Multipoint correlation functions in the 1+1 dimensional quantum Sinh-Gordon model

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K. Kozlowski, A. S., *On multipoint correlation functions in the Sinh-Gordon 1+1 dimensional quantum field theory*, arXiv:2502.03894.

K. Kozlowski, A. S., *Wightman axioms for the 1+1 dimensional Sinh-Gordon model*, to appear.

Outline

Sinh-Gordon model

Classical Field Theory: Ex. 1+1d Sinh-Gordon $\varphi = \varphi(\boldsymbol{x})$, $\boldsymbol{x} = (t, x)$, $\boldsymbol{x}^2 = t^2 - x^2$

$$(\partial_t^2 - \partial_x^2)\varphi + \frac{m^2}{g}\sinh(g\varphi) = 0 \ , \quad m, g > 0 \ .$$

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Construction of measurable quantities, ex. correlation functions:

$$(\Omega, \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_k(\boldsymbol{x}_k) \cdot \Omega) \ , \quad \mathcal{O}_1(\boldsymbol{x}_1), \dots, \mathcal{O}_k(\boldsymbol{x}_k) \in \mathcal{L}(\mathcal{H}) \ .$$

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- $\Phi(\boldsymbol{x}), \mathcal{H}, \Omega \to$ correlation functions (direct computation);
- Correlation functions + properties $\to$ QFT ($\to \Phi(\boldsymbol{x}), \ldots$) (Wightman's reconstruction theorem)

$\to$ construct QFT: find correlation functions with properties, ex:

- local commutativity (causality):

$$(\Omega, \mathcal{O}_1(\boldsymbol{x}_1) \ldots [\mathcal{O}_j(\boldsymbol{x}_j), \mathcal{O}_{j+1}(\boldsymbol{x}_{j+1})] \ldots \mathcal{O}_k(\boldsymbol{x}_k) \cdot \Omega) = 0 \;, \quad (\boldsymbol{x}_j - \boldsymbol{x}_{j+1})^2 < 0 \;;$$

- hermiticity:

$$(\Omega, \mathcal{O}_1(\boldsymbol{x}_1) \ldots \mathcal{O}_k(\boldsymbol{x}_k) \cdot \Omega) = \overline{(\Omega, \mathcal{O}_k(\boldsymbol{x}_k) \ldots \mathcal{O}_1(\boldsymbol{x}_1) \cdot \Omega)}\,;$$

- positivity, cluster decomposition...

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- Stochastic quantization: Gubinelli, Hairer...
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Output $\to$ correlation functions:

$$(\Omega, \mathcal{O}_1(\boldsymbol{x}_1)\dots\mathcal{O}_k(\boldsymbol{x}_k)\cdot\Omega) = \mathbb{E}_\mu[o_1(\boldsymbol{x}_1)\dots o_k(\boldsymbol{x}_k)]$$

with $o_1,\dots,o_k$ regular functions $\hookrightarrow \mathcal{O}_1,\dots,\mathcal{O}_k$ operators, and a measure μ:

- well-defined (ex: fixed point $\xi[\mu]=\mu$);
- not explicit, perturbative

How to construct QFTs?

Free theories well understood, ex. Sine-Gordon at free fermion point:

$$(\Omega, \mathcal{O}(\boldsymbol{x})\mathcal{O}(\boldsymbol{0}) \cdot \Omega) = \mathrm{Det}(1 + K^{(x)})$$

with $K^{(x)}$ explicit 2x2 matrix of integral operators [Bernard, Leclair '94]

$$K^{(x)}(u, v) = \begin{pmatrix} 0 & \frac{e_1(u)e_2(v)}{u+v} \\ \frac{e_2(u)e_1(v)}{u+v} & 0 \end{pmatrix} \tag{1}$$

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→ PDE for correlation functions // Painlevé III.
→ correlation = series of multiple integrals:

$$(\Omega, \mathcal{O}(\boldsymbol{x})\mathcal{O}(\mathbf{0}) \cdot \Omega) = \sum_{N \in \mathbb{N}} \frac{1}{N!} \sum_{\varepsilon_1, \dots, \varepsilon_N = \pm} \int_{\mathbb{R}^N} \mathrm{d}^N \boldsymbol{u}_N \, \det(K_{\varepsilon_i, \varepsilon_j}(u_i, u_j))_{1 \le i,j \le N}$$

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Heuristics (path integrals) not enough $\longrightarrow$ Need explicit + rigorous QFTs.

Goals

Our goals:

- rigorous construction;
- explicit expressions;
- analysis of universality.

$\longrightarrow$ integrable models

In this talk: construction of multipoint functions.

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Construction of Integrable QFTs in 1+1d

↪ Bootstrap program [Kirillov and Smirnov, Karowski and Weisz, Khamitov '70s-90s...].

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Construction of Integrable QFTs in 1+1d

$\hookrightarrow$ Bootstrap program [Kirillov and Smirnov, Karowski and Weisz, Khamitov '70s-90s...].

Output: explicit expressions for correlation functions.

Focus on simplest integrable model: 1+1d Sinh-Gordon.

Goals

Bootstrap program → 2 point function well understood [Lukyanov, Zamolodchikov '90s]:

$$(\Omega, \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\cdot\Omega) = \sum_{N=0}^{+\infty} \frac{1}{(2\pi)^N N!} \int_{\mathbb{R}^N} \mathrm{d}^N \boldsymbol{\beta}_N \underbrace{\prod_{a=1}^{N} \mathrm{e}^{-mr\cosh(\beta_a)} \, |\mathcal{F}_N(\boldsymbol{\beta}_N)|^2}_{\neq \det(K(\beta_i,\beta_j))\,!} \; , \; r = \sqrt{-(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2} \; .$$

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Convergent series [Kozlowski '20]

Multipoint functions? Only 3,4 points [Niedermaier et al '00, Caselle et al '06, Babujian et al '16].

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$\hookrightarrow$ here, closed expressions for 1+1d Sinh-Gordon (truncated) multipoint functions:

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+ conjecture for the actual multipoint functions:

$$(\Omega, \mathcal{O}_1(\boldsymbol{x}_1)\mathcal{O}_2(\boldsymbol{x}_2)\dots\mathcal{O}_k(\boldsymbol{x}_k)\cdot\Omega) = \lim_{r_1,\dots,r_{k-1}\to+\infty} \mathcal{W}_{\boldsymbol{r}}^{(k)}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k)$$

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Quantum Sinh-Gordon model

- $\mathcal{H} = \bigoplus_{n=0}^{+\infty} L^2(\mathbb{R}^n_>)$, with $\mathbb{R}^n_> = \{\boldsymbol{\alpha}_n = (\alpha_1, \dots, \alpha_n),\ \alpha_1 > \cdots > \alpha_n\}$;
- Vacuum state $\Omega = (1, 0, 0, \dots) \in \mathcal{H}$;
- Minkowski scalar product: $\boldsymbol{x} \cdot \boldsymbol{y} = x_0 y_0 - x_1 y_1$ if $\boldsymbol{x} = (x_0, x_1)$ and $\boldsymbol{y} = (y_0, y_1)$;
- Main building block: S-matrix [Gryanik, Vergeles '76]:

$$S(u) = \frac{\tanh(u/2 - \mathrm{i}\pi\mathfrak{b})}{\tanh(u/2 + \mathrm{i}\pi\mathfrak{b})}\ , \qquad \mathfrak{b} = \frac{1}{2}\cdot\frac{g^2}{8\pi + g^2}\ .$$

Operator content

State parametrization:

$$\mathcal{H} = \bigoplus_{n=0}^{+\infty} L^2(\mathbb{R}^n_>) \,, \quad f = (f^{(0)}, \dots, f^{(n)}, \dots) \in \mathcal{H} \,, \quad f^{(p)} \in L^2(\mathbb{R}^p_>) \,.$$

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Quantum field $\mathcal{O}(\boldsymbol{x}) : \mathcal{H} \to \mathcal{H}$

$$\Big(\mathcal{O}(\boldsymbol{x}) \cdot f\Big)^{(n)}(\boldsymbol{\alpha}_n) \;=\; \sum_{m \geq 0} \mathtt{M}_{\mathcal{O}}^{(n;m)}\big(\boldsymbol{x} \mid \boldsymbol{\alpha}_n\big)\big[f^{(m)}\big] \,, \quad \boldsymbol{\alpha}_n = (\alpha_1, \dots, \alpha_n) \,,$$

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Action of quantum fields

L^2 structure $\to$ integral operators:

$$\mathrm{M}_{\mathcal{O}}^{(n;m)}(\boldsymbol{x} \mid \boldsymbol{\alpha}_n)\big[f^{(m)}\big] = \int_{\mathbb{R}^m_>} \mathrm{d}^m\beta \, \mathcal{M}^{(\mathcal{O})}_{n;m}(\boldsymbol{\alpha}_n; \boldsymbol{\beta}_m) \cdot \mathrm{e}^{\mathrm{i}\,\big[\boldsymbol{p}(\boldsymbol{\alpha}_n) - \boldsymbol{p}(\boldsymbol{\beta}_m)\big] \cdot \boldsymbol{x}} \cdot f^{(m)}(\boldsymbol{\beta}_m) \,,$$

$$\boldsymbol{p}(\boldsymbol{\alpha}_n) = m \sum_{k=1}^{n} \big(\cosh \alpha_k, \sinh \alpha_k\big) \in \mathbb{R}^{1,1} \,.$$

$n = 0$: Matrix elements = + boundary values on $\mathbb{R}^m$ of meromorphic functions:

$$\mathcal{M}^{(\mathcal{O})}_{0;m}(\emptyset; \boldsymbol{\beta}_m) = \mathcal{F}^{(\mathcal{O})}_{m,+}(\boldsymbol{\beta}_m) = \lim_{\varepsilon_1,\ldots,\varepsilon_m \to 0^+} \mathcal{F}^{(\mathcal{O})}_m(\beta_1 + \mathrm{i}\varepsilon_1, \ldots, \beta_m + \mathrm{i}\varepsilon_m)\,.$$

Form factors

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Form factors:

- meromorphic w.r.t. each β_a on $0 < \Im(\beta_a) < 2\pi$,
- C^0 + boundary value on $\mathbb{R}^m$;
- C^0 - boundary value on $(\mathbb{R} + 2\mathrm{i}\pi)^m$;
- solve coupled Riemann-Hilbert problems.

Recall

$$S(u) = \frac{\tanh(u/2 - \mathrm{i}\pi\mathfrak{b})}{\tanh(u/2 + \mathrm{i}\pi\mathfrak{b})} .$$

Riemann-Hilbert Problem (m=2)

Find $(\beta_1, \beta_2) \mapsto \mathcal{F}_2^{(\mathcal{O})}(\beta_1, \beta_2)$ meromorphic in each β_a on $0 < \Im(\beta_a) < 2\pi$:

i) $\mathcal{F}_2^{(\mathcal{O})}(\beta_1, \beta_2) \; = \; S(\beta_1 - \beta_2) \cdot \mathcal{F}_2^{(\mathcal{O})}(\beta_2, \beta_1)$;

ii) $\mathcal{F}_{2,-}^{(\mathcal{O})}(\beta_1 + 2\mathrm{i}\pi, \beta_2) \; = \; \mathcal{F}_{2,+}^{(\mathcal{O})}(\beta_2, \beta_1)$;

iii) $\mathcal{F}_2^{(\mathcal{O})}(\beta_1 + \theta, \beta_2 + \theta) \; = \; \mathcal{F}_2^{(\mathcal{O})}(\beta_1, \beta_2)$ with $\theta \in \mathbb{R}$.

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$\longrightarrow$ explicit solution $\mathcal{F}_2^{(\mathcal{O})}(\beta_1, \beta_2) = C \cdot \mathtt{F}(\beta_1 - \beta_2)$:

$$\mathtt{F}(\beta) \; = \; \frac{1}{\Gamma(1+\mathfrak{z})\Gamma(-\mathfrak{z})} \frac{G(1-\mathfrak{b}-\mathfrak{z})G(2-\mathfrak{b}+\mathfrak{z})G(1-\hat{\mathfrak{b}}-\mathfrak{z})G(2-\hat{\mathfrak{b}}+\mathfrak{z})}{G(\mathfrak{b}-\mathfrak{z})G(1+\mathfrak{b}+\mathfrak{z})G(\hat{\mathfrak{b}}-\mathfrak{z})G(1+\hat{\mathfrak{b}}+\mathfrak{z})} \,,$$

$\mathfrak{z} = \frac{\mathrm{i}\beta}{2\pi}$, $\hat{\mathfrak{b}} = \frac{1}{2} - \mathfrak{b}$, G = Barnes G-function.

General Riemann-Hilbert Problem

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iv) Recurrent coupling via poles:

$$-\mathrm{iRes}\Big(\mathcal{F}_{m+2}^{(\mathcal{O})}(\alpha + \mathrm{i}\pi, \beta, \boldsymbol{\beta}_m) \cdot \mathrm{d}\alpha\,,\ \alpha = \beta\Big) \;=\; \Big\{1 - \prod_{a=1}^{m} S(\beta - \beta_a)\Big\} \cdot \mathcal{F}_m^{(\mathcal{O})}(\boldsymbol{\beta}_m)\,.$$

Form factors

Long history of devising solutions: Zamolodchikov, Lukyanov, Brazhnikov, Babujian, Fring, Karowski, Zapletal '70s-90s

Theorem

Form factors are of the form

$$\mathcal{F}_m^{(\mathcal{O})}(\boldsymbol{\beta}_m) = \prod_{a<b}^{m} F(\beta_{ab}) \cdot \mathcal{K}_m\left[p_m^{(\mathcal{O})}\right](\boldsymbol{\beta}_m) ,$$

with

$$\mathcal{K}_m\left[p_m^{(\mathcal{O})}\right](\boldsymbol{\beta}_m) = \sum_{\boldsymbol{\ell}_m \in \{0,1\}^m} (-1)^{\ell_1+\cdots+\ell_m} \prod_{k<s}^{m} \left\{ 1 - \mathrm{i} \frac{\ell_{ks} \cdot \sin[2\pi \mathfrak{b}]}{\sinh(\beta_{ks})} \right\} \cdot p_m^{(\mathcal{O})}(\boldsymbol{\beta}_m \mid \boldsymbol{\ell}_m) ,$$

and $p_m^{(\mathcal{O})}$ *= explicit function depending on* $\mathcal{O}$.

Form factors

Long history of devising solutions: Zamolodchikov, Lukyanov, Brazhnikov, Babujian, Fring, Karowski, Zapletal '70s-90s

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and $p_m^{(\mathcal{O})}$ *= explicit function depending on* $\mathcal{O}$.

Bootstrap for $\mathcal{F}_m^{(\mathcal{O})} \longleftrightarrow$ system of eq for the $p_m^{(\mathcal{O})}$.
Example: field Φ

$$p_m^{(\Phi)}(\boldsymbol{\beta}_m \mid \boldsymbol{\ell}_m) = C^m \cdot \frac{2\pi\mathrm{i}\mathfrak{b}}{g} \cdot \sum_{p=1}^{m} (-1)^{\ell_p} .$$

Build the correlation functions

Recursion for the matrix elements (axiom):

$$\mathcal{M}^{(\mathcal{O})}_{n;m}(\boldsymbol{\alpha}_n;\boldsymbol{\beta}_m) = \mathcal{M}^{(\mathcal{O})}_{n-1;m+1}(\boldsymbol{\alpha}'_n;(\alpha_1+\mathrm{i}\pi,\boldsymbol{\beta}_m)) + 2\pi\sum_{a=1}^{m}\delta(\alpha_1-\beta_a)\prod_{k=1}^{a-1}S(\beta_k-\beta_a)\cdot\mathcal{M}^{(\mathcal{O})}_{n-1;m-1}(\boldsymbol{\alpha}'_n;\widehat{\boldsymbol{\beta}_m}^{(a)})$$

with $\boldsymbol{\alpha}'_n = (\alpha_2,\ldots,\alpha_n)$; $\widehat{\boldsymbol{\beta}_m}^{(a)} = (\beta_1,\ldots,\beta_{a-1},\beta_{a+1},\ldots,\beta_m)$.

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- solved by "tree-like" combinatorics [Kirillov, Smirnov '87];
- initialization $n=0$: form factors;
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Regularization → well-defined operations!

→ Matrix elements = combinatorial sums of form factors.
Correlation functions built from $\mathcal{M}^{(\mathcal{O})}_{n;m}(\boldsymbol{\alpha}_n;\boldsymbol{\beta}_m)$, ex 2 point:

$$(\Omega, \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\cdot\Omega) = \sum_{m\in\mathbb{N}} \int_{\mathbb{R}^m_>} \frac{\mathrm{d}^m \boldsymbol{\beta}_m}{(2\pi)^m} \mathcal{M}^{(\mathcal{O})}_{0;m}(\emptyset;\boldsymbol{\beta}_m)\mathcal{M}^{(\mathcal{O})}_{m;0}(\boldsymbol{\beta}_m;\emptyset)\mathrm{e}^{-\mathrm{i}\,\boldsymbol{p}(\boldsymbol{\beta}_m)\cdot(\boldsymbol{x}-\boldsymbol{y})}$$

$(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 < 0 \rightarrow$ convergent series

$$\mathcal{W}^{(2)}(\boldsymbol{x}_1,\boldsymbol{x}_2) = (\Omega, \mathcal{O}(\boldsymbol{x}_1)\mathcal{O}(\boldsymbol{x}_2)\cdot\Omega) = \sum_{m\in\mathbb{N}} \int_{\mathbb{R}^m_>} \frac{\mathrm{d}^m \boldsymbol{\beta}_m}{(2\pi)^m} |\mathcal{F}^{(\mathcal{O})}_m(\boldsymbol{\beta}_m)|^2 \prod_{a=1}^{N} \mathrm{e}^{-mr\cosh(\beta_a)}$$

Distributional definition in general.

Avoid the series problem $\rightarrow$ truncature parameter $r \in \mathbb{N}$ and truncated operator:

$$\mathcal{O}^{(r)}(\boldsymbol{x}) = \pi_r \circ \mathcal{O}(\boldsymbol{x})$$

with π_r the canonical projection $\mathcal{H} \mapsto L^2(\mathbb{R}^r_>)$.

$\rightarrow$ Truncated correlation functions:

$$\mathcal{W}^{(k)}_{\boldsymbol{r}}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k) = \left(\Omega, \mathcal{O}^{(0)}_1(\boldsymbol{x}_1)\mathcal{O}^{(r_1)}_2(\boldsymbol{x}_2) \dots \mathcal{O}^{(r_{k-1})}_k(\boldsymbol{x}_k) \cdot \cdot \Omega\right) \ ,$$

with $\boldsymbol{r} = (r_1, \dots, r_{k-1}) \in \mathbb{N}^{k-1}$.

Theorem (Kozlowski, S. 2024)

Let $\boldsymbol{r} = (r_1, \ldots, r_{k-1}) \in \mathbb{N}^{k-1}$ *and* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ *such that*

$$\boldsymbol{x}_{ba}^2 < 0\,, \qquad x_{a;1} > x_{b;1} \qquad \text{for any} \quad \boldsymbol{x}_a = (x_{a;0}, x_{a;1}) \qquad \text{and} \quad b > a\,.$$

Then, the k-point truncated correlation function is well-defined:

$$\mathcal{W}_{\boldsymbol{r}}^{(k)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) \;=\; \sum_{\boldsymbol{n} \in \mathcal{N}_{\boldsymbol{r}}} \frac{1}{\prod\limits_{b>a}^{k} n_{ba}!} \cdot \mathcal{I}_{\boldsymbol{n}}^{(k)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)\,,$$

with $\mathcal{N}_{\boldsymbol{r}} = \Big\{ \boldsymbol{n} = \big(n_{21}, n_{31}, n_{32}, n_{41}, \ldots, n_{kk-1}\big) \;:\; \sum\limits_{u=p+1}^{k} \sum\limits_{s=1}^{p} n_{us} \;=\; r_p\,, \quad p = 1, \ldots, k-1 \Big\}.$

For general $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$: true in distributional sense.

Vectorial notations

Let $\boldsymbol{\alpha}_n = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ and $\boldsymbol{\beta}_m = (\beta_1, \dots, \beta_m) \in \mathbb{C}^m$. Denote:

– reverse $\overleftarrow{\boldsymbol{\alpha}_n} = (\alpha_n, \dots, \alpha_1)$;

– $\overline{\boldsymbol{e}} = (1, \dots, 1)$ with "fitting size", i.e. $\boldsymbol{\alpha}_n + \mathrm{i}\pi\overline{\boldsymbol{e}} = (\alpha_1 + \mathrm{i}\pi, \dots, \alpha_n + \mathrm{i}\pi)$;

– concatenation $\boldsymbol{\alpha}_n \cup \boldsymbol{\beta}_m = (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m)$.

Theorem (Kozlowski, S. 2024)

Let $\boldsymbol{n} = (n_{21}, n_{31}, n_{32}, \ldots, n_{kk-1}) \in \mathbb{N}^{\frac{k(k-1)}{2}}$ *and* $0 < \eta^{(21)} < \eta^{(31)} < \cdots < \eta^{(kk-1)}$ *be small. For* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ *such that*

$$\boldsymbol{x}_{ba}^2 < 0\,, \qquad x_{a;1} > x_{b;1} \qquad \text{for any} \quad \boldsymbol{x}_a = (x_{a;0}, x_{a;1}) \qquad \text{with} \quad b > a\,.$$

the following integrals exist:

$$\mathcal{I}_{\boldsymbol{n}}^{(k)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) = \prod_{b>a}^{k} \int\limits_{\left\{\mathbb{R}+\mathrm{i}\eta^{(ba)}\right\}^{n_{ba}}} \frac{\mathrm{d}^{n_{ba}}\boldsymbol{\gamma}^{(ba)}}{(2\pi)^{n_{ba}}} \cdot \prod_{b>a}^{k} \mathrm{e}^{\mathrm{i}\boldsymbol{p}(\boldsymbol{\gamma}^{(ba)})\cdot \boldsymbol{x}_{ba}} \cdot \mathcal{T}(\boldsymbol{\gamma})$$

$$\times \prod_{p=1}^{k} \mathcal{F}^{(\mathcal{O})}\Big(\overleftarrow{\boldsymbol{\gamma}^{(pp-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(p1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}, \boldsymbol{\gamma}^{(kp)} \cup \cdots \cup \boldsymbol{\gamma}^{(p+1p)} \Big)\,,$$

with

$$\mathcal{T}(\boldsymbol{\gamma}) = \prod_{\substack{v>p \\ p\geq 3}}^{k-1} \prod_{u>s}^{p-1} \prod_{r=1}^{n_{vu}} \prod_{t=1}^{n_{ps}} S\big(\gamma_r^{(vu)} - \gamma_t^{(ps)}\big)\,.$$

Conjecture (Kozlowski, S. 2024)

The series defining the k-point correlation function is convergent:

$$\mathcal{W}^{(k)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) = \sum_{\boldsymbol{n} \in \mathbb{N}^{\frac{k(k-1)}{2}}} \frac{1}{\prod_{b>a}^{k} n_{ba}!} \cdot \mathcal{I}_{\boldsymbol{n}}^{(k)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) ,$$

Truncated functions well-defined in general (distribution), but CV problem still open!

Heavy expressions, but can be used!

Assuming CV (2 points OK) $\rightarrow$ Wightman axioms explicitly checked, ex:

- local commutativity (causality):

$$\mathcal{W}^{(k)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_j, \boldsymbol{x}_{j+1}, \ldots, \boldsymbol{x}_k) = \mathcal{W}^{(k)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{j+1}, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_k) \,, \quad (\boldsymbol{x}_j - \boldsymbol{x}_{j+1})^2 < 0 \,;$$

- hermiticity:

$$\mathcal{W}^{(k)}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) = \overline{\mathcal{W}^{(k)}(\boldsymbol{x}_k, \ldots, \boldsymbol{x}_1)} \,.$$

All Wightman axioms satisfied $\longrightarrow$ well-defined QFT ✓(if convergence!)

Conclusion and open problems

Computation of k-point functions in 1+1d Sinh-Gordon:

- rigorous + explicit for truncated functions;
- satisfy Wightman axioms (assuming CV)

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Computation of k-point functions in 1+1d Sinh-Gordon:

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Open problems:

- series CV? use of large deviations?
- universality questions?
- generalize to more theories (ex. Sine-Gordon)?

Thank you!