Strange pre- and post-Lie structures on rooted trees

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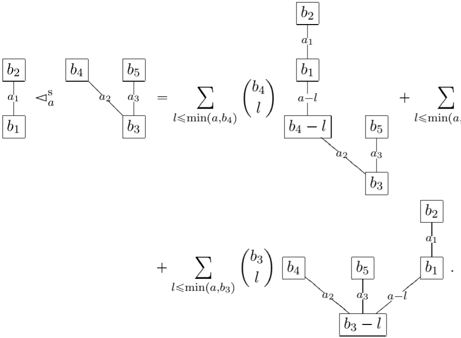
Abstract

We present a construction of pre-Lie on rooted trees whose edges and vertices are decorated, with a grafting product twisted by an action of a map acting on both edges and vertices. We show that this construction indeed gives a pre-Lie algebra if, and only if, a certain commutation relation is satisfied. Then, this pre-Lie algebra can be extended as a post-Lie algebra through a semi-direct product.
A particular example is used for normal forms in the study of stochastics PDEs. Here, the set of decorations of edges and vertices is $\mathbb{N}^{d+1}$ and the acting map is the exponentiation of a simpler map.

In the theory of regularity structures, strange pre-Lie products defined on decorated rooted trees occur.



Here, $a, a_1, a_2, a_3, b_1, \ldots, b_5$ are integers. The game between the decorations of the edges and of the vertices is rather unusual.

Aim

Try to understand this game on the decorations and to insert it in a more classical settings of decorated rooted trees.

Definition

A pre-Lie algebra is a pair $(V, \lhd)$, where V is a vector space and $\lhd : V \otimes V \longrightarrow V$ such that, for any $x, y, z \in V$,

$$x \lhd (y \lhd z) - (x \lhd y) \lhd z = y \lhd (x \lhd z) - (y \lhd x) \lhd z.$$

These objects are also called left-symmetric, Vinberg or Gerstenhaber algebras. They appear in numerical analysis (Runge-Kutta methods and Butcher's series) Quantum Field Theory and Renormalization (structures on trees and Feynman graphs), Ecalle's mould calculus (arborification's process), etc.

Example

The Lie algebra of derivations of $\mathbb{K}[X]$ is a pre-Lie algebra, with the pre-Lie product defined by

$$P(X)\frac{d}{dX} \lhd Q(X)\frac{d}{dX} = P\frac{dQ}{dX}\frac{d}{dX}.$$

Example

Let A be a commutative algebra and D be a derivation of A. Then A is a pre-Lie algebra with the product defined by

$$a \lhd b = aD(b).$$

Example

Let $(\mathcal{P}, \circ)$ be an operad. Then $P = \bigoplus_{n=1}^{\infty} \mathcal{P}(n)$ is a pre-Lie algebra, with, for any $p \in \mathcal{P}(n)$ and $q \in P$,

$$q \lhd p = \sum_{i=1}^{n} p \circ_i q.$$

Example

Let $\mathfrak{g}$ be a graded Lie algebra, with $\mathfrak{g}_0 = (0)$.
For any $x \in \mathfrak{g}_k$ and $y \in \mathfrak{g}_l$, with $k, l \geqslant 1$, we put

$$x \lhd y = \frac{l}{k+l}[x, y].$$

Then $(\mathfrak{g}, \lhd)$ is pre-Lie. The induced Lie bracket is $[-, -]$.

Example [Loday and Ronco, 2010]

Let B be a bialgebra, such that:

1. $B = S(V)$ as an algebra, for a particular subspace V of the augmentation ideal of B.
2. B is left-sided: for any $v \in V$,

$$\Delta(v) - v \otimes 1 - 1 \otimes v \in B \otimes V.$$

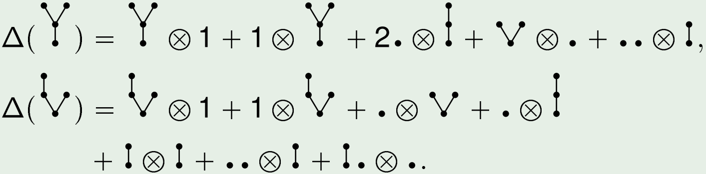
Then V^* is a pre-Lie algebra, with the product defined by

$$\forall v \in V,\ (f \triangleleft g)(v) = (f \otimes g) \circ \Delta(v).$$

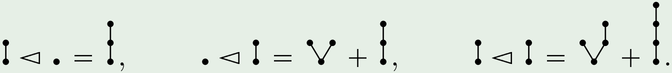
The elements of V^* are extended to B by making then vanishing on any $S^k(V)$ with $k \neq 1$.

The Butcher, Connes and Kreimer bialgebra

This bialgebra is defined on rooted forests, with a coproduct given by admissible cuts.

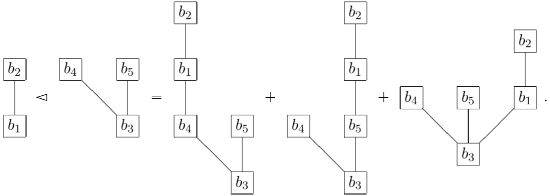


Dually, the space of rooted trees admits a pre-Lie product, given by graftings.



Theorem [Ermolaev, 1994, Chapoton and Livernet, 2001]

Let D_V be a vector space. The free pre-Lie algebra is the space of rooted trees whose vertices are decorated by elements of D_V, with the grafting product.



Here, $b_1, \ldots, b_5$ are elements of D_V.

If $(V, \lhd)$ is a pre-Lie algebra, then it is a Lie algebra, with

$$[x, y] = x \lhd y - y \lhd x.$$

Its enveloping algebra admits a beautiful description:

Guin-Oudom construction [Guin and Oudom, 2005]

Let $(V, \lhd)$ be a pre-Lie algebra. The pre-Lie product is extended to $S(V) \otimes V$ as follows:

1. $1 \lhd v = 0$.
2. If $k \geqslant 2$,

$$v_1 \ldots v_k \lhd v = v_1 \lhd (v_2 \ldots v_k \lhd v) - \sum_{i=2}^{k} (v_2 \ldots (v_1 \lhd v_i) \ldots v_k) \lhd v.$$

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This is then extended to $S(V) \otimes S(V)$ with the help of the usual coproduct of $S(V)$:

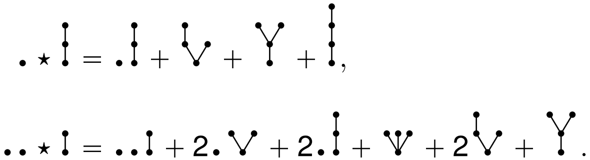
$$\forall x, y, z \in S(V), \qquad x \lhd (yz) = \sum \left(x^{(1)} \lhd y\right) \left(x^{(2)} \lhd y\right)$$

We define $\star$ on $S(V)$ by $x \star y = \sum x^{(1)} \left(x^{(2)} \lhd y\right)$.
Then $(S(V), \star, \Delta)$ is a Hopf algebra, isomorphic to the enveloping algebra of $(V, [-,-])$.

Example

When applied to the (free) pre-Lie algebra of decorated rooted trees, we obtain the Grossman-Larson Hopf algebra, with its grafting products of forests. It is in duality with the Butcher-Connes-Kreimer Hopf algebra of rooted trees, with the coproduct of admissible cuts.

Grossman-Larson product:



In order to take in account the decorations on the edges, we shall use multiple pre-Lie algebras. Let us fix a space D_E, which will be used to decorate the edges.

Definition

A D_E-multiple pre-Lie algebra is a pair $(V, \lhd)$, where

$$\lhd : \left\{ \begin{array}{rcl} D_E \otimes V \otimes V & \longrightarrow & V \\ a \otimes x \otimes y & \longmapsto & x \lhd_a y \end{array} \right.$$

with, for any $a, a' \in D_E$,

$$x \lhd_a (y \lhd_{a'} z) - (x \lhd_a y) \lhd_{a'} z = y \lhd_{a'} (x \lhd_a z) - (y \lhd_{a'} x) \lhd_a z.$$

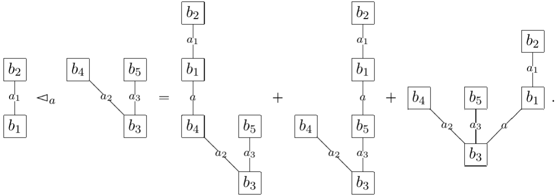
Example

Let A be a commutative algebra and $D_1, \ldots, D_N$ be derivations on A which pairwise commute. Then A is a $\mathbb{K}^N$-multiple pre-Lie algebra, with

$$a \triangleleft_{(\lambda_1,\ldots,\lambda_N)} b = \sum_{i=1}^{N} \lambda_i a D_i(b).$$

Theorem

Let D_V be a vector space. The free D_E-multiple pre-Lie algebra is the space of rooted trees whose vertices are decorated by elements of D_V and the edges by elements of D_E, with the grafting product.



Here, $b_1, \ldots, b_5$ are elements of D_V, and a_1, a_2, a_3, a elements of D_E.

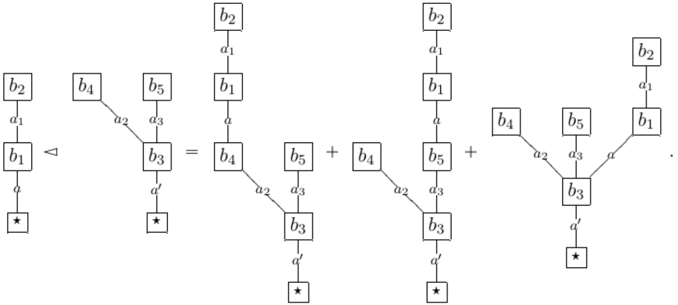
Multiple pre-Lie algebras are related to homogeneous D_E-graded pre-Lie algebras:

Proposition

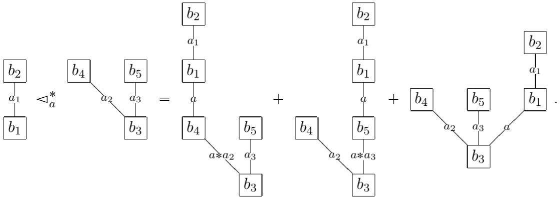
Let $(V, \triangleleft)$ be a D_E-multiple pre-Lie algebra. Then $D_E \otimes V$ is a pre-Lie algebra, with

$$a \otimes x \triangleleft a' \otimes y = a' \otimes x \triangleleft_a y.$$

We apply this to free D_E-multiple pre-Lie algebras.
We identify the tensor $a \otimes T$ with the planted tree obtained by grafting T on a undecorated root, with an edge decorated by a.
This gives a pre-Lie product on planted trees, with the product given by identification of the root of the tree on the left with a vertex of the tree on the right.



Other generalizations of pre-Lie algebras, based on trees with decorations, can be found in the literature. For example, if $(D_E, *)$ is a commutative and associative algebra, in a free $(D_E, *)$-family pre-Lie algebra,

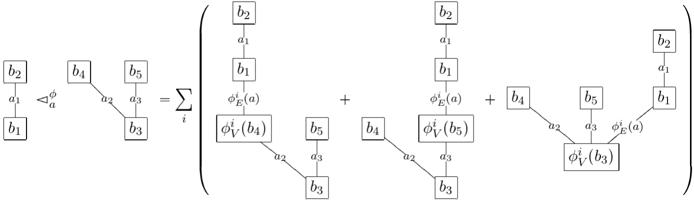


In all these examples, the decorations of the vertices are "inert" and are never modified.

In order to formalize the interaction between the decorations of the vertices and of the edges, we use a linear map

$$\phi : \left\{ \begin{array}{rcl} D_E \otimes D_V & \longrightarrow & D_E \otimes D_V \\ a \otimes b & \longmapsto & \displaystyle\sum_i \phi_E^i(a) \otimes \phi_V^i(b). \end{array} \right.$$

This map is used to deform the grafting multiple pre-Lie product, making ϕ act on the edge of the added edge and on the vertex which holds the grafting.



Here, $b_1, \ldots, b_5$ are elements of D_V, and a, a_1, a_2, a_3 elements of D_E.

We denote by ϕ_{23} and ϕ_{13} the endomorphisms of $D_E \otimes D_E \otimes D_V$ defined by

$$\phi_{13}(a \otimes a' \otimes b) = \sum_i \phi_E^i(a) \otimes a' \otimes \phi_V^i(b),$$
$$\phi_{23}(a \otimes a' \otimes b) = \sum_i a \otimes \phi_E^i(a') \otimes \phi_V^i(b).$$

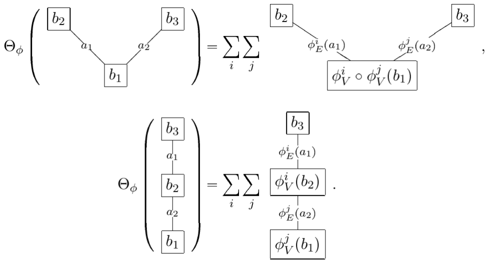
Theorem

$(\mathcal{T}(D_E, D_V), \triangleleft^\phi)$ is a D_E-multiple pre-Lie algebra if, and only if,

$$\phi_{13} \circ \phi_{23} = \phi_{23} \circ \phi_{13}.$$

Such a map ϕ will called tree-compatible.

If ϕ is tree-admissible, we define an endomorphism Θ_ϕ of $\mathcal{T}(D_E, D_V)$ by the action of ϕ on all the pairs (edge, source of the edge) of any tree.



The tree-compatibility of ϕ is required to prove that Θ_ϕ is well-defined, as shown for the first tree in this example.

Proposition

The map Θ_ϕ is a D_E multiple pre-Lie algebra from $(\mathcal{T}(D_E, D_V), \triangleleft)$ to $(\mathcal{T}(D_E, D_V), \triangleleft^\phi)$.

Corollary

The D_E-multiple pre-Lie algebra $(\mathcal{T}(D_E, D_V), \triangleleft^\phi)$ is generated by the trees with a single vertex if, and only if, ϕ is surjective.

Corollary

The D_E-multiple pre-Lie algebra $(\mathcal{T}(D_E, D_V), \triangleleft^\phi)$ is freely generated by the trees with a single vertex if, and only if, ϕ is bijective.

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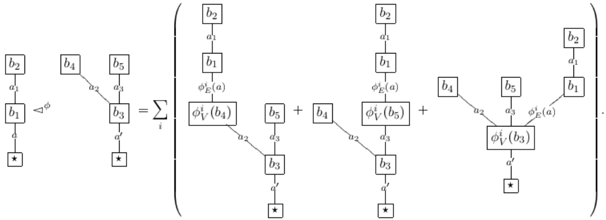
When ϕ is tree-compatible, we can apply the Guin-Oudom construction to the pre-Lie algebra $D_E \otimes \mathcal{T}(D_E, D_V)$. The elements of this pre-Lie algebra are identified with planted trees.



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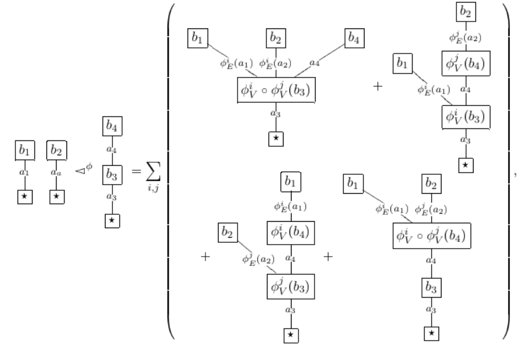

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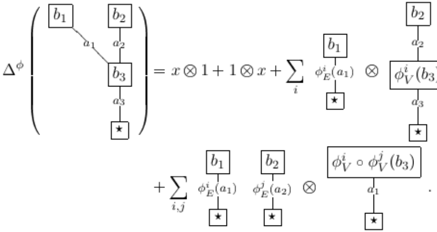
Dually, we obtain a Butcher-Connes-Kreimer-like Hopf algebra of planted rooted trees.



Example

If f is an endomorphism of D_E and g an endomorphism of D_V, then $f \otimes g$ is tree-compatible.

Let D_E and D_V be finite-dimensional spaces, with bases $\mathcal{B}_E = (a_1, \ldots, a_m)$ and $\mathcal{B}_V = (b_1, \ldots, b_n)$. The basis $\mathcal{B}_E \otimes \mathcal{B}_V$ of $D_E \otimes D_V$ is

$$\mathcal{B}_E \otimes \mathcal{B}_V = (a_1 \otimes b_1, \ldots, a_1 \otimes b_n, \ldots, a_m \otimes b_1, \ldots, a_m \otimes b_n).$$

If ϕ is an endomorphism of $D_E \otimes D_V$, its matrix in the basis $\mathcal{B}_E \otimes \mathcal{B}_V$ is written under the form of a matrix with m^2 blocks of size $n \times n$:

$$\begin{pmatrix} A_{11} & \ldots & A_{1m} \\ \vdots & & \vdots \\ A_{m1} & \ldots & A_{mm} \end{pmatrix}.$$

Proposition

ϕ is tree-compatible if, and only if, for any $i, j, k, l \in [\![1; m]\!]$, $A_{ij} A_{kl} = A_{kl} A_{ij}$.

Example

If D_E or D_V is 1-dimensional, then any endomorphism ϕ of $D_E \otimes D_V$ is tree-compatible.

Example ($\mathbb{K} = \mathbb{C}$)

Let us assume that D_V is 2-dimensional. Then ϕ is tree-compatible if, in convenient bases of D_E and D_V, the matrix of ϕ has one of the following forms:

$$\left(\begin{array}{cc|c|cc} a_{11} & b_{11} & \cdots & a_{1n} & b_{1n} \\ 0 & a_{11} & & 0 & a_{1n} \\ \hline \vdots & & & & \vdots \\ \hline a_{n1} & b_{n1} & \cdots & a_{nn} & b_{nn} \\ 0 & a_{n1} & & 0 & a_{nn} \end{array}\right) \text{ or } \left(\begin{array}{cc|c|cc} a_{11} & 0 & \cdots & a_{1n} & 0 \\ 0 & b_{11} & & 0 & b_{1n} \\ \hline \vdots & & & & \vdots \\ \hline a_{n1} & 0 & \cdots & a_{nn} & 0 \\ 0 & b_{n1} & & 0 & b_{nn} \end{array}\right).$$

Sum and composition

If ϕ and ψ are tree-compatible, and if $\phi_{13} \circ \psi_{23} = \psi_{23} \circ \phi_{13}$, then $\psi \circ \phi$ and $\psi + \phi$ are tree-compatible.

Polynomials

If ϕ is tree-compatible, then for any $P \in \mathbb{K}[X]$, $P(\phi)$ is tree-compatible.

Formal series

If ϕ is tree-compatible and locally nilpotent, that is to say

$$\forall x \in D_E \otimes D_V,\ \exists n \geqslant 1,\ \phi^n(x) = 0,$$

then, for any $P \in \mathbb{K}[[X]]$, $P(\phi)$ is tree-compatible.

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Direct sums

Let $\phi_1 : D_E^1 \otimes D_V^1 \longrightarrow D_E^1 \otimes D_V^1$ and $\phi_2 : D_E^1 \otimes D_V^2 \longrightarrow D_E^2 \otimes D_V^2$ be two tree-compatible maps, and let $\lambda, \mu \in \mathbb{K}$. We define an endomorphism $\Phi = \phi_1 \oplus_{\lambda,\mu} \phi_2$ of $(D_E^1 \oplus D_E^2) \otimes (D_V^1 \oplus D_V^2)$ by

$$\Phi(a_1 \otimes b_1) = \phi_1(a_1 \otimes b_1), \qquad \Phi(a_2 \otimes b_2) = \phi_2(a_2 \otimes b_2),$$
$$\Phi(a_1 \otimes b_2) = \lambda a_1 \otimes b_2, \qquad \Phi(a_2 \otimes b_1) = \mu a_2 \otimes b_1.$$

Then Φ is tree-compatible.

We fix $d \in \mathbb{N}$ and put

$$D^{\mathrm{s}}_E = D^{\mathrm{s}}_V = \mathrm{Vect}\left(\mathbb{N}^{d+1}\right).$$

The canonical basis of $\mathbb{N}^{d+1}$ is denoted by $\left(\epsilon^{(0)}, \ldots, \epsilon^{(d)}\right)$.

Proposition

For any i, we put

$$\partial^{(j)}: \begin{cases} D^{\mathrm{s}}_E \otimes D^{\mathrm{s}}_V & \longrightarrow & D^{\mathrm{s}}_E \otimes D^{\mathrm{s}}_V \\ a \otimes b & \longmapsto & b_j \left(a - \epsilon^{(j)}\right) \otimes \left(b - \epsilon^{(j)}\right), \end{cases}$$

Then any linear span of $\partial^{(j)}$'s is tree-compatible and locally nilpotent.

As a consequence, any formal series in a linear span of $\partial^{(j)}$'s is tree-compatible.

Proposition

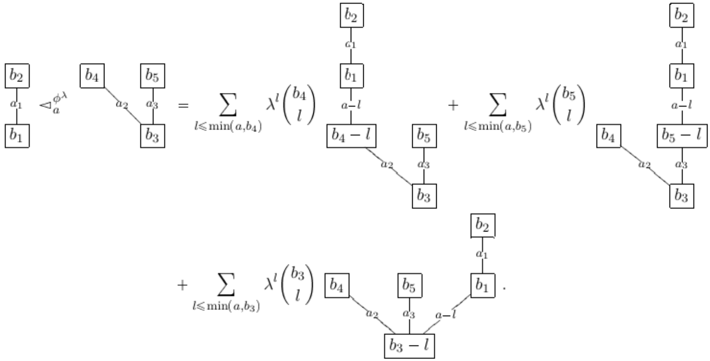
For any $\lambda \in \mathbb{K}^{d+1}$, we put

$$\phi^\lambda = \exp\left(\lambda_0 \partial^{(0)}\right) \circ \cdots \circ \exp\left(\lambda_d \partial^{(d)}\right).$$

Then ϕ^λ is tree-compatible. Moreover, for any $a, b \in \mathbb{N}^{d+1}$,

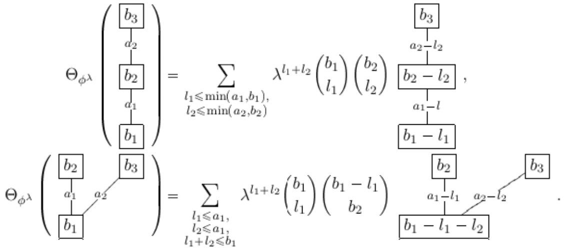
$$\phi^\lambda(a \otimes b) = \sum_{l \leqslant \min(a,b)} \lambda^l \binom{b}{l} (a-l) \otimes (b-l).$$

The map ϕ^λ is invertible, of inverse $\phi^{-\lambda}$.



In particular, the multiple pre-Lie product of the introduction corresponds to $\lambda = (1, \ldots, 1)$.

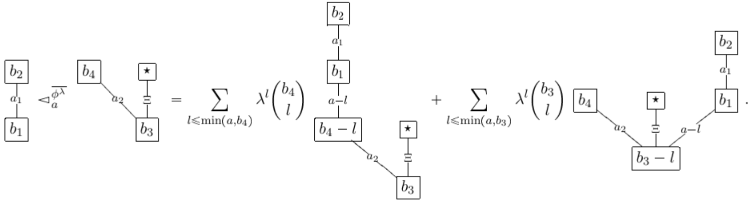
For any $\lambda \in \mathbb{K}^{d+1}$, Θ_{ϕ^λ} is invertible, of inverse $\Theta_{\phi^{-\lambda}}$.



Proposition

The map Θ_{ϕ^λ} is a multiple pre-Lie algebra isomorphism from $(\mathcal{T}(D^s_E, D^s_V), \triangleleft^{\phi^\mu})$ to $(\mathcal{T}(D^s_E, D^s_V), \triangleleft^{\phi^{\lambda+\mu}})$.
In particular, it is an isomorphism from $(\mathcal{T}(D^s_E, D^s_V), \triangleleft)$ to $(\mathcal{T}(D^s_E, D^s_V), \triangleleft^{\phi^\lambda})$.

To take care of noises, we need to add a special decoration $\star$ (on leaves uniquely), on which it is not possible to graft, and Ξ on edges.



We need to extend D_E^s and D_V^s.

$$\overline{D_E^s} = \mathrm{Vect}(\mathbb{N}^{d+1} \sqcup \{\Xi\}), \qquad \overline{D_V^s} = \mathrm{Vect}(\mathbb{N}^{d+1} \sqcup \{\star\}).$$

We extend ϕ^λ:

$$\overline{\phi}^\lambda(a \otimes \star) = 0, \qquad \overline{\phi}^\lambda(\Xi \otimes b) = \Xi \otimes b, \qquad \overline{\phi}^\lambda(\Xi \otimes \star) = 0.$$

This was described earlier as

$$\phi^\lambda \oplus_{0,1} \phi',$$

where ϕ' is the zero map on $\mathrm{Vect}(\Xi) \otimes \mathrm{Vect}(\star)$.

Corollary

The map $\overline{\phi}^\lambda$ is tree-compatible. The pre-Lie product for stochastic PDEs with noises corresponds to $\lambda = (1, \ldots, 1)$.

Definition [Vallette, 2007]

A post-Lie algebra is a triple $(\mathfrak{g}, \{-,-\}, \triangleleft)$ where $\mathfrak{g}$ is a vector space and $\{-,-\}, \triangleleft$ are bilinear products on $\mathfrak{g}$ such that

$$\begin{aligned}
0 &= \{\{x,y\},z\} + \{\{y,z\},x\} + \{\{z,x\},y\}.\\
x \triangleleft \{y,z\} &= \{x \triangleleft y, z\} + \{y, x \triangleleft z\}.\\
\{x,y\} \triangleleft z &= x \triangleleft (y \triangleleft z) - (x \triangleleft y) \triangleleft z\\
&\quad - y \triangleleft (x \triangleleft z) + (y \triangleleft x) \triangleleft z.
\end{aligned}$$

In particular, pre-Lie algebras are post-Lie algebras with a zero bracket $\{-,-\}$.
The Guin-Oudom construction is extended to post-Lie algebras.

Let ϕ be a tree-compatible map and P a post-Lie algebra. We want to extend the pre-Lie product $\triangleleft^\phi$ on $D_E \otimes \mathcal{T}(D_E, D_V)$ to a semi-direct product $P \otimes D_E \otimes \mathcal{T}(D_E, D_V)$. We fix two maps

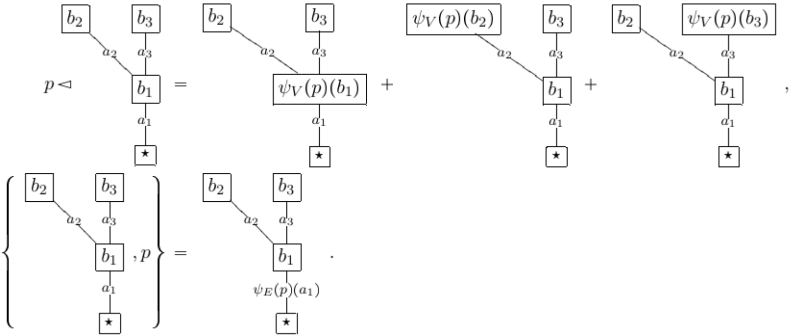
$$\psi_V : \left\{ \begin{array}{rcl} P & \longrightarrow & \mathrm{End}(D_V) \\ p & \longmapsto & \left\{ \begin{array}{rcl} D_V & \longrightarrow & D_V \\ b & \longmapsto & \psi_V(p)(b), \end{array} \right. \end{array} \right.$$

$$\psi_E : \left\{ \begin{array}{rcl} P & \longrightarrow & \mathrm{End}(D_E) \\ p & \longmapsto & \left\{ \begin{array}{rcl} D_E & \longrightarrow & D_E \\ a & \longmapsto & \psi_E(p)(a), \end{array} \right. \end{array} \right.$$

which represent the action of P on decorations of edges and vertices.

We put, for any trees T, T', any $a, a' \in D_E$, any $p, p' \in P$,

$$\begin{aligned}
a \otimes T \triangleleft a' \otimes T' &= a' \otimes T \triangleleft_a^{\phi} T',\\
\{a \otimes T, a' \otimes T'\} &= 0,\\
p \triangleleft a' \otimes T' &= \sum_{v_0 \in V(T')} a' \otimes \psi_V(p)_{v_0}(T'),\\
a \otimes T \triangleleft p' &= 0,\\
\{a \otimes T, p'\} &= \psi_E(p')(a) \otimes T.
\end{aligned}$$


p ◁
=
+
+
,
,p
=
.
b2
b3
b1
a1
a2
a3
ψV(p)(b1)
ψV(p)(b2)
ψV(p)(b3)
ψE(p)(a1)
⋆

Proposition

$P \otimes D_E \otimes \mathcal{T}(D_{,E}, D_V)$ is a post-Lie algebra if, and only if,

$$\begin{aligned}
\psi_E(\{p,p'\}) &= \psi_E(p') \circ \psi_E(p) - \psi_E(p) \circ \psi_E(p'), \\
\psi_E(p \lhd p') &= 0, \\
\psi_V(\{p,p'\}) &= \psi_V(p) \circ \psi_V(p') - \psi_V(p') \circ \psi_V(p) \\
&\quad - \psi_V(p \lhd p') + \psi_V(p' \lhd p), \\
\phi \circ (\psi_E(p) \otimes \mathrm{Id}_{D_V}) &= \phi \circ (\mathrm{Id}_{D_E} \otimes \psi_V(p)) - (\mathrm{Id}_{D_E} \otimes \psi_V(p)) \circ \phi.
\end{aligned}$$

In particular, ψ_E should be a Lie algebra morphism from $(P, -\{-,-\})$ to $\mathrm{End}(D_E)$ and ψ_V a Lie algebra morphism from $(P, \{-,-\}_\lhd)$ to $\mathrm{End}(D_V)$, with

$$\{p,p'\}_\lhd = \{p,p'\} + p \lhd p' - p' \lhd p.$$

When P is trivial (that is to say, both $\lhd$ and $\{-,-\}$ are zero), this simplifies:

Corollary

If P is pre-Lie, then $P \otimes D_E \otimes \mathcal{T}(D_{,E}, D_V)$ is a post-Lie algebra if, and only if,

$$\begin{aligned} \psi_E(p') \circ \psi_E(p) &= \psi_E(p) \circ \psi_E(p'),\\ \psi_V(p) \circ \psi_V(p') &= \psi_V(p') \circ \psi_V(p),\\ \phi \circ (\psi_E(p) \otimes \mathrm{Id}_{D_V}) &= \phi \circ (\mathrm{Id}_{D_E} \otimes \psi_V(p)) - (\mathrm{Id}_{D_E} \otimes \psi_V(p)) \circ \phi. \end{aligned}$$

In the case of $\phi^{(1,\ldots,1)}$, we take $P = \mathrm{Vect}(X_i \mid 0 \leqslant i \leqslant d)$ with the trivial post-Lie structure.

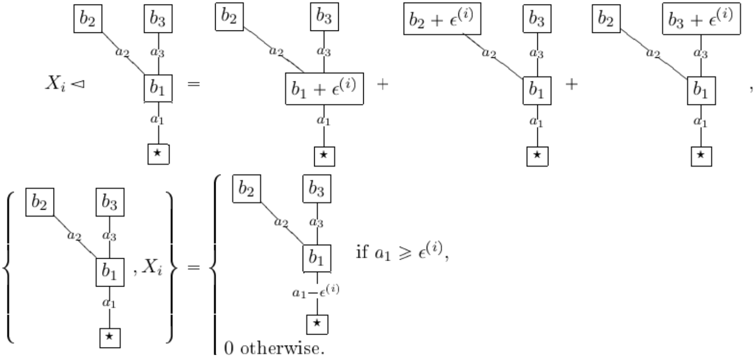
Example for stochastic PDEs

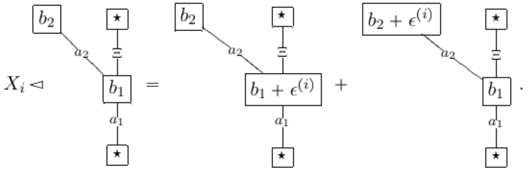
We obtain a post-Lie algebra on $P \otimes D_E^s \otimes \mathcal{T}(D_E^s, D_V^s)$ with

$$\psi_V(X_i)(a) = a + \epsilon^{(i)}, \qquad \psi_E(X_i)(a) = a - \epsilon^{(i)}.$$

This is extended to the case with noise by

$$\overline{\psi}_V(X_i)(\star) = 0, \qquad \overline{\psi}_E(X_i)(\Xi) = 0.$$


$X_i \triangleleft$
b_1
b_2
b_3
a_1
a_2
a_3
$b_1 + \epsilon^{(i)}$
$b_2 + \epsilon^{(i)}$
$b_3 + \epsilon^{(i)}$
X_i
$a_1 - \epsilon^{(i)}$
if $a_1 \geqslant \epsilon^{(i)}$,
0 otherwise.


$X_i \lhd$
b_2
a_2
b_1
a_1
$=$
$b_1 + \epsilon^{(i)}$
$+$
$b_2 + \epsilon^{(i)}$

References

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Thank you for your attention!