Orders and shuffles on nestohedra

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Conférence "Algèbres de Hopf combinatoires et Analyse"
en l'honneur de Dominique Manchon
Clermont-Ferrand, 27-28 Novembre 2025

Outline

Shuffles of packed words and planar trees

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1. Shuffles of packed words and planar trees
2. Hypergraph polytopes (a.k.a. nestoedra)
3. Algebraic structures on faces of nestohedra

Shuffles of permutations

Let $\mathcal{A}$ be a finite alphabet and $\mathcal{A}^*$ its set of (possibly empty) words.
For any $a, b \in \mathcal{A}$ and $m, n \in \mathcal{A}^*$ the shuffle product is:

$$a.m \sqcup\!\sqcup\, b.n = a.(m \sqcup\!\sqcup\, b.n) + b.(a.m \sqcup\!\sqcup\, n),$$

with $\varepsilon \sqcup\!\sqcup\, m = m \sqcup\!\sqcup\, \varepsilon = m$, where ε is the empty word.
Identify a permutation $\pi \in \mathfrak{S}_n$ with the word $\pi(1)\dots\pi(n)$.
For $\sigma \in \mathfrak{S}_q$, we denote by $\sigma + p$ the permutation of $\mathfrak{S}_{p+1,\dots,p+q}$ satisfying $(\sigma + p)(i) = \sigma(i-p) + p$
Given two permutations $\pi \in \mathfrak{S}_p$ and $\sigma \in \mathfrak{S}_q$, Malvenuto-Reutenauer define the (horizontal) shuffle of permutations as:

$$\pi \sqcup\!\sqcup_h \sigma = \pi \sqcup\!\sqcup\, (\sigma + p)$$

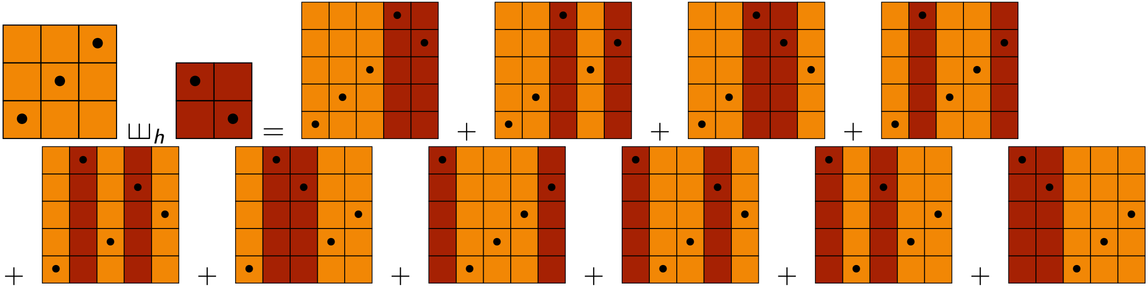
Example:

$123 \sqcup\!\sqcup_h 21 = 12354 + 12534 + 12543 + 15234 + 15243 + 15423 + 51234 + 51243 + 51423 + 54123$

Horizontal shuffle

Example:

$123 \sqcup\!\sqcup_h 21 = 12354 + 12534 + 12543 + 15234 + 15243 + 15423 + 51234 + 51243 + 51423 + 54123$



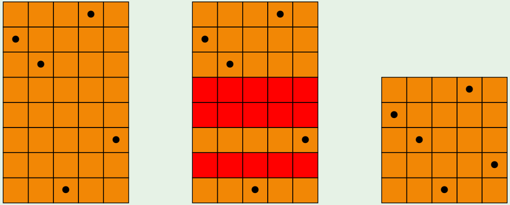
Malvenuto-Reutenauer Hopf algebra [$\mathfrak{S}Sym$: Malvenuto-Reutenauer 1995, "FQSym" : Duchamp-Hivert-Thibon, 2002]

$$\Delta(\sigma) = \sum_{p=0}^{n} \mathrm{std}\left(\sigma(1)\ldots\sigma(p)\right) \otimes \mathrm{std}\left(\sigma(p+1)\ldots\sigma(n)\right),$$

where std is the standardisation, i.e. $\mathrm{std}(t) = \phi \circ t$ with ϕ the unique increasing bijection from $\Im(t)$ to $[\![|\Im(t)|]\!]$.

Example:

$$\mathrm{std}(76183) = 43152$$



Example:

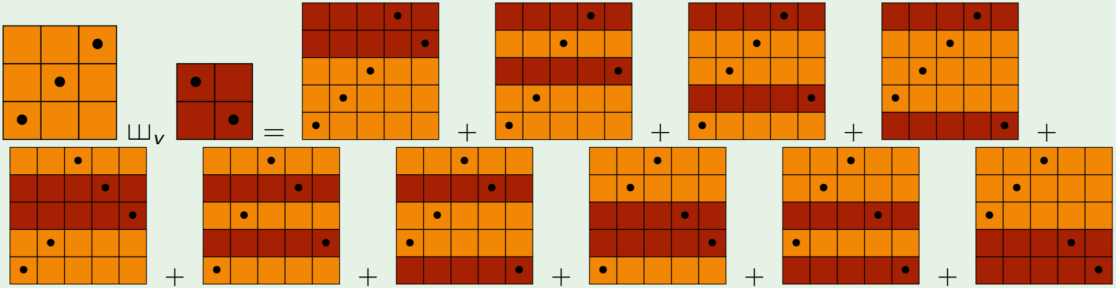
$$\Delta(132) = \varepsilon \otimes 132 + 1 \otimes 21 + 12 \otimes 1 + 132 \otimes \varepsilon.$$

Vertical shuffle

$$\sigma \sqcup\!\sqcup_v \tau = \sum_{\substack{\mathrm{std}(s)=\sigma \\ \mathrm{std}(t)=\tau}} s.t,$$

Example:

$123 \sqcup\!\sqcup_v 21 = 12354 + 12453 + 13452 + 23451 + 12543 + 13542 + 23541 + 14532 + 24531 + 34521$



Shuffles of packed words/surjections [WQSym: Hivert-Novelli-Thibon ≡2000, NQSym* : Bergeron-Zabrocki, 2005, Perm : Chapoton 2000]

Definition

A packed word is a word w on $\mathbb{N}$ such that the set of letters in w is an interval $[\![1;k]\!]$. Equivalently, it is a surjection.

$$u \sqcup\!\sqcup v = \sum_{\substack{\mathrm{pack}(\alpha)=u \\ \mathrm{pack}(\beta)=v}} \alpha\beta,$$

Example :

$$11 \sqcup\!\sqcup 221 = 22221 + 33221 + 22331 + 11221 + 11332$$

Shuffles of packed words/surjections [WQSym: Hivert-Novelli-Thibon ≡2000, NQSym* : Bergeron-Zabrocki, 2005, Perm : Chapoton 2000]

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Example :

$$11 \sqcup\!\sqcup 221 = 22221 + 33221 + 22331 + 11221 + 11332$$

In fact, more than a shuffle product : tridendriform products (WQSym free tridendriform algebra on infinitely many generators [Vong, Burgunder-Curien-Ronco, 2015])

Hopf algebra of packed words/surjections [WQSym: Hivert-Novelli-Thibon ≡2000, NQSym* : Bergeron-Zabrocki, 2005, Perm : Chapoton 2000]

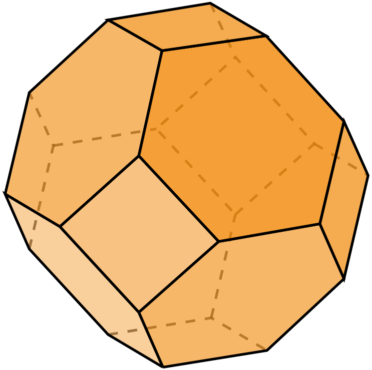
Given a packed word u, define:

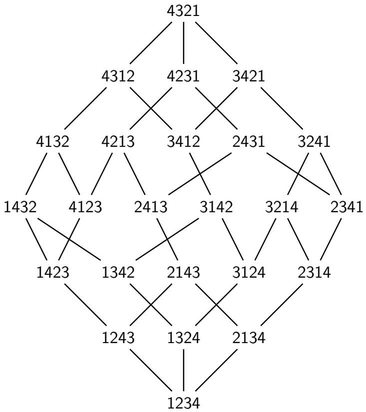
$$\Delta(u) = \sum_{k=0}^{n} u_{|[\![1;k]\!]} \otimes u_{|[\![k+1;l]\!]}$$

Example:

$\Delta(11332) = \varepsilon \otimes 11332 + 11 \otimes 221 + 112 \otimes 11 + 11332 \otimes \varepsilon$

Link with permutohedra [Schoute 1911]





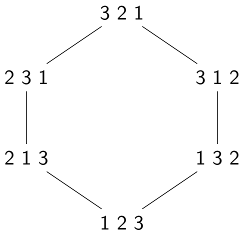
Weak Bruhat order [Verma 1968]

1234

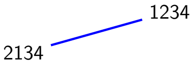
Covering relations :

$\ldots ab \ldots \lhd \ldots ba \ldots$

with $a < b$



Weak Bruhat order [Verma 1968]



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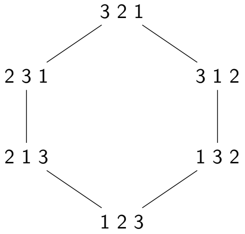

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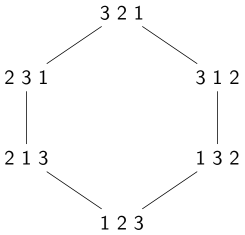

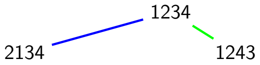

Weak Bruhat order [Verma 1968]

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$$\ldots ab \ldots \lessdot \ldots ba \ldots$$

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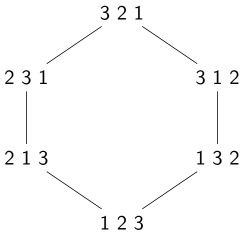

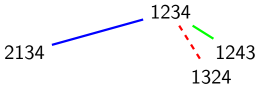

Weak Bruhat order [Verma 1968]

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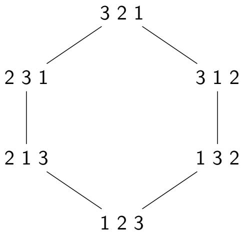
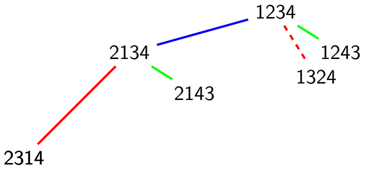
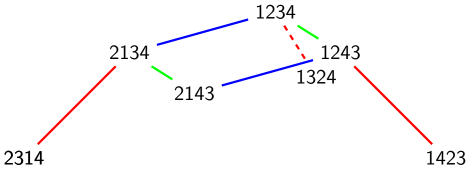




Weak Bruhat order [Verma 1968]



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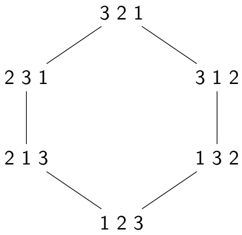

Weak Bruhat order [Verma 1968]

Covering relations :

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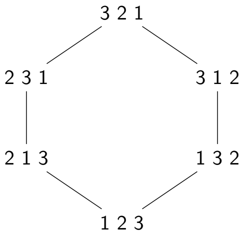
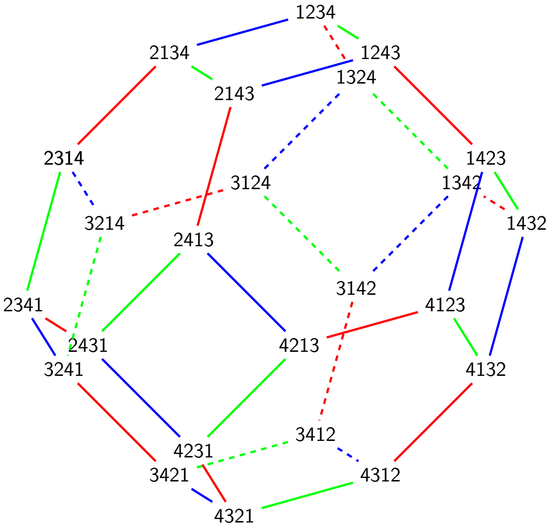




Link between the shuffle of permutations and weak Bruhat order

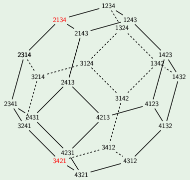
Proposition (Loday-Ronco 02)

There exists two (duplicial) products on permutations $\triangleleft$ and $\triangleright$ s.t. for any permutations $\sigma \in \mathfrak{S}_p$ and $\tau \in \mathfrak{S}_q$:

$$\sigma \sqcup\!\sqcup_h \tau = \sum_{\sigma \triangleleft \tau \leqslant p \leqslant \sigma \triangleright \tau} p,$$

where $\leqslant$ is the weak Bruhat order and $\sigma \triangleleft \tau = \sigma.(\tau + p)$ and $\sigma \triangleright \tau = (\tau + p).\sigma$.

Example:



$$21 \sqcup\!\sqcup 12 = \sum_{2134 \leqslant p \leqslant 3421} p$$

$$= 2134 + 2314 + 2341 + 3214 + 3241 + 3421$$

Link between the shuffle of packed words and an order [Palacios-Ronco 2006]

Let us represent a packed word/surjection f as the composition $\left(f^{-1}(1), \ldots, f^{-1}(n)\right)$.

Palacios and Ronco introduced the following order for $A_i < A_{i+1}$ (i.e. for all $x \in A_i$ and $y \in A_{i+1}$, $x < y$):

$$(A_1, \ldots, A_{i-1}, A_i, A_{i+1}, \ldots, A_n) \leqslant (A_1, \ldots, A_{i-1}, A_i \cup A_{i+1}, \ldots, A_n)$$
$$(A_1, \ldots, A_{i-1}, A_i \cup A_{i+1}, \ldots, A_n) \leqslant (A_1, \ldots, A_{i-1}, A_{i+1}, A_i, \ldots, A_n)$$

Proposition (Palacios-Ronco 02)

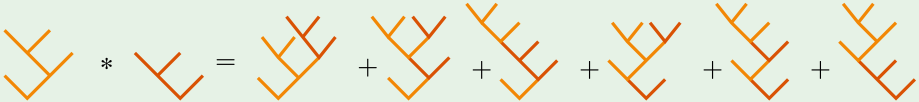
The shuffle product of packed words is the sum of element in an interval of this order.

Shuffles of planar binary trees [Loday-Ronco 1998]

Denoted by $T = \vee(T_l, T_r)$ a planar binary tree, Loday-Ronco introduced the following product in 1998:

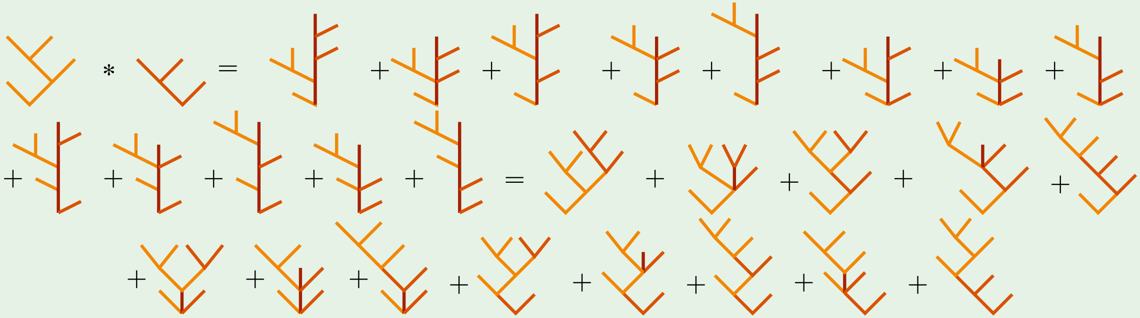
$$T * S = \vee(T_l, T_r * S) + \vee(T * S_l, T_r).$$

Example



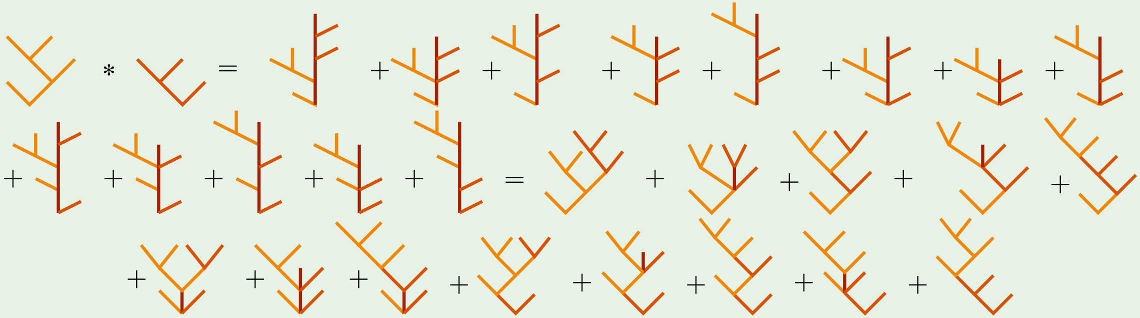
Shuffles of planar trees [Loday-Ronco, 2004; Chapoton, 2002]

Example



Shuffles of planar trees [Loday-Ronco, 2004; Chapoton, 2002]

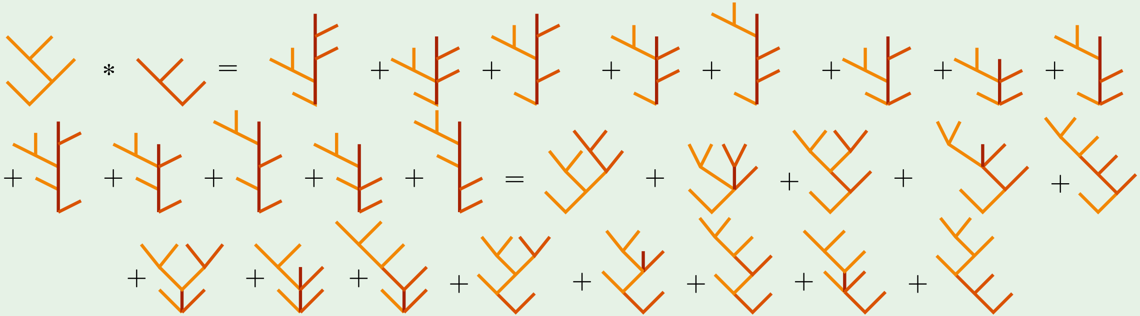
Example



Product $*$ is associative, free associated algebra but generated by infinitely many generators [Loday-Ronco, 1998]

Shuffles of planar trees [Loday-Ronco, 2004; Chapoton, 2002]

Example

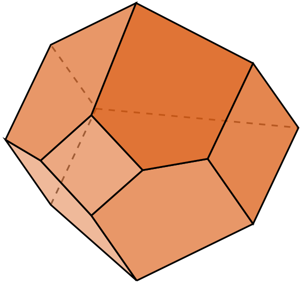


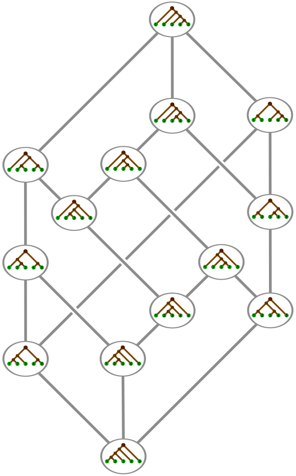
Product $*$ is associative, free associated algebra but generated by infinitely many generators [Loday-Ronco, 1998]

But:

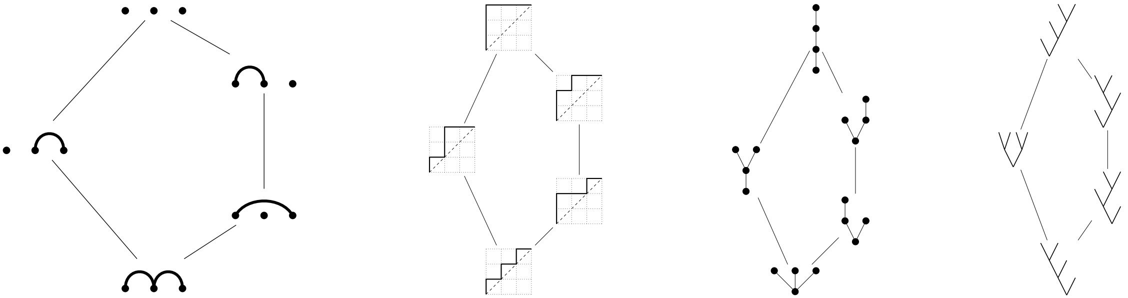
Free tridendriform algebra on one generator !

Link with associahedra



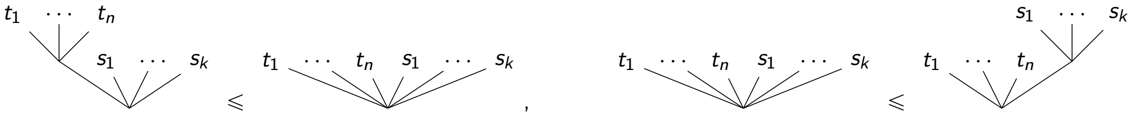


Tamari order



Link between the shuffle of planar trees and generalised Tamari order

Palacios and Ronco introduced the following order:



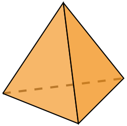
Proposition (Palacios-Ronco 02)

The shuffle product of planar trees is the sum of element in an interval of this order.

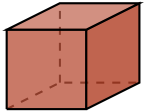
- The analog of this theorem on planar binary trees was proven by Loday and Ronco in 2002.

Hypergraph polytopes (a.k.a. nestoedra)

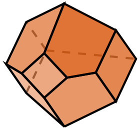
Outline



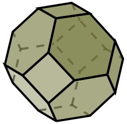
Simplices



Hypercubes



Associahedra



Permutohedra

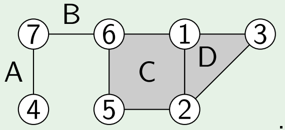
Hypergraphs

Definition

A hypergraph (on vertex set V) is a pair (V, E) where:

- V is a finite set, (the vertex set)
- E is a set of sets of size at least 2, $E \subset \mathcal{P}(V)$.

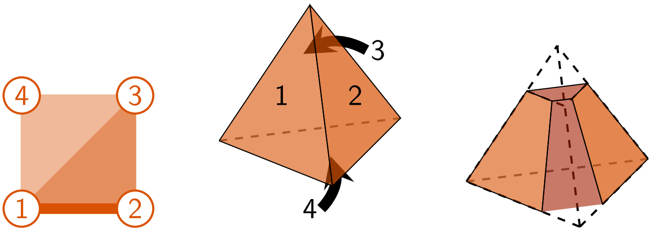
Example of an hypergraph on $[1;7]$



Warning:

All the hypergraphs considered in this talk are connected !

Hypergraph polytope [Došen, Petrić] (=nestohedra [Postnikov])



Constructs [Postnikov; Curien-Ivanovic-Obradović]

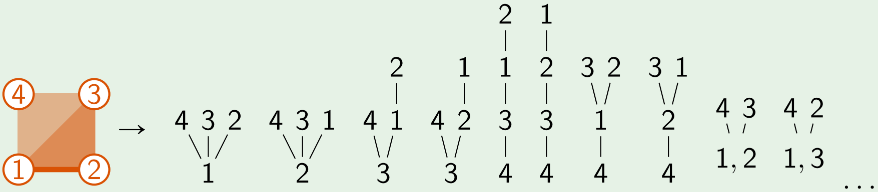
Constructs

A construct of a hypergraph H is defined inductively. For $E \subset V(H)$ (the set of vertices of H),

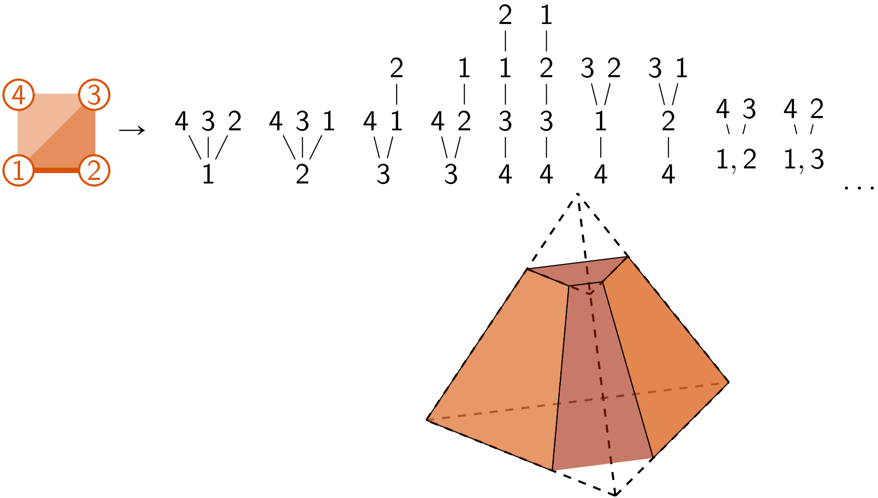
- If $E = V(H)$, the construct is the rooted tree with only one node labelled by E,
- Otherwise, denoting by $(T_1, \ldots, T_n)$ constructs on every connected component in $H - E$, a construct of H can be obtained by grafting these trees on a node labelled by E.

The set of constructs of a given hypergraph labels faces of the associated polytope.

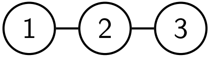
First example:



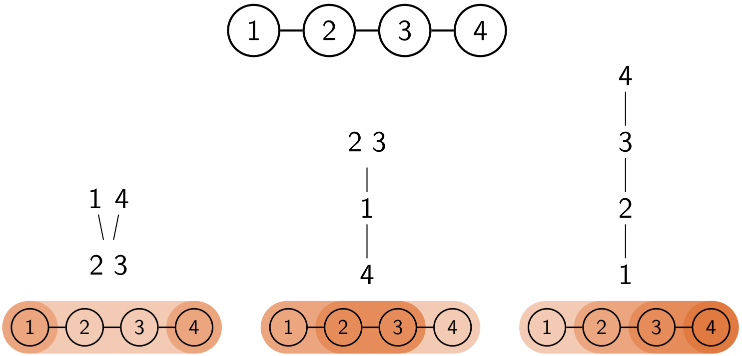
First example geometrically

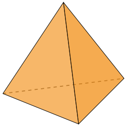
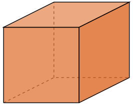
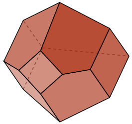
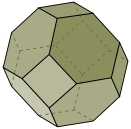
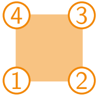
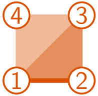
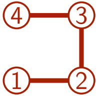
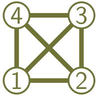


Let's practice



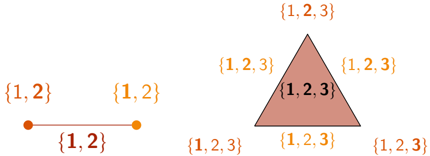
Correspondence Tubings = Constructs = Spines



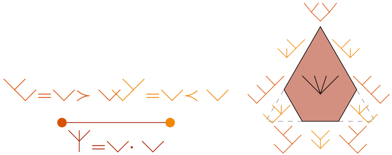
Polytope	Simplex	Hypercube	Associahedron	Permutohedron
Photo				
Associated hypergraph				
Combinatorial objects	multipointed sets	left-comb tree	planar trees	packed words
Cardinality	$2^{n+1}-1$ (A074909)	3^n (A013609)	Super-Catalan (A001003)	Fubini nbrs (A000670)

Combinatorial interpretation of constructs

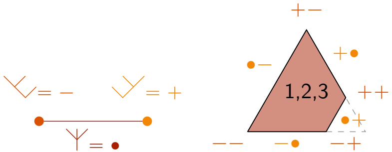
Simplex To a k-dimensional face $\{a_1, \dots, a_k\}$ is associated the multipointed set $(V(H), \{a_1, \dots, a_k\})$



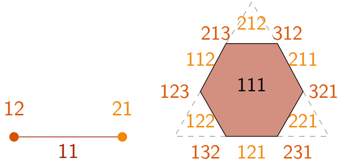
Associahedron To a k-dimensional faceis associated a planar tree on $n - k$ nodes.



Cube To a k-dimensional face is associated the set of words of length $n - 1$ on $+$, $-$ and $\bullet$ with k $\bullet$ (or left-comb trees)



Permutohedron To a k-dimensional face is associated a surjection of height k, i.e., a packed word on $\{1, \dots, n - k\}$



Algebraic structures on faces of nestohedra

Outline

Heuristics for a shuffle product

Let $\mathbf{H}^{\mathcal{X}}$ be a family of hypergraph polytopes, indexed by some finite sets $\mathcal{X}$ (sets of vertices of the associated hypergraphs).
For $S = A(S_1, \ldots, S_m)$ and $T = B(T_1, \ldots, T_n)$ two constructs of $\mathbf{H}^{\mathcal{X}}$ and $\mathbf{H}^{\mathcal{Y}}$ respectively ($\mathcal{X}, \mathcal{Y}$ disjoint), we would like to define the following operations

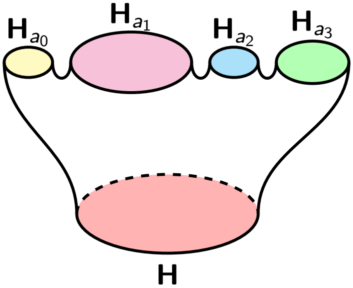
$S * T$ as a linear combination of constructs of $\mathbf{H}^{\mathcal{X} \cup \mathcal{Y}}$ having **root** A**,** or **root** B**,** or **root** $A \cup B$**.**

Universe and preteam

The considered hypergraphs belong to a set of hypergraphs $\mathfrak{U}$, called universe.

A preteam is a pair $\tau = (\{\mathbf{H}_a | a \in A\}, \mathbf{H})$ where

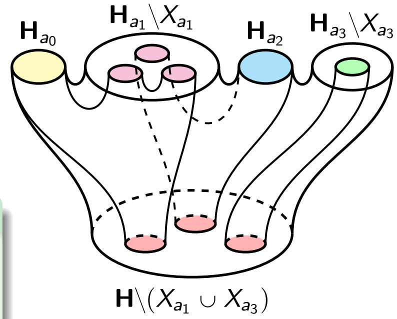
- $\{\mathbf{H}_a | a \in A, \mathbf{H}_a \in \mathfrak{U}\}$ is a set of pairwise disjoint hypergraphs, called participating hypergraphs
- $\mathbf{H} \in \mathfrak{U}$ is a hypergraph such that $H = \bigcup_{a \in A} H_a$, called supporting hypergraph.



Strict and quasi-strict teams [Curien–D.O.–Obradovic, 25]

A preteam is a (resp. quasi-strict) strict team
if the connected components obtained by deleting a subset X_a to every hypergraph $\mathbf{H}_a$ are

- in $\mathfrak{U}$
- and included in the connected components of $\mathbf{H} \setminus \left(\bigcup_{a \in A} X_a\right)$ (resp. or totally disconnected).



$(X_{a_0} = X_{a_2} = \varnothing)$

Examples:

- Strict teams : Associahedra, Permutohedra, Restrictohedra, ...
- Quasi-strict teams : Simplices, Hypercubes, Erosohedra, ...

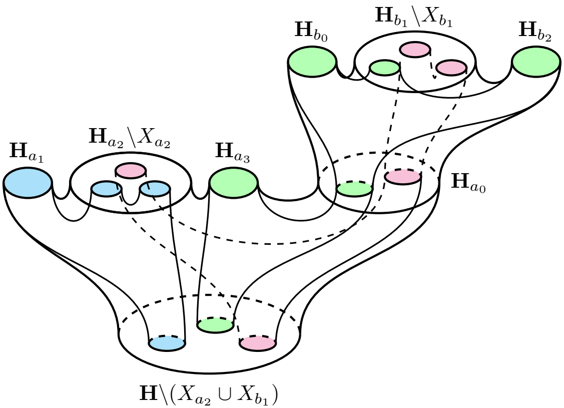
Shuffle product

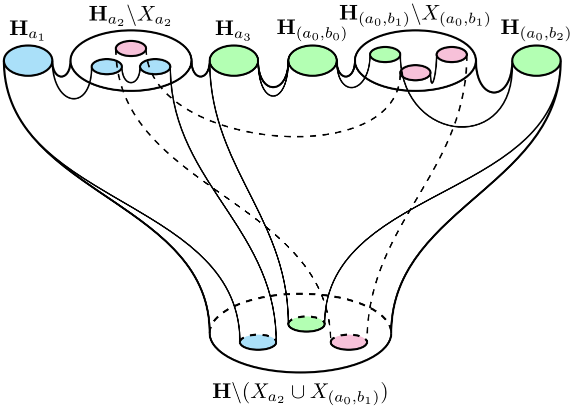
Considering a team E and denoting by δ a tuple of constructs of the team's participating hypergraphs, we inductively associate to δ a sum of constructs of the supporting hypergraph:

$$*(\delta) = \sum_{\emptyset \subset B \subseteq A} q^{|B|-1} (\bigcup_{b \in B} X_b)(*(\delta_1^B), \ldots, *(\delta_{n_B}^B)),$$

Associative clan

A set of (resp. quasi-strict) strict team with "good" closure properties is called strict clan (each connected component obtained from the supporting hypergraph is itself a supporting hypergraph of a team).





Associativity of $*$

Theorem (Curien-D.O.-Obradović, 25)

Consider a clan $\mathcal{C}$. The product $*$ is associative if

- *$\mathcal{C}$ is strict,*
- *or $\mathcal{C}$ is quasi-strict and $q = -1$.*

- Strict clans: Associahedra, Permutohedra, Restrictohedra, ...
- quasi-strict clans: Simplices, Hypercubes, ...

Order on faces of nestohedra : generalised flip order [Curien–Laplante-Anfossi, Curien–D.O.–Obradovic]

- Hypergraphs on $\mathbb{Z}$
- For $X_1, X_2 \subseteq \mathbb{Z}$, we write $X_1 < X_2$ if $\max(X_1) < \min(X_2)$.
- Every considered set of hypergraphs (preteam, decomposition) is ordered

Definition

Let us consider two constructs S and T of an ordered hypergraph $\mathbf{H}$.
$S \lessdot T$ if and only if there exists

- either two adjacent vertices U and V in S such that $\min(U) > \max(V)$, U is the parent of V and T can be obtained from S by a contraction of the edge between U and V which merges both vertices into a unique vertex $U \cup V$, (contraction)
- or two adjacent vertices U and V in T such that $\min(U) > \max(V)$, V is the parent of U and S can be obtained from T by a contraction of the edge between U and V which merges both vertices into a unique vertex $U \cup V$. (split)

What about the coproduct ? [Work in progress]

Question: Would the following coproduct work ?

$$\Delta(S) = \sum_{\text{c cut}} R_c(S) \otimes *(F_c(S)) + 1 \otimes S$$

Problem: renormalisation

Possible for :

- associahedra
- permutohedra
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Thank you very much for your attention !