Random Fibonacci Words via Clone Schur Functions

Jeanne Scott - joint work with Leonid Petrov

(based on arXiv:2412.21126)

University of Minnesota, Minneapolis

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Today: Understanding clone Schur functions probabilistically

Outline of the Talk

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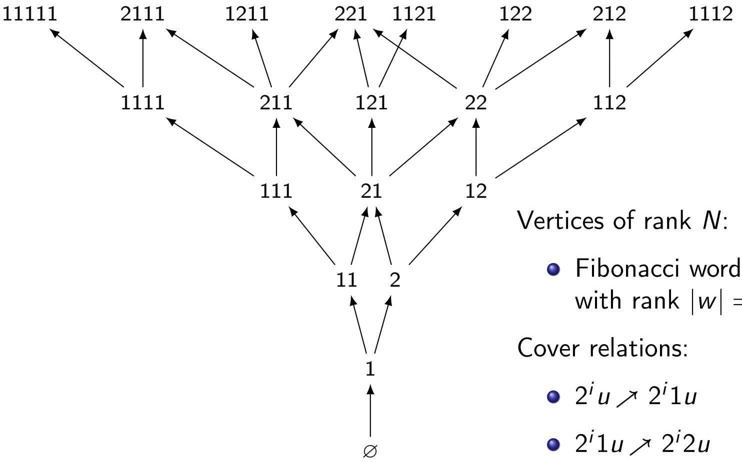
Lemma

The number Fibonacci words w of rank $|\boldsymbol{w}| = \boldsymbol{n}$ is exactly the **Fibonacci number** $\boldsymbol{F_n}$, *with the convention that* $F_0 = F_1 = 1$.

Young-Fibonacci lattice $\mathbb{YF}$ (Stanley 1988)



Vertices of rank N:

- Fibonacci words w with rank $|w| = N$

Cover relations:

- $2^i u \nearrow 2^i 1 u$
- $2^i 1 u \nearrow 2^i 2 u$

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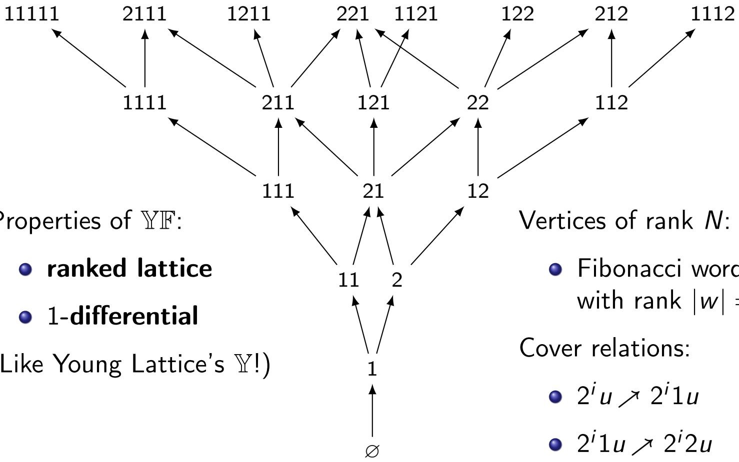

Properties of $\mathbb{YF}$:

- **ranked lattice**
- 1-**differential**

(Like Young Lattice's $\mathbb{Y}$!)

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Differential posets

Definition

The Up and Down operators for a locally finite poset $\mathbb{P}$ are

$$U(v) := \sum_{v \nearrow w} w \qquad D(v) := \sum_{w \nearrow v} w$$

for any $v \in \mathbb{P}$. The notation $\boldsymbol{v \nearrow w}$ means v is **covered** by w.

Definition (Stanley 1988)

A **differential poset** is a locally finite, ranked poset with a unique minimum element $\varnothing$ whose **Up** and **Down** operators satisfy

$$DU - UD = \mathrm{Id}\,.$$

Properties of differential posets

Theorem (Stanley 1988)

If $\mathbb{P}$ is differential with minimum element $\varnothing$ then

$$D^n U^n(\varnothing) = n!\, \varnothing$$

Otherwise said:

$$\sum_{w \in \mathbb{P}_n} \dim_{\mathbb{P}}^2(w) = n!$$

where $\dim_{\mathbb{P}}(w)$ *is the number of* **saturated chains** *from $\varnothing$ to $w \in \mathbb{P}$ and where $\mathbb{P}_n$ is the subset of $w \in \mathbb{P}$ with rank $|w| = n$.*

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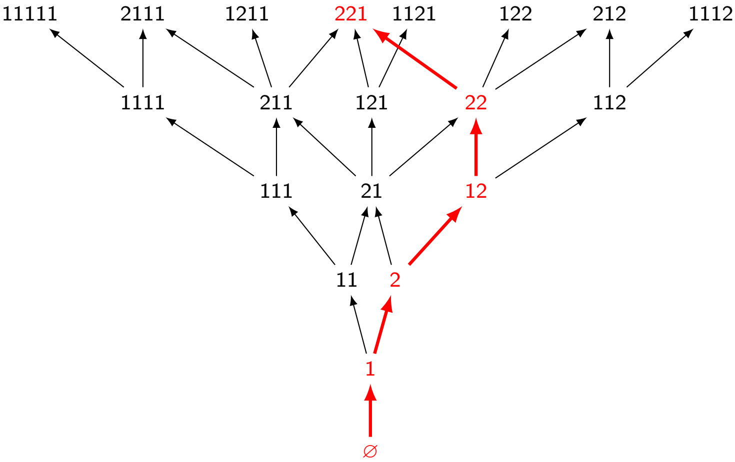
Corollary (Plancherel Measures)

$\boldsymbol{M_n(w) := \dim^2(w)/n!}$ *for $w \in \mathbb{P}_n$ defines a* **coherent system** *of probability measures on $\mathbb{P}$.*

Saturated chain from $\varnothing$ to $w = 221$ in $\mathbb{YF}$-lattice



Coherent Systems

Definition

Let $\mathbb{P}$ be a ranked poset with a unique minimal element $\varnothing$. A **coherent system** is a sequence $\{M_n\}_{n \in \mathbb{N}}$ where

- M_n is a probability measure on $\mathbb{P}_n$ for each $n \in \mathbb{N}$
- $M_n(v) = \sum_{v \nearrow w} \frac{\dim_{\mathbb{P}}(v)}{\dim_{\mathbb{P}}(w)} M_{n+1}(w)$ for each $v \in \mathbb{P}_n$

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Remark (Transition Probabilities)

$\{M_n\}_{n \in \mathbb{N}}$ *is a coherent system if and only if*

$$\mu(v \nearrow w) := \frac{\dim_{\mathbb{P}}(v)}{\dim_{\mathbb{P}}(w)} \frac{M_{n+1}(w)}{M_n(v)}$$

restricts to a probability measure on all covering relations $(v \nearrow w)$ *for each* $v \in \mathbb{P}$.

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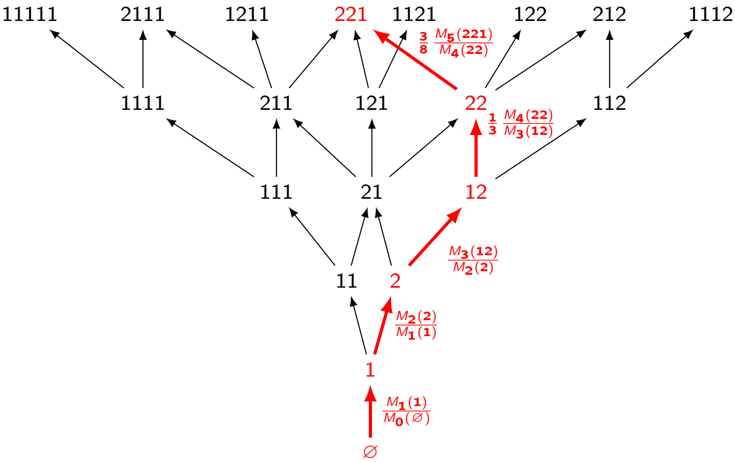
restricts to a probability measure on all covering relations $(v \nearrow w)$ *for each* $v \in \mathbb{P}$. **Defines Markov process evolving along saturated chains.**

Picture: Transition Probabilities from a Coherent System



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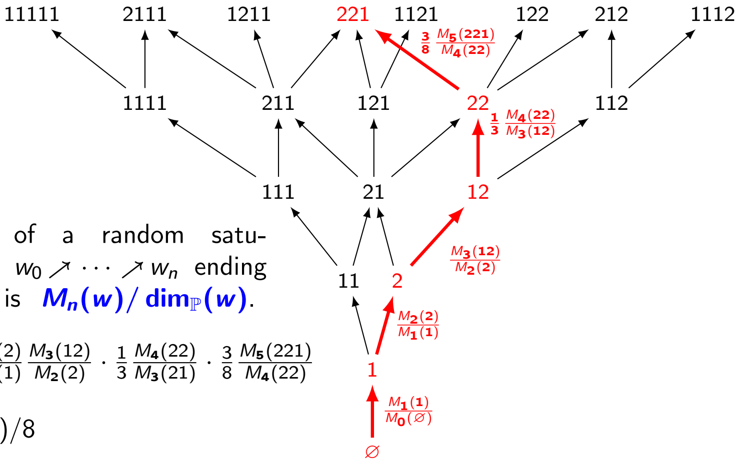

Probability of a random saturated chain $w_0 \nearrow \cdots \nearrow w_n$ ending at $w_n = w$ is $\boldsymbol{M_n(w)/\dim_{\mathbb{P}}(w)}$.

$$\frac{M_1(1)}{M_0(\varnothing)} \cdot \frac{M_2(2)}{M_1(1)}\,\frac{M_3(12)}{M_2(2)} \cdot \frac{1}{3}\,\frac{M_4(22)}{M_3(21)} \cdot \frac{3}{8}\,\frac{M_5(221)}{M_4(22)}$$

$$= M_5(221)/8$$

Coherent Systems: The Young lattice $\mathbb{Y}$

Let:

- $\vec{x} = (x_1, x_2, x_3, \dots)$ and $\vec{y} = (y_1, y_2, y_3, \dots)$ be two sequences of variables, γ an auxiliary variable
- λ be a partition of n and $\square$ the unique partition of 1
- $s_\lambda(\vec{x}; \vec{y}; \gamma)$ be the γ-super-symmetric Schur function

Pieri Rule: $\boldsymbol{s_\square(\vec{x}; \vec{y}; \gamma) \cdot s_\lambda(\vec{x}; \vec{y}; \gamma) = \sum_{\lambda \nearrow \nu} s_\nu(\vec{x}; \vec{y}; \gamma)}$

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Theorem (Thoma)

Specializing $x_k, y_k, \gamma \in \mathbb{R}_{\geq 0}$ *such that* $\gamma + \sum_{k\geq 0}^{\infty} (x_k + y_k) = 1$ *then*

$$M_n(\lambda) := \dim_{\mathbb{Y}}(\lambda) \, \frac{s_\lambda(\vec{x}; \vec{y}; \gamma)}{s_\square^n(\vec{x}; \vec{y}; \gamma)}$$

for $\lambda \in \mathbb{Y}_n$ *defines a* **minimal coherent system** *on the* **Young lattice** $\mathbb{Y}$. *Moreover all minimal coherent systems on* $\mathbb{Y}$ *are of this form.*

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$\gamma = 1$ gives Plancherel

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Biserial Clone Schur Functions

Let $\vec{x} = (x_1, x_2, x_3, \ldots)$ and $\vec{y} = (y_1, y_2, y_3, \ldots)$ be two sequences of variables. Introduce the following tridiagonal determinants

$$A_\ell = \det \underbrace{\begin{pmatrix} x_1 & y_1 & 0 & \cdots \\ 1 & x_2 & y_2 & \\ 0 & 1 & x_3 & \\ \vdots & & & \ddots \end{pmatrix}}_{\ell \times \ell \text{ tridiagonal matrix}} \quad B_{\ell-1}(r) = \det \underbrace{\begin{pmatrix} y_{r+1} & x_{r+1}y_{r+2} & 0 & \cdots \\ 1 & x_{r+3} & y_{r+3} & \\ 0 & 1 & x_{r+4} & \\ \vdots & & & \ddots \end{pmatrix}}_{\ell \times \ell \text{ tridiagonal matrix}}$$

Definition (Okada 1994)

For $w \in \mathbb{YF}$ the **clone Schur function $s_w(\vec{x} \mid \vec{y})$** is given recursively by

$$s_w(\vec{x} \mid \vec{y}) := \begin{cases} A_k & \text{if } w = 1^k \text{ for some } k \geq 0, \\ B_k(|v|) \cdot s_v(\vec{x} \mid \vec{y}) & \text{if } w = 1^k 2v \text{ for some } k \geq 0. \end{cases}$$

Clone Pieri Rule and Coherent Systems on $\mathbb{YF}$

Proposition (Okada 1994)

Clone Pieri Rule: $$x_{n+1} \cdot s_v(\vec{x} \mid \vec{y}) = \sum_{v \nearrow w} s_w(\vec{x} \mid \vec{y}) \quad \textit{for } v \in \mathbb{YF}_n$$

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Corollary/Defintion: Clone Schur Coherent Systems (PS 2024)

Let $\vec{x} = (x_1, x_2, x_3, \dots)$ *and* $\vec{y} = (y_1, y_2, y_3, \dots)$ *be two sequences of real numbers and assume* $\boldsymbol{s_w(\vec{x} \mid \vec{y})} > 0$ *for each* $w \in \mathbb{YF}$, *then*

$$M_n(w) := \dim_{\mathbb{YF}}(w) \, \frac{s_w(\vec{x} \mid \vec{y})}{x_1 \cdots x_n}$$

for $w \in \mathbb{YF}_n$ *defines a* **coherent system** *on the* $\mathbb{YF}$**-lattice**.

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for $w \in \mathbb{YF}_n$ *defines a* **coherent system** *on the* $\mathbb{YF}$-**lattice**.

Definition: Fibonacci Positivity (PS 2024)

A pair of sequences $(\vec{x}, \vec{y})$ *of real numbers is* **Fibonacci positive** *if, upon specializing,* $\boldsymbol{s_w(\vec{x} \mid \vec{y})} > 0$ *for all* $w \in \mathbb{YF}$.

Fibonacci Positivity and Tridiagonal Total Positivity

Lemma

A pair of sequences $(\vec{x}, \vec{y})$ *is* **Fibonacci positive** *if and only if the infinite tridiagonal matrices defined (for* $r \geq 1$*) by*

$$\mathcal{A}(\,\vec{x}\,|\,\vec{y}\,) = \begin{pmatrix} x_1 & y_1 & 0 & \cdots \\ 1 & x_2 & y_2 & \\ 0 & 1 & x_3 & \\ \vdots & & & \ddots \end{pmatrix} \quad \mathcal{B}_r(\,\vec{x}\,|\,\vec{y}\,) = \begin{pmatrix} y_r & x_r y_{r+1} & 0 & \cdots \\ 1 & x_{r+2} & y_{r+2} & \\ 0 & 1 & x_{r+3} & \\ \vdots & & & \ddots \end{pmatrix}$$

are are **totally positive***: All matrix minors, which are not identically zero in virtue of the tridiagonal structure, must be real and positive.*

Reason

$\mathcal{A}(\,\vec{x}\,|\,\vec{y}\,)$ *and* $\mathcal{B}_r(\,\vec{x}\,|\,\vec{y}\,)$ *are* **totally positive** *if and only if their* **leading, principal minors** *are positive, i.e.* $A_\ell > 0$ *and* $B_{\ell-1}(r-1) > 0$ *for* $\ell \geq 1$.

Fibonacci Positivity: Characterization

Theorem (PS 2024)

Each **Fibonacci positive** *pair of sequences* $(\vec{x}, \vec{y})$ *is uniquely expressed as*

$$x_k = c_k(1 + t_{k-1}) \qquad y_k = c_k c_{k+1} t_k \qquad \text{for } k \geq 1$$

where $\vec{c}$ *is an arbitrary positive real sequence, and* $\vec{t} = (t_1, t_2, t_3, \ldots)$ *is a positive real sequence of one of the following two types:*

- *(***divergent type***) The infinite series*

  $$(\dagger) \;\; 1 + t_1 + t_1 t_2 + t_1 t_2 t_3 + \ldots$$

  diverges*, and* $t_{m+1} \geq 1 + t_m$ *for all* $m \geq 1$

- *(***convergent type***) The series* $(\dagger)$ **converges***, and for all* $m \geq 0$

  $$1 + t_{m+3} + t_{m+3}t_{m+4} + t_{m+3}t_{m+4}t_{m+5} + \ldots \geq \frac{t_{m+1}}{t_{m+2}(1 + t_m - t_{m+1})}$$

Fibonacci Positivity: Examples

Remark

A sequence $\vec{t} = (t_1, t_2, t_3, \ldots)$ *of* **divergent type** *can be expressed as*

$$t_k = k + \varepsilon_1 + \ldots + \varepsilon_k$$

for unique real parameters $\varepsilon_k \geq 0$ *with* $k \geq 1$. *Treating* $\vec{\boldsymbol{c}}$ *and* $\vec{\varepsilon}$ *as formal variables, we can express the clone Schur function* $\boldsymbol{s_w(\vec{x} \mid \vec{y})}$ *as a non-zero polynomial in* $\mathbb{Z}[\vec{\boldsymbol{c}}, \vec{\varepsilon}]$ *with* **non-negative** *coefficients for each* $w \in \mathbb{YF}$.

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Examples: Fibonacci Positive pairs of sequences $(\vec{x}, \vec{y})$

- **Plancherel**: $\boldsymbol{x_n = y_n = n}$ (*i.e.* $M_n(w) = \dim^2_{\mathbb{YF}}(w)/n!$) **divergent**

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- **shifted-Plancherel**: $\boldsymbol{x_n = y_n = n - 1 + \sigma}$ with $\sigma > 1$ **convergent**

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- **shifted-Plancherel**: $\boldsymbol{x_n = y_n = n - 1 + \sigma}$ with $\sigma > 1$ **convergent**
- **Power specializations**: $\boldsymbol{x_n = 1 + t_{n-1}}$, $\boldsymbol{y_n = t_n}$ with $\boldsymbol{t_n = \varkappa_p / n^p}$ and $p \geq 1$ and sufficiently small values $\varkappa_p > 0$ **convergent**

Fibonacci Positivity: Stieltjes Moment Sequences

Recall that a sequence $\vec{a} = (a_0, a_1, a_2, \dots)$ of real numbers is called a **strong Stieltjes moment sequence** if there exists a **Borel measure $\nu(dt)$** ≥ 0 on $[0, \infty)$ with infinite support such that $\boldsymbol{a_n = \int_0^\infty t^n \nu(dt)}$ for $n \geq 0$.

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Theorem (Flajolet, Viennot, Sokal ...)

$\boldsymbol{\vec{a} = (a_0, a_1, a_2, \ldots)}$ *is a* **strong Stieltjes moment sequence** *if and only if there exist a pair of sequences* $\boldsymbol{(\vec{x}, \vec{y})}$ *such that the matrix* $\mathcal{A}(\,\vec{x}\,|\,\vec{y}\,)$ *is* **totally positive** *and the moment generating function* $\boldsymbol{M(z)}$ *of* $\boldsymbol{\vec{a}}$ *is given by the following Jacobi continued fraction associated to* $\boldsymbol{(\vec{x}, \vec{y})}$*:*

$$M(z) := \sum_{n \geq 0} \frac{a_n}{a_0} z^n = \cfrac{1}{1 - x_1 z - \cfrac{y_1 z^2}{1 - x_2 z - \cfrac{y_2 z^2}{1 - x_3 z - \cfrac{y_3 z^2}{\ddots}}}}$$

Fibonacci Positivity: Stieltjes Moment Sequences

Theorem (continuation ...)

The pair of sequences $(\vec{x}, \vec{y})$ *is also responsible for the recursion*

$$P_{n+1}(t) = (t - x_{n+1})P_n(t) - y_n P_{n-1}(t) \quad \text{with } P_0(t) = 1\,,\ P_1(t) = t - x_1$$

generating the polynomials $P_n(t)$ *which are* **orthogonal** *with respect to the* **Borel measure** $\nu(dt)$ *on* $[0, \infty)$ *whose moment sequence is* $\vec{a}$*.*

Fibonacci Positivity: Stieltjes Moment Sequences

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Remark (Fibonacci Positivity and the (q)-Askey Scheme)

The recursion parameters $(\vec{x}, \vec{y})$ *for the following systems of orthogonal polynomials are* **Fibonacci positive** *of* **divergent type**:

- *Charlier polynomials*
- *Type-I Al-Salam-Carlitz polynomials*
- *Al-Salam-Chihara polynomials*
- *q-Charlier polynomials*

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- *q-Charlier polynomials*

In all cases the Borel measure $\nu(dt)$ is **discretely** supported on $[0, \infty)$. Moments are (q-analogues of) Bell polynomials.

Fibonacci Positivity: Borel Measures and Moment Sequences

Question (Fibonacci Positivity vis-à-vis Borel measures)

Fibonacci positive *sequences* $(\vec{x}, \vec{y})$ *give rise to a* **proper subclass** *of Borel measures* $\nu(dt)$ *on* $[0, \infty)$*. Can these measures, and/or their moments be characterized in an informative way?*

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Proposition (PS 2024)

Let $(\vec{x}, \vec{y})$ *be a Fibonacci positive specialization of* **divergent type** *with* $x_k = c_k(1 + t_{k-1})$ *and* $y_k = c_k c_{k+1} t_k$ *for all* $k \geq 1$ *and where*

$$t_k = k + \varepsilon_1 + \cdots + \varepsilon_k \quad \textit{with} \quad \varepsilon_k \geq 0 \quad \textit{for all } k \geq 1$$

then the moments $\vec{a} = (a_0, a_1, a_2, \ldots)$ *of the Borel measure* $\nu(dt)$ *on* $[0, \infty)$ *are given by (novel)* **$(\vec{c}, \vec{\varepsilon})$-deformed Bell polynomials**:

$$a_n = B_n(\vec{c}, \vec{\varepsilon}) := \sum_{\pi \in \Pi(n)} \prod_{k \geq 1} c_k^{\mathrm{height}_k(\pi)} (1 + \varepsilon_k)^{\mathcal{C}_k(\pi)}$$

where the sum is taken over all **set partitions** π *of* n.

Fibonacci Positivity: Borel Measures and Moment Sequences

Question (Fibonacci Positivity vis-à-vis Borel measures)

How about **Fibonacci positive** *sequences* $(\vec{x}, \vec{y})$ *of* **convergent type**?

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Example: Fibonacci Positive specialization $(\vec{x}, \vec{y})$ of Convergent Type

To this end, we examine the **shifted-Charlier** *specialization* $(\vec{x}, \vec{y})$:

$$x_n = n - 2 + \rho + \sigma \quad y_n = \rho(n - 1 + \sigma) \quad \rho \in (0, 1] \text{ and } \sigma \geq 1$$

It is simultaneously a deformation of both the **Charlier** *specialization (when $\sigma = 1$) and the* **shifted-Plancherel** *specialization (when $\rho = 1$).*

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Remark (M. Ismail et. al. 1988)

The orthogonal polynomials generated by the **shifted-Charlier** *parameters* $(\vec{x}, \vec{y})$ *were studied by M. Ismail in connection to* **linear birth and death models**. *The associated Borel measure* $\nu(dt)$ *was shown to be discretely supported on* $[0, \infty)$.

Fibonacci Positivity: Shifted-Charlier specialization

Proposition (M. Ismail et. al. 1988, PS 2024)

The moment generating function $\boldsymbol{M(z)}$ *of the Borel measure associated to the* **shifted-Charlier** *specialization* $(\vec{x}, \vec{y})$ *is given by*

$$M(z) = \frac{{}_1F_1\left(\sigma;\, \sigma - \frac{1}{z};\, -\rho\right)}{{}_1F_1\left(\sigma - 1;\, \sigma - \frac{1}{z};\, -\rho\right) - z(\sigma - 1)\, {}_1F_1\left(\sigma;\, \sigma - \frac{1}{z};\, -\rho\right)}$$

Moreover the corresponding moments $\vec{a} = (a_0, a_1, a_2, \dots)$ *are given by the following (novel)* $\boldsymbol{\sigma}$**-deformed Bell polynomials***:*

$$a_n = B^{\mathrm{ntc}}_{\sigma,n}(\rho) := \sum_{\pi \in \Pi(n)} \rho^{\#\mathrm{blocks}^\star(\pi)}\, \sigma^{\boldsymbol{ntc}(\pi)}\, \left(\rho + \sigma - 1\right)^{\#\mathcal{S}(\pi)}$$

where the sum is taken over all **set partitions** π *of* $\boldsymbol{n}$.

Asymptotics of Clone Coherent Measures: Runs and Hikes

Parsing Fibonacci words **w** into consecutive groups of **1's** and **2's**:

- **Runs:** $w = 1^{r_1} 2\, 1^{r_2} 2\, 1^{r_3} 2 \cdots 2\, 1^{r_p}$
  where $r_k \geq 0$ is an integer called the **k-th run** and $p = \#\mathbf{2's} + 1$
- **Hikes:** $w = 2^{h_1} 1\, 2^{h_2} 1\, 2^{h_3} 1 \cdots 1\, 2^{h_q}$
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Definition (Modified and Scaled Runs and Hikes)

$$\tilde{\boldsymbol{r}}_{\boldsymbol{k}} := \begin{cases} r_k + 2 & k \leq p - 1 \\ r_p & k = p \end{cases} \qquad \tilde{\boldsymbol{h}}_{\boldsymbol{k}} := \begin{cases} 2h_k + 1 & k \leq q - 1 \\ 2h_m & k = q \end{cases}$$

$$\boldsymbol{R}_{\boldsymbol{k}}(w) = \frac{\tilde{r}_k(w)}{|w| - \sum_{i=1}^{k-1} \tilde{r}_j(w)} \leq 1 \qquad \boldsymbol{H}_{\boldsymbol{k}}(w) = \frac{\tilde{h}_k(w)}{|w| - \sum_{i=1}^{k-1} \tilde{h}_j(w)} \leq 1$$

Asymptotics: Runs and Charlier specialization

Recall the **Charlier specialization $(\vec{x}, \vec{y})$**:

$$\boldsymbol{x_n} = \boldsymbol{n} - 1 + \boldsymbol{\rho}\,, \quad \boldsymbol{y_n} = \boldsymbol{n\rho} \quad \text{where } \rho \in (0,1]$$

Definition: Convex mixture of $\mathrm{beta}(1,\rho)$ and point mass

For any $\rho \in (0,1]$ let η_ρ be a random variable distributed in $[0,1]$ by

$$\rho\delta_0(\alpha) + (1-\rho)\rho(1-\alpha)^{\rho-1}d\alpha \qquad \alpha \in [0,1]$$

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Theorem (PS 2024)

Let $w \in \mathbb{YF}_n$ be a random Fibonacci word distributed according to the coherent measure $\boldsymbol{M_n(w)}$ associated to the **Charlier specialization** *with $\rho \in (0,1]$. For any $k \geq 1$ the scaled runs jointly converge in distribution:*

$$(R_1(w), \dots, R_k(w)) \xrightarrow[n\to\infty]{d} (\alpha_1, \dots, \alpha_k)$$

where $\alpha_1, \dots, \alpha_k \in [0,1]$ are **iid.** *random variables sampled with respect to η_ρ.*

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where $\alpha_1, \ldots, \alpha_k \in [0,1]$ are **iid.** *random variables sampled with respect to η_ρ.* **Note: the $R_k(w)$'s are not iid. prior to taking the limit.**

Asymptotics: Hikes and shifted-Plancherel specialization

Definition (*σ*-model for random sequences)

Fix $\sigma \in [1, \infty)$ and construct random variables $\boldsymbol{\xi_{\sigma;k}} \in [0, 1]$ as follows:

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• Toss a sequence of independent coins with probabilities of success $1, \sigma^{-1}, \sigma^{-2}, \ldots$ and let $\boldsymbol{N}$ be the (random) number of successes until the first failure. Notice

$$\mathrm{Prob}(\boldsymbol{N} = n) \;=\; \sigma^{-\binom{n}{2}}(1 - \sigma^{-n})$$

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• Now independently sample $\boldsymbol{\xi_{\sigma;k}} \in [0, 1]$ for $k = 1, \ldots, \boldsymbol{N}$ with respect to the $\mathbf{beta}(1, \boldsymbol{\sigma}/2)$ measure. Set $\boldsymbol{\xi_{\sigma;k} = 0}$ for all $k > \boldsymbol{N}$.

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Remark

For $\sigma \neq 1$ the $\boldsymbol{\xi_{\sigma;k}}$'s are **not iid.** *The marginal distribution of $\boldsymbol{\xi_{\sigma;k}}$ is*

$$(1 - \sigma^{-k+1})\delta_0(\alpha) + \frac{1}{2}\sigma^{-k+2}(1 - \alpha)^{\frac{\sigma}{2} - 1} d\alpha$$

When $\sigma = 1$ the $\boldsymbol{\xi_{\sigma;k}}$'s are $\mathbf{beta}(1, 1/2)$-sampled **iid.** *random variables.*

Asymptotics: Hikes and shifted-Plancherel specialization

Recall the **shifted-Plancherel specialization $(\vec{x}, \vec{y})$**:

$$\boldsymbol{x_n = y_n = n - 1 + \sigma} \quad \text{where } \sigma \in [1, \infty)$$

Theorem (Gnedin-Kerov 2000, PS 2024)

Let $w \in \mathbb{YF}_n$ be a random Fibonacci word distributed according to the coherent measure $\boldsymbol{M_n(w)}$ associated to the **shifted-Plancherel specialization** *with $\sigma \in [1, \infty)$. For $k \geq 1$ the scaled hikes jointly converge in distribution:*

$$(H_1(w), \dots, H_k(w)) \xrightarrow[n \to \infty]{d} (\xi_{\sigma;1}, \cdots, \xi_{\sigma;k})$$

where $\boldsymbol{\xi_{\sigma;k}} \in [0, 1]$ are sampled according to the **$\boldsymbol{\sigma}$-model**.

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where $\boldsymbol{\xi_{\sigma;k}} \in [0, 1]$ are sampled according to the **σ-model**. **When $\sigma = 1$ we recover Gnedin and Kerov's result: In the $n \to \infty$ limit, the H_k's are $\mathrm{beta}(1, 1/2)$-sampled iid. random variables in $[0, 1]$.**

Asymptotics: Type-I Observables

Definition (Type-I Fibonacci words)

A **Type-I Fibonacci word** is an infinite Fibonacci word of the form $\mathbf{1}^{\infty}$ or $\mathbf{1}^{\infty}\mathbf{2}\boldsymbol{w}$ for some finite suffix $\boldsymbol{w} \in \mathbb{YF}$.

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Remark (Goodman and Kerov 2000)

The set $\mathbf{1}^{\infty}\mathbb{YF}$ of all **Type-I Fibonacci words** *appears in Goodman and Kerov's description of the* **Martin boundary** *of $\mathbb{YF}$ — i.e. the space of all coherent systems on $\mathbb{YF}$ obtained by finite rank approximation.*

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Definition (Extending Coherent Systems to $\mathbf{1}^{\infty}\mathbb{YF}$)

Given a coherent system $\{\boldsymbol{M_n}\}_{\boldsymbol{n}\in\mathbb{N}}$ on $\mathbb{YF}$ define $\boldsymbol{M}_{\mathrm{I}}$ on $\mathbf{1}^{\infty}\mathbb{YF}$ by

$$\boldsymbol{M}_{\mathrm{I}}(\mathbf{1}^{\infty}) = \lim_{n\to\infty} M_n(1^n) \qquad \boldsymbol{M}_{\mathrm{I}}(\mathbf{1}^{\infty}\mathbf{2}\boldsymbol{w}) = \lim_{n\to\infty} M_{n+2+|w|}(1^n 2w)$$

In general $\boldsymbol{M}_{\mathrm{I}}$ defines a **sub-probability distribution** on $\mathbf{1}^{\infty}\mathbb{YF}$, i.e.

$$\boldsymbol{M}_{\mathrm{I}}(\mathbf{1}^{\infty}) + \sum_{w\in\mathbb{YF}} \boldsymbol{M}_{\mathrm{I}}(\mathbf{1}^{\infty}\mathbf{2}\boldsymbol{w}) \leq \mathbf{1}$$

Asymptotics: Type-I Observables and Convergent Type

Theorem (PS 2024)

Let $(\vec{x}, \vec{y})$ *be a Fibonacci positive specialization of* **covergent type**, *i.e.*

- $A_\infty(m) := 1 + t_m + t_m t_{m+1} + t_m t_{m+1} t_{m+2} + \ldots$ **converges**
- $B_\infty(m) := t_{m+1} + t_{m+2}\big(t_{m+1} - t_m - 1\big)A_\infty(m+3) \geq 0$

for all $m \geq 0$ *where* $x_k = c_k(1 + t_{k-1})$ *and* $y_k = c_k c_{k+1} t_k$ *and* $k \geq 1$. *Let* $\{M_n\}_{n\in\mathbb{N}}$ *be the coherent system associated to* $(\vec{x}, \vec{y})$.

M_{I} **is a probability measure on** $1^\infty\mathbb{YF}$ **iff** $\prod_{k=0}^{\infty}(1 + t_k)^{-1} \neq 0$.

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M_{I} **is a probability measure on** $1^\infty\mathbb{YF}$ **iff** $\prod_{k=0}^{\infty}(1 + t_k)^{-1} \neq 0$.
When this is the case we have:

- $M_{\mathrm{I}}(1^\infty) = \prod_{k=0}^{\infty}(1 + t_k)^{-1}$
- $M_{\mathrm{I}}(1^\infty 2w) = (n+1) M_n(w) B_\infty(n) \prod_{k=n}^{\infty}(1 + t_k)^{-1}$

for all $w \in \mathbb{YF}_n$. **If this is not the case,** M_{I} **vanishes on** $1^\infty\mathbb{YF}$.

Asymptotics: Power Specializations ($p = 1$)

Example: Power specialization $p = 1$

Let $(\vec{x}, \vec{y})$ *be the* **power specialization** *with* $p = 1$*:*

$$x_n = 1 + t_{n-1} \; , \; y_n = t_n \quad \text{where } t_n = \varkappa / n$$

where $0 < \varkappa < \varkappa_1 \approx 0.844637$*. In this case* $\prod_{k=0}^{\infty} (1 + t_k)^{-1}$ **vanishes** *and consequently* M_{I} **vanishes on** $1^{\infty}\mathbb{YF}$*.*

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This specialization behaves like the **Charlier specialization***: For any* $k \geq 1$ *the scaled runs of* $M_n(w)$*-distributed random Fibonacci works* $w \in \mathbb{YF}_n$ *jointly converge in distribution:*

$$(R_1(w), \ldots, R_k(w)) \xrightarrow[n \to \infty]{d} (\alpha_1, \ldots, \alpha_k)$$

where $\alpha_1, \ldots, \alpha_k \in [0, 1]$ *are* **iid.** η_{κ}*-distribued random variables.*

Asymptotics: Power Specializations ($p = 2$)

Example: Power specialization $p = 2$

Let $(\vec{x}, \vec{y})$ *be the* **power specialization** *with* $p = 2$*:*

$$x_n = 1 + t_{n-1} \ , \ y_n = t_n \quad \text{where } t_n = \varkappa / n^2$$

where $0 < \varkappa < \varkappa_2 \approx 1.41056$*. In this case*

$$\prod_{k=0}^{\infty} (1 + t_k)^{-1} = \pi\sqrt{\varkappa}\,\mathrm{csch}(\pi\sqrt{\varkappa}) > 0$$

and consequently $M_{\mathbb{I}}$ **is a probability measure on** $1^{\infty}\mathbb{YF}$.

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and consequently M_{I} **is a probability measure on** $1^{\infty}\mathbb{YF}$*. Note that:*

- $M_{\mathrm{I}}(1^{\infty}) = \pi\sqrt{\varkappa}\,\mathrm{csch}(\pi\sqrt{\varkappa})$
- $M_{\mathrm{I}}(1^{\infty}2w) = \pi\sqrt{\varkappa}\,\mathrm{csch}(\pi\sqrt{\varkappa})(n+1)M_n(w)B_{\infty}(n)\prod_{k=1}^{n-1}\left(1 + \frac{\varkappa}{k^2}\right)$

where $|w| = n$ *and* $B_{\infty}(n)$ *is expressed using* ${}_1F_2(1; n+3, n+3; \varkappa)$.