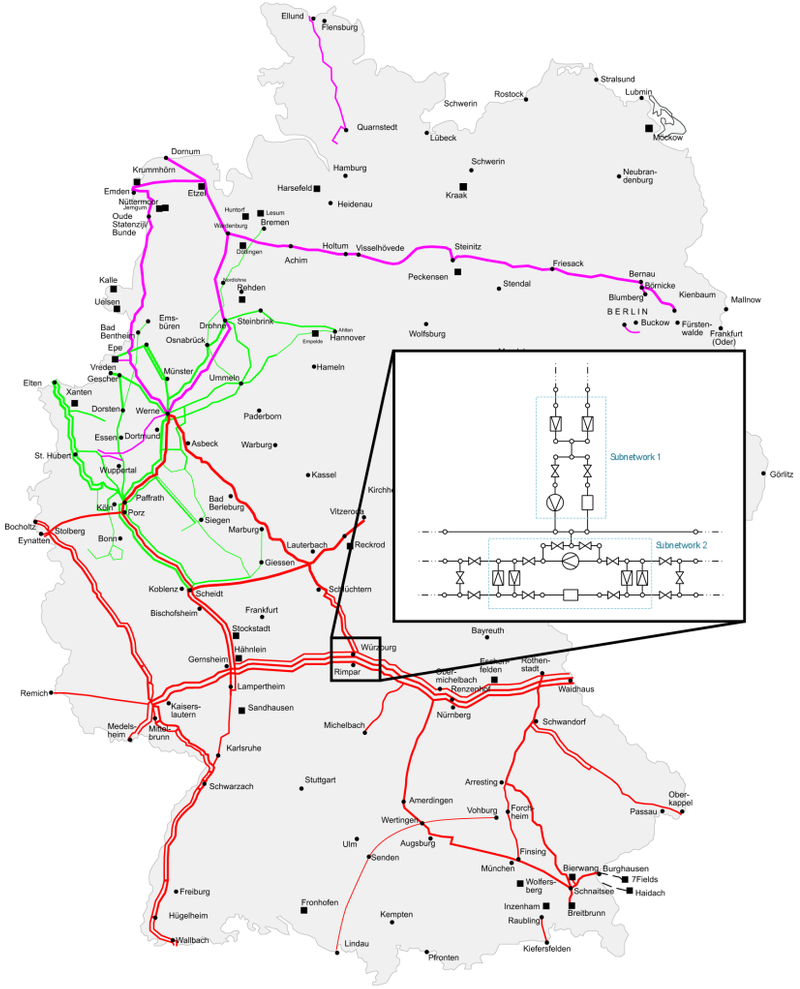

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$$s.t. \quad \mathbb{P}(\text{pressure in bounds}) \geq 0.9 \quad \forall\, \mathbb{P} \in \mathcal{U}$$

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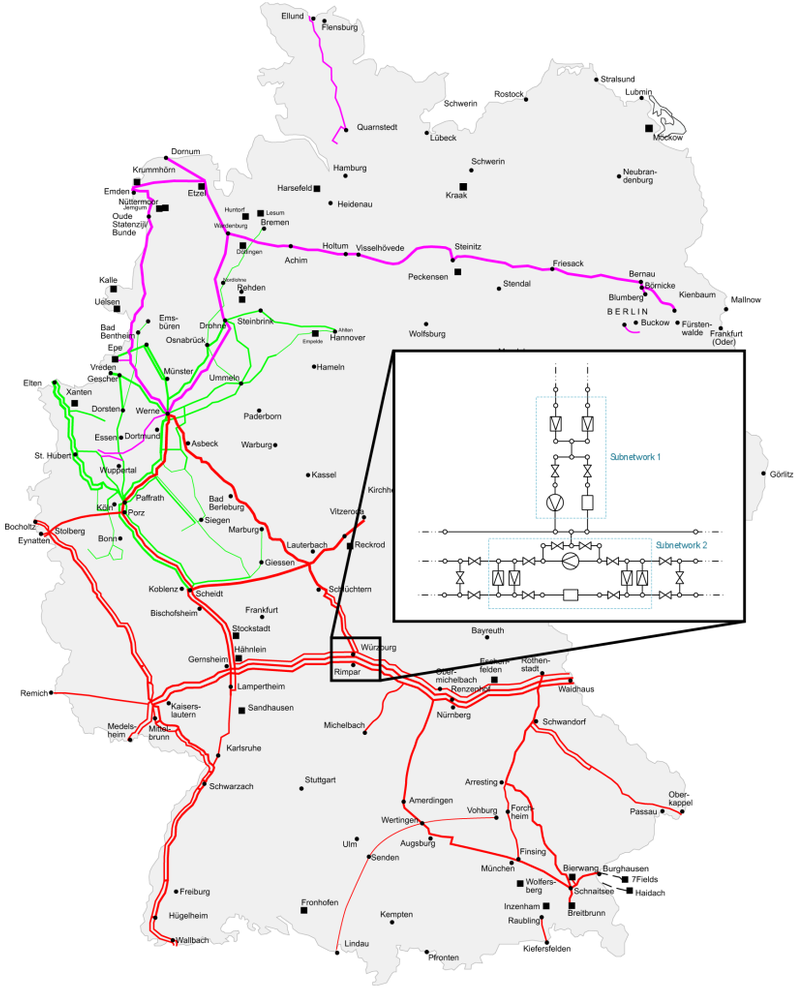

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- x: control of active elements
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- π: squared pressure variables
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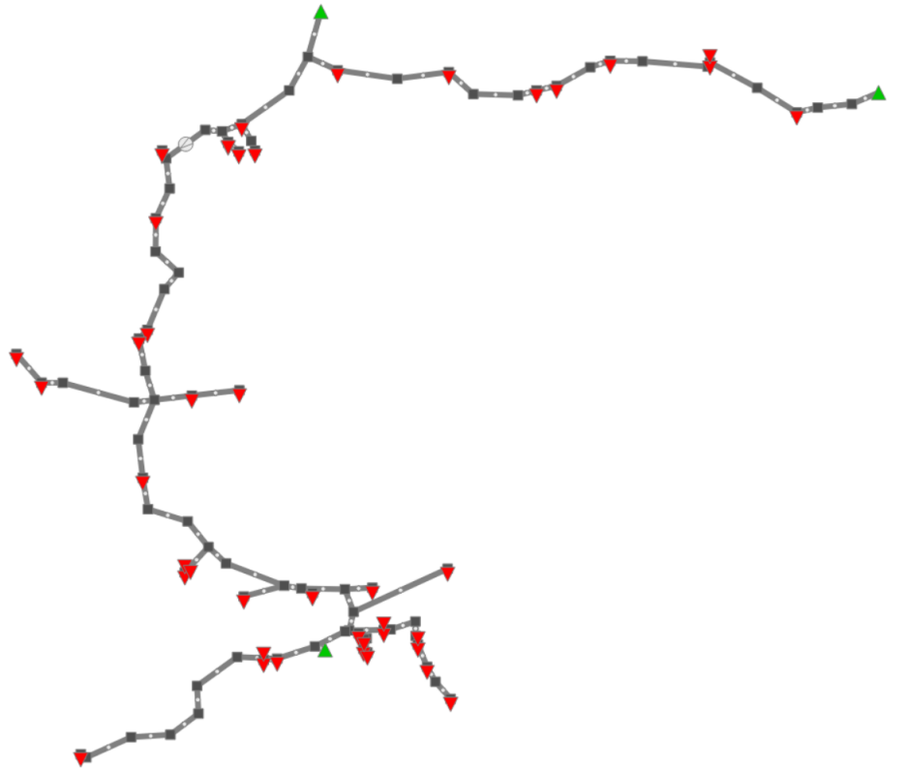
Numerical results for GasLib 134 (Schmidt et al. (2017))



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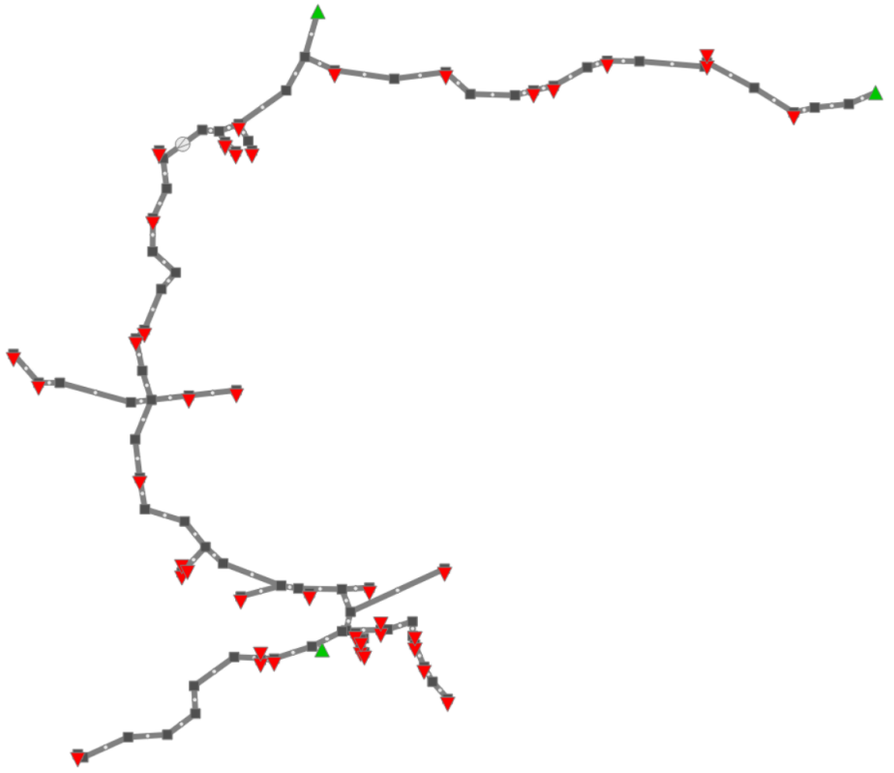

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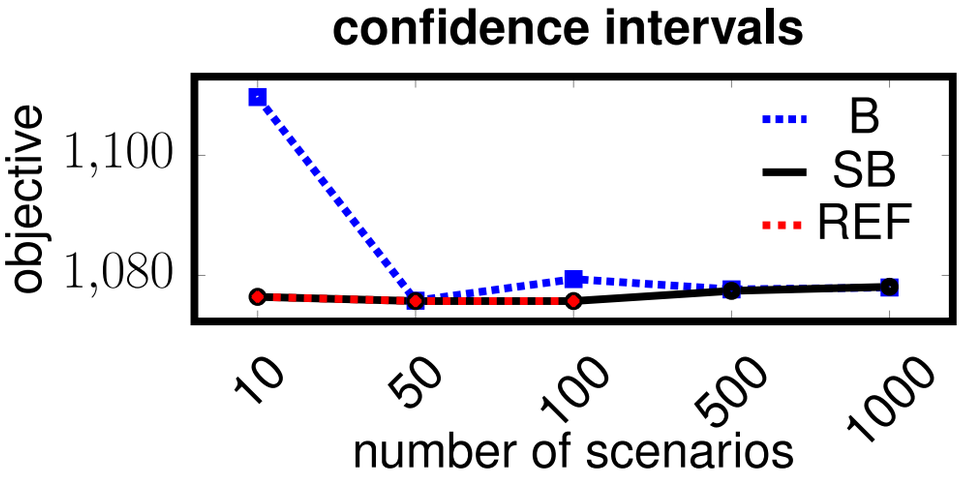
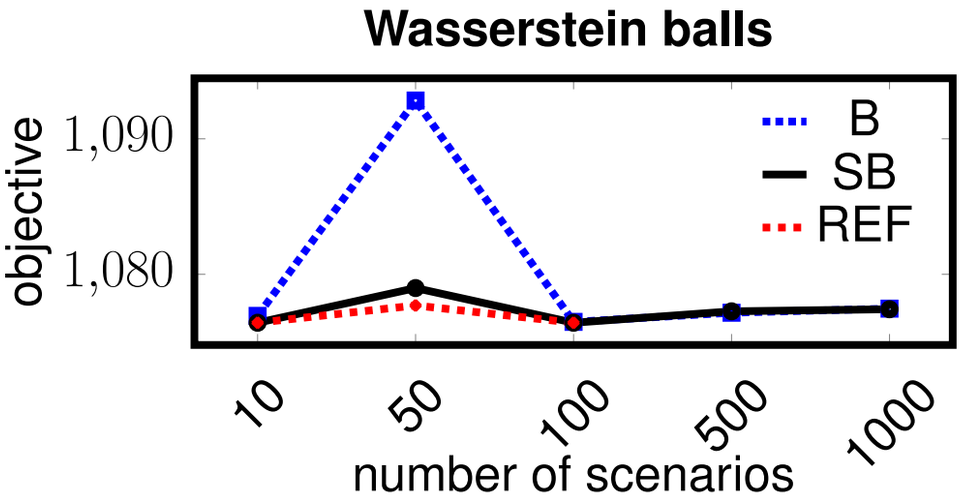
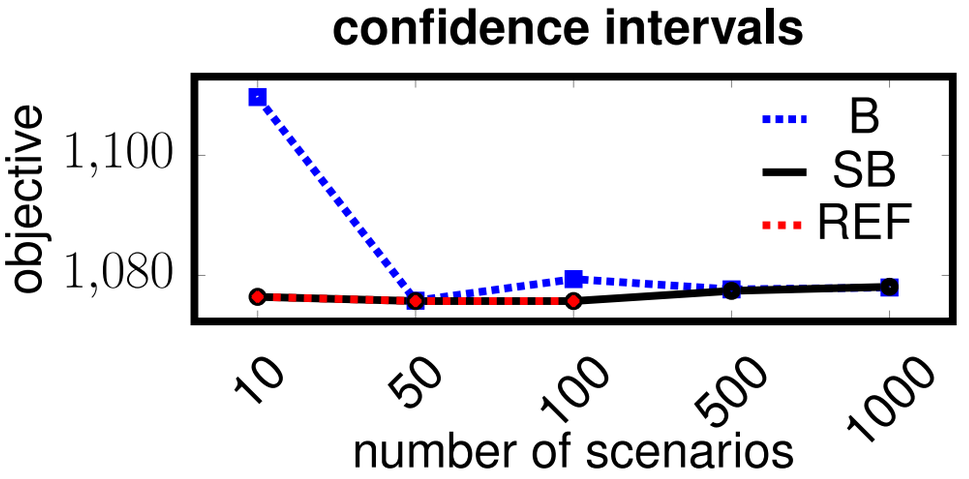


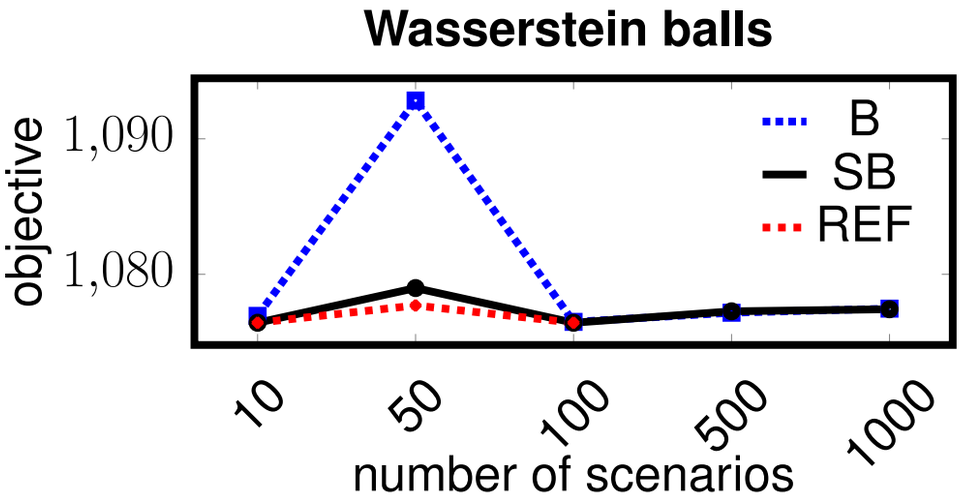


Numerical results for GasLib 134 (Schmidt et al. (2017))

observations

- REF only solvable up to 100 scenarios
- REF faster, if solvable
- pure bundle solution often suboptimal
- scenario-based bundle solution mostly globally optimal





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$\rightarrow$ **SB produces high-quality solutions for gas problems with robust chance constraints**

Conclusion and outlook

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- derive approximation problem for robust joint chance-constrained optimization with discrete distributions
- develop convergence theory for smoothing approximation
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- apply scenario-based postprocessing afterwards
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Current work and outlook

- extend setting to mixture ambiguity sets[1]
- test approach for different applications
- include mixed-integer design variables with spatial branch-and-bound techniques

[1] B., Liers, Stingl: https://opus4.kobv.de/opus4-trr154/602