Deformation quantization with branes and coloured MZVs

Clermont-Ferrand, en l'honneur de Dominique Manchon

Damien Calaque (Université de Montpellier)

27 November 2025

Deformation quantization

Given a Poisson bracket $\{-,-\}$ on a commutative algebra A_0, does there exist an associative formal deformation of the commutative product $\cdot$ of the form $\star = \cdot + \hbar\{-,-\} + o(\hbar)$?

Given a Poisson bracket $\{-,-\}$ on a commutative algebra A_0, does there exist an associative formal deformation of the commutative product $\cdot$ of the form $\star = \cdot + \hbar\{-,-\} + o(\hbar)$?

The general answer is "NO" [Mathieu]

Given a Poisson bracket $\{-,-\}$ on a commutative algebra A_0, does there exist an associative formal deformation of the commutative product $\cdot$ of the form $\star = \cdot + \hbar\{-,-\} + o(\hbar)$?

The general answer is "NO" [Mathieu], but it is "YES" whenever the A_0 is reasonnable enough [Kontsevich]

Given a Poisson bracket $\{-,-\}$ on a commutative algebra A_0, does there exist an associative formal deformation of the commutative product $\cdot$ of the form $\star = \cdot + \hbar\{-,-\} + o(\hbar)$?

The general answer is "NO" [Mathieu], but it is "YES" whenever the A_0 is reasonnable enough [Kontsevich], that is to say:

1. $A_0 = k[\![x_1, \ldots, x_n]\!]$ (k field of char. 0);
2. $A_0 = C^\infty(M)$, M being a smooth manifold;
3. $A_0 = k[X]$, X being a smooth affine algebraic variety over k a field of char. 0.

Given a Poisson bracket $\{-,-\}$ on a commutative algebra A_0, does there exist an associative formal deformation of the commutative product $\cdot$ of the form $\star = \cdot + \hbar\{-,-\} + o(\hbar)$?

The general answer is "NO" [Mathieu], but it is "YES" whenever the A_0 is reasonnable enough [Kontsevich], that is to say:

1. $A_0 = k[\![x_1, \dots, x_n]\!]$ (k field of char. 0);
2. $A_0 = C^\infty(M)$, M being a smooth manifold;
3. $A_0 = k[X]$, X being a smooth affine algebraic variety over k a field of char. 0.

Actually, (2) and (3) are obtained from (1) by globalization techniques that we are not going to discuss here. Kontsevich formula for (1) is remarkably elegant.

$$f \star g = \sum_{n \geq 0} \hbar^n \sum_{\Gamma \in \mathcal{G}_{n,2}} c_\Gamma B_{\Gamma,\alpha}(f,g)$$

$$f \star g = \sum_{n \geq 0} \hbar^n \sum_{\Gamma \in \mathcal{G}_{n,2}} c_\Gamma B_{\Gamma,\alpha}(f, g)$$

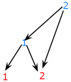
- $\mathcal{G}_{n,2}$ is a set of directed graphs with
  - vertex set $\{1, \ldots, n, \bar{1}, \bar{2}\}$,
  - no loops and no multiple edges,
  - exactly two outgoing edges from every blue vertex,
  - no outgoing edge from red vertices,

$$f \star g = \sum_{n \geq 0} \hbar^n \sum_{\Gamma \in \mathcal{G}_{n,2}} c_\Gamma B_{\Gamma,\alpha}(f, g)$$

- $\mathcal{G}_{n,2}$ is a set of directed graphs with
  - vertex set $\{1, \ldots, n, \bar{1}, \bar{2}\}$,
  - no loops and no multiple edges,
  - exactly two outgoing edges from every blue vertex,
  - no outgoing edge from red vertices,
- $B_{\Gamma,\alpha}$ is a bidifferential operator built from Γ and the Poisson tensor $\alpha = \alpha^{ij}\partial_i \wedge \partial_j$, where $\alpha^{ij} = \{x_i, x_j\}$.

$$f \star g = \sum_{n \geq 0} \hbar^n \sum_{\Gamma \in \mathcal{G}_{n,2}} c_\Gamma B_{\Gamma,\alpha}(f, g)$$

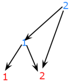
- $\mathcal{G}_{n,2}$ is a set of directed graphs with
  - vertex set $\{1, \ldots, n, \bar{1}, \bar{2}\}$,
  - no loops and no multiple edges,
  - exactly two outgoing edges from every blue vertex,
  - no outgoing edge from red vertices,
- $B_{\Gamma,\alpha}$ is a bidifferential operator built from Γ and the Poisson tensor $\alpha = \alpha^{ij}\partial_i \wedge \partial_j$, where $\alpha^{ij} = \{x_i, x_j\}$.



$$B_{\Gamma,\alpha}(f, g) = (\partial_k \alpha^{ij})\alpha^{kl}(\partial_i f)(\partial_l \partial_j g)$$

$$f \star g = \sum_{n \geq 0} \hbar^n \sum_{\Gamma \in \mathcal{G}_{n,2}} c_\Gamma B_{\Gamma,\alpha}(f,g)$$

- $\mathcal{G}_{n,2}$ is a set of directed graphs with
  - vertex set $\{1, \dots, n, \bar{1}, \bar{2}\}$,
  - no loops and no multiple edges,
  - exactly two outgoing edges from every blue vertex,
  - no outgoing edge from red vertices,
- $B_{\Gamma,\alpha}$ is a bidifferential operator built from Γ and the Poisson tensor $\alpha = \alpha^{ij}\partial_i \wedge \partial_j$, where $\alpha^{ij} = \{x_i, x_j\}$.



$$B_{\Gamma,\alpha}(f,g) = (\partial_k \alpha^{ij})\alpha^{kl}(\partial_i f)(\partial_l \partial_j g)$$

- coefficients $c_\Gamma \in \mathbb{R}$ are of transcendental nature.

Moduli of marked disks

$C_{n,2}$ is the moduli of holomorphic (closed) disks D with an embedding $\{1, \ldots, n\} \hookrightarrow D \backslash \partial D$, and a cyclic order preserving embedding $\{\bar{1}, \bar{2}, \infty\} \hookrightarrow \partial D$.

Moduli of marked disks

$C_{n,2}$ is the moduli of holomorphic (closed) disks D with an embedding $\{1, \ldots, n\} \hookrightarrow D \backslash \partial D$, and a cyclic order preserving embedding $\{\bar{1}, \bar{2}, \infty\} \hookrightarrow \partial D$.

$$C_{n,2} \simeq \left(Conf_n(\mathbb{H}) \times Conf_{2,+}(\mathbb{R}) \right) / \mathbb{R}_{>0} \ltimes \mathbb{R}.$$

Moduli of marked disks

$C_{n,2}$ is the moduli of holomorphic (closed) disks D with an embedding $\{1, \ldots, n\} \hookrightarrow D \backslash \partial D$, and a cyclic order preserving embedding $\{\bar{1}, \bar{2}, \infty\} \hookrightarrow \partial D$.

$$C_{n,2} \simeq \left(Conf_n(\mathbb{H}) \times Conf_{2,+}(\mathbb{R})\right)/\mathbb{R}_{>0} \ltimes \mathbb{R}.$$

Kontsevich weight of $\Gamma \in \mathcal{G}_{n,2}$

$$c_\Gamma := \int_{C_{n,2}} \omega_\Gamma\,, \text{ with } \qquad \omega_\Gamma := \bigwedge_{(i,j)\in E(\Gamma)} \frac{dArg\left((z_j - z_i)(z_j - \bar{z}_i)\right)}{2\pi}\,.$$

Moduli of marked disks

$C_{n,2}$ is the moduli of holomorphic (closed) disks D with an embedding $\{1, \dots, n\} \hookrightarrow D \backslash \partial D$, and a cyclic order preserving embedding $\{\bar{1}, \bar{2}, \infty\} \hookrightarrow \partial D$.

$$C_{n,2} \simeq \left(Conf_n(\mathbb{H}) \times Conf_{2,+}(\mathbb{R}) \right) / \mathbb{R}_{>0} \ltimes \mathbb{R}.$$

Kontsevich weight of $\Gamma \in \mathcal{G}_{n,2}$

$$c_\Gamma := \int_{C_{n,2}} \omega_\Gamma \,, \text{ with } \qquad \omega_\Gamma := \bigwedge_{(i,j) \in E(\Gamma)} \frac{dArg\left((z_j - z_i)(z_j - \bar{z}_i)\right)}{2\pi} \,.$$

These integrals converge and satisfy algebraic relations ensuring the associativity of $\star$ [Kontsevich].

There is a TFT, the Poisson σ-model [Ikeda,Schaller–Strobl], from which one can derive Kontsevich formula [Cattaneo–Felder]:

There is a TFT, the Poisson σ-model [Ikeda,Schaller–Strobl], from which one can derive Kontsevich formula [Cattaneo–Felder]:

- fields are maps $\phi : D \to M$ together with connection 1-form $\eta \in \Omega^1(D, \phi^* T^* M)$.

There is a TFT, the Poisson σ-model [Ikeda,Schaller–Strobl], from which one can derive Kontsevich formula [Cattaneo–Felder]:

- fields are maps $\phi : D \to M$ together with connection 1-form $\eta \in \Omega^1(D, \phi^* T^* M)$.
- action functional is $S(\phi, \eta) := \int_D \left(\langle \eta, d\phi \rangle + \frac{1}{2} \langle \eta \wedge \eta, \phi^* \alpha \rangle \right)$.

There is a TFT, the Poisson σ-model [Ikeda,Schaller–Strobl], from which one can derive Kontsevich formula [Cattaneo–Felder]:

- fields are maps $\phi : D \to M$ together with connection 1-form $\eta \in \Omega^1(D, \phi^* T^* M)$.
- action functional is $S(\phi, \eta) := \int_D \left(\langle \eta, d\phi \rangle + \frac{1}{2} \langle \eta \wedge \eta, \phi^* \alpha \rangle \right)$.
- the star products reads as

$$(f \star g)(x) = \int_{\text{fields}} f\left(\phi(\bar{1})\right) g\left(\phi(\bar{2})\right) \delta_{x=\phi(\infty)} e^{\frac{S(\phi,\eta)}{\hbar}} D\phi D\eta \,.$$

There is a TFT, the Poisson σ-model [Ikeda,Schaller–Strobl], from which one can derive Kontsevich formula [Cattaneo–Felder]:

- fields are maps $\phi : D \to M$ together with connection 1-form $\eta \in \Omega^1(D, \phi^* T^* M)$.
- action functional is $S(\phi,\eta) := \int_D \left(\langle \eta, d\phi\rangle + \frac{1}{2}\langle \eta\wedge\eta, \phi^*\alpha\rangle\right)$.
- the star products reads as

$$(f\star g)(x) = \int_{\text{fields}} f\big(\phi(\bar{1})\big) g\big(\phi(\bar{2})\big)\delta_{x=\phi(\infty)} e^{\frac{S(\phi,\eta)}{\hbar}} D\phi D\eta\,.$$

Topological invariance guaranties the associativity of $\star$

Both $\big((f\star g)\star h\big)(x)$ and $\big(f\star(g\star h)\big)(x)$ equal

$$\int_{\text{fields}} f\big(\phi(\bar{1})\big) g\big(\phi(\bar{2})\big) h\big(\phi(\bar{3})\big)\delta_{x=\phi(\infty)} e^{\frac{S(\phi,\eta)}{\hbar}} D\phi D\eta\,.$$

- More general observables: one is led to replace 2 with any positive integer m ($\Rightarrow$ Kontsevich formality theorem).

- More general observables: one is led to replace 2 with any positive integer m ($\Rightarrow$ Kontsevich formality theorem).
- There is a consistent choice of orientations on compactified configurations spaces that make all signs correct:

D. Arnal, **D. Manchon**, M. Masmoudi, Choix des signes pour la formalité de M. Kontsevich, Pacific Journal of Mathematics **203** (2002), no. 1, 23–66.

- More general observables: one is led to replace 2 with any positive integer m ($\Rightarrow$ Kontsevich formality theorem).
- There is a consistent choice of orientations on compactified configurations spaces that make all signs correct:

D. Arnal, **D. Manchon**, M. Masmoudi, Choix des signes pour la formalité de M. Kontsevich, Pacific Journal of Mathematics **203** (2002), no. 1, 23–66.

- There is an additional compatibility with cup-products:

D. Manchon, C. Torossian, Cohomologie tangente et cup-produit pour la quantification de Kontsevich, Annales mathématiques Blaise Pascal **10** (2003) no. 1, 75–106.

$\Rightarrow$ New proof of the Duflo isomorphism [Kontsevich] and algebraic geometry analog [C-Van den Bergh].

- More general observables: one is led to replace 2 with any positive integer m ($\Rightarrow$ Kontsevich formality theorem).
- There is a consistent choice of orientations on compactified configurations spaces that make all signs correct:

D. Arnal, **D. Manchon**, M. Masmoudi, Choix des signes pour la formalité de M. Kontsevich, Pacific Journal of Mathematics **203** (2002), no. 1, 23–66.

- There is an additional compatibility with cup-products:

D. Manchon, C. Torossian, Cohomologie tangente et cup-produit pour la quantification de Kontsevich, Annales mathématiques Blaise Pascal **10** (2003) no. 1, 75–106.

$\Rightarrow$ New proof of the Duflo isomorphism [Kontsevich] and algebraic geometry analog [C-Van den Bergh].

- Different gauge fixing: one obtains variants of c_Γ's where $dArg$ is replaced by $dlog$.

- Boundary condition: require that $\phi(\partial D) \subset C$, where $C \subset M$ is a coisotropic submanifold (a "brane"); $\Rightarrow$ A_∞-deformation of $\Gamma(C, \wedge^\bullet NC)$; $\Rightarrow$ quantization of reduced spaces [Cattaneo–Felder].

- Boundary condition: require that $\phi(\partial D) \subset C$, where $C \subset M$ is a coisotropic submanifold (a "brane"); $\Rightarrow$ A_∞-deformation of $\Gamma(C, \wedge^\bullet NC)$; $\Rightarrow$ quantization of reduced spaces [Cattaneo–Felder].
- Several branes ($\Rightarrow$ Fukaya-type category):
  - two branes [Cattaneo–Felder]: two A_∞-algebras together with an invertible A_∞-bimodule realizing a Koszul/Morita duality/equivalence [C–Felder–Ferrario–Rossi] (conjectured by Shoikhet).

- Boundary condition: require that $\phi(\partial D) \subset C$, where $C \subset M$ is a coisotropic submanifold (a "brane"); $\Rightarrow$ A_∞-deformation of $\Gamma(C, \wedge^\bullet NC)$; $\Rightarrow$ quantization of reduced spaces [Cattaneo–Felder].
- Several branes ($\Rightarrow$ Fukaya-type category):
  - two branes [Cattaneo–Felder]: two A_∞-algebras together with an invertible A_∞-bimodule realizing a Koszul/Morita duality/equivalence [C–Felder–Ferrario–Rossi] (conjectured by Shoikhet).
    $\Rightarrow$ Applications to Lie theory [Cattaneo–Torossian] and algebraic geometry [C–Vitanov] (in progress).

- Boundary condition: require that $\phi(\partial D) \subset C$, where $C \subset M$ is a coisotropic submanifold (a "brane"); $\Rightarrow$ A_∞-deformation of $\Gamma(C, \wedge^\bullet NC)$; $\Rightarrow$ quantization of reduced spaces [Cattaneo–Felder].
- Several branes ($\Rightarrow$ Fukaya-type category):
  - two branes [Cattaneo–Felder]: two A_∞-algebras together with an invertible A_∞-bimodule realizing a Koszul/Morita duality/equivalence [C–Felder–Ferrario–Rossi] (conjectured by Shoikhet).
    $\Rightarrow$ Applications to Lie theory [Cattaneo–Torossian] and algebraic geometry [C–Vitanov] (in progress).
  - three branes: composition up to homotopy of A_∞-bimodules [Ferrario].
  - more...?

- Boundary condition: require that $\phi(\partial D) \subset C$, where $C \subset M$ is a coisotropic submanifold (a "brane"); $\Rightarrow$ A_∞-deformation of $\Gamma(C, \wedge^\bullet NC)$; $\Rightarrow$ quantization of reduced spaces [Cattaneo–Felder].
- Several branes ($\Rightarrow$ Fukaya-type category):
  - two branes [Cattaneo–Felder]: two A_∞-algebras together with an invertible A_∞-bimodule realizing a Koszul/Morita duality/equivalence [C–Felder–Ferrario–Rossi] (conjectured by Shoikhet).
    $\Rightarrow$ Applications to Lie theory [Cattaneo–Torossian] and algebraic geometry [C–Vitanov] (in progress).
  - three branes: composition up to homotopy of A_∞-bimodules [Ferrario].
  - more...?

Spoiler: already with two branes, the weights (and graphs) involved are more general.

Multiple zeta values

Definition

Let $s_1, \ldots, s_\ell$ be positive integers, with $s_1 > 1$:

$$\zeta(s_1, \ldots, s_\ell) := \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} \cdots n_\ell^{s_\ell}} .$$

Definition

Let $s_1, \ldots, s_\ell$ be positive integers, with $s_1 > 1$:

$$\zeta(s_1, \ldots, s_\ell) := \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} \cdots n_\ell^{s_\ell}} .$$

These numbers also have an integral representation:

$$\zeta(s_1, \ldots, s_\ell) = \int_{\Delta^k} \omega_0(t_1) \ldots \omega_0(t_{s_1-1}) \omega_1(t_{s_1}) \omega_0(t_{s_1+1}) \ldots \omega_1(t_k)$$

where

- $\omega_0(t) = dt/t$ and $\omega_1(t) = dt/(1-t)$,
- $\Delta^k = \{(t_1, \ldots, t_k) \in [0,1]^k | t_1 \geq \cdots \geq t_k\}$.

Definition

Let $s_1, \ldots, s_\ell$ be positive integers, with $s_1 > 1$:

$$\zeta(s_1, \ldots, s_\ell) := \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{1}{n_1^{s_1} \cdots n_\ell^{s_\ell}}.$$

These numbers also have an integral representation:

$$\zeta(s_1, \ldots, s_\ell) = \int_{\Delta^k} \omega_0(t_1) \ldots \omega_0(t_{s_1-1}) \omega_1(t_{s_1}) \omega_0(t_{s_1+1}) \ldots \omega_1(t_k)$$

where

- $\omega_0(t) = dt/t$ and $\omega_1(t) = dt/(1-t)$,
- $\Delta^k = \{(t_1, \ldots, t_k) \in [0,1]^k | t_1 \geq \cdots \geq t_k\}$.

They are iterated integrals of $dlog(c.r.)$ on $\mathcal{M}_{0,4}$.

- [Broadhurst–Kreimer]: a lot of Feynman amplitudes in QFT are (linear combinations of) MZVs.

- [Broadhurst–Kreimer]: a lot of Feynman amplitudes in QFT are (linear combinations of) MZVs.
- [Brown]: periods of $\mathcal{M}_{0,n}$ are $\mathbb{Q}[(2\pi\mathrm{i})^{-1}]$-linear combinations of MZVs.

- [Broadhurst–Kreimer]: a lot of Feynman amplitudes in QFT are (linear combinations of) MZVs.
- [Brown]: periods of $\mathcal{M}_{0,n}$ are $\mathbb{Q}[(2\pi \mathrm{i})^{-1}]$-linear combinations of MZVs.
- **Warning** (life isn't simple): there are amplitudes in ϕ^4 at high loop orders, which are related to modular forms (e.g. [Brown–Schnetz]), and not expected to be expressible as multiple zeta values (contrary to what may have been believed in the past).

- [Broadhurst–Kreimer]: a lot of Feynman amplitudes in QFT are (linear combinations of) MZVs.
- [Brown]: periods of $\mathcal{M}_{0,n}$ are $\mathbb{Q}[(2\pi\mathrm{i})^{-1}]$-linear combinations of MZVs.
- **Warning** (life isn't simple): there are amplitudes in ϕ^4 at high loop orders, which are related to modular forms (e.g. [Brown–Schnetz]), and not expected to be expressible as multiple zeta values (contrary to what may have been believed in the past).

What about the Kontsevich weights c_Γ, that are Feynman amplitudes for the Poisson σ-model?

- [Broadhurst–Kreimer]: a lot of Feynman amplitudes in QFT are (linear combinations of) MZVs.
- [Brown]: periods of $\mathcal{M}_{0,n}$ are $\mathbb{Q}[(2\pi\mathrm{i})^{-1}]$-linear combinations of MZVs.
- **Warning** (life isn't simple): there are amplitudes in ϕ^4 at high loop orders, which are related to modular forms (e.g. [Brown–Schnetz]), and not expected to be expressible as multiple zeta values (contrary to what may have been believed in the past).

What about the Kontsevich weights c_Γ, that are Feynman amplitudes for the Poisson σ-model?

Theorem [Banks–Panzer–Pym]

The coefficients c_Γ are $\mathbb{Q}[(2\pi\mathrm{i})^{-1}]$-linear combinations of MZVs.

Define the sheaf $\mathcal{U}^{\bullet}_{n,m}$ of *polylogarithmic forms* on $C_{n,m}$:

Define the sheaf $\mathcal{U}^\bullet_{n,m}$ of *polylogarithmic forms* on $C_{n,m}$:

1. Consider the map $\iota : C_{n,m} \hookrightarrow C_{2n+m} \simeq \mathcal{M}_{0,2n+m+1}$ that "double" the interior marked points.

Define the sheaf $\mathcal{U}^\bullet_{n,m}$ of *polylogarithmic forms* on $C_{n,m}$:

1. Consider the map $\iota : C_{n,m} \hookrightarrow C_{2n+m} \simeq \mathcal{M}_{0,2n+m+1}$ that "double" the interior marked points.
2. Define the sheaf $\mathcal{U}^\bullet$ of polylogarithmic forms on $\mathcal{M}_{0,2n+m+1}$

Define the sheaf $\mathcal{U}^{\bullet}_{n,m}$ of *polylogarithmic forms* on $C_{n,m}$:

1. Consider the map $\iota : C_{n,m} \hookrightarrow C_{2n+m} \simeq \mathcal{M}_{0,2n+m+1}$ that "double" the interior marked points.
2. Define the sheaf $\mathcal{U}^{\bullet}$ of polylogarithmic forms on $\mathcal{M}_{0,2n+m+1}$: linear combinations of *dlog* of cross-ratios with coefficients being polylogs (period integrals on the universal curve).

Define the sheaf $\mathcal{U}^{\bullet}_{n,m}$ of *polylogarithmic forms* on $C_{n,m}$:

1. Consider the map $\iota : C_{n,m} \hookrightarrow C_{2n+m} \simeq \mathcal{M}_{0,2n+m+1}$ that "double" the interior marked points.
2. Define the sheaf $\mathcal{U}^{\bullet}$ of polylogarithmic forms on $\mathcal{M}_{0,2n+m+1}$: linear combinations of *dlog* of cross-ratios with coefficients being polylogs (period integrals on the universal curve).
3. Restrict: $\mathcal{U}^{\bullet}_{n,m} := \iota^{*}\mathcal{U}^{\bullet}$.

Define the sheaf $\mathcal{U}^\bullet_{n,m}$ of *polylogarithmic forms* on $C_{n,m}$:

1. Consider the map $\iota : C_{n,m} \hookrightarrow C_{2n+m} \simeq \mathcal{M}_{0,2n+m+1}$ that "double" the interior marked points.
2. Define the sheaf $\mathcal{U}^\bullet$ of polylogarithmic forms on $\mathcal{M}_{0,2n+m+1}$: linear combinations of *dlog* of cross-ratios with coefficients being polylogs (period integrals on the universal curve).
3. Restrict: $\mathcal{U}^\bullet_{n,m} := \iota^* \mathcal{U}^\bullet$.

Theorem [Banks–Panzer–Pym]

Fiber-integrating along "forgetting-a-point" maps sends polylogarithmic forms to polylogarithmic forms.

Define the sheaf $\mathcal{U}^\bullet_{n,m}$ of *polylogarithmic forms* on $C_{n,m}$:

1. Consider the map $\iota : C_{n,m} \hookrightarrow C_{2n+m} \simeq \mathcal{M}_{0,2n+m+1}$ that "double" the interior marked points.
2. Define the sheaf $\mathcal{U}^\bullet$ of polylogarithmic forms on $\mathcal{M}_{0,2n+m+1}$: linear combinations of *dlog* of cross-ratios with coefficients being polylogs (period integrals on the universal curve).
3. Restrict: $\mathcal{U}^\bullet_{n,m} := \iota^*\mathcal{U}^\bullet$.

Theorem [Banks–Panzer–Pym]

Fiber-integrating along "forgetting-a-point" maps sends polylogarithmic forms to polylogarithmic forms.

This is essentially the same strategy as for Brown's result, with a specific difficulty for when one forgets an interior point.

Alternating MZVs

Let $s_1, \ldots, s_\ell$ be non-zero integers, with $s_1 \neq 1$:

$$\zeta(s_1, \ldots, s_\ell) := \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{\epsilon(s_1)^{n_1} \cdots \epsilon(s_\ell)^{n_\ell}}{n_1^{|s_1|} \cdots n_\ell^{|s_\ell|}},$$

where $\epsilon(s) = s/|s|$.

Alternating MZVs

Let $s_1, \ldots, s_\ell$ be non-zero integers, with $s_1 \neq 1$:

$$\zeta(s_1, \ldots, s_\ell) := \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{\epsilon(s_1)^{n_1} \cdots \epsilon(s_\ell)^{n_\ell}}{n_1^{|s_1|} \cdots n_\ell^{|s_\ell|}},$$

where $\epsilon(s) = s/|s|$.

New period in the game: $\zeta(-1) = \sum_{n \geq 1} \frac{(-1)^n}{n} = -\log(2)$.

Alternating MZVs

Let $s_1, \ldots, s_\ell$ be non-zero integers, with $s_1 \neq 1$:

$$\zeta(s_1, \ldots, s_\ell) := \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{\epsilon(s_1)^{n_1} \cdots \epsilon(s_\ell)^{n_\ell}}{n_1^{|s_1|} \cdots n_\ell^{|s_\ell|}},$$

where $\epsilon(s) = s/|s|$.

New period in the game: $\zeta(-1) = \sum_{n \geq 1} \frac{(-1)^n}{n} = -\log(2)$.

Generalization: N-coloured MZVs

$s_1, \ldots, s_\ell \in \mathbb{N}_{>0}$ and $\xi_1, \ldots \xi_\ell \in \mu_N$, with $(s_1, \xi_1) \neq (1, 1)$:

$$\zeta(s_1, \ldots, s_\ell | \xi_1, \ldots, \xi_\ell) := \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{\xi_1^{n_1} \cdots \xi_\ell^{n_\ell}}{n_1^{s_1} \cdots n_\ell^{s_\ell}},$$

Alternating MZVs

Let $s_1, \ldots, s_\ell$ be non-zero integers, with $s_1 \neq 1$:

$$\zeta(s_1, \ldots, s_\ell) := \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{\epsilon(s_1)^{n_1} \cdots \epsilon(s_\ell)^{n_\ell}}{n_1^{|s_1|} \cdots n_\ell^{|s_\ell|}},$$

where $\epsilon(s) = s/|s|$.

New period in the game: $\zeta(-1) = \sum_{n \geq 1} \frac{(-1)^n}{n} = -\log(2)$.

Generalization: N-coloured MZVs

$s_1, \ldots, s_\ell \in \mathbb{N}_{>0}$ and $\xi_1, \ldots \xi_\ell \in \mu_N$, with $(s_1, \xi_1) \neq (1, 1)$:

$$\zeta(s_1, \ldots, s_\ell | \xi_1, \ldots, \xi_\ell) := \sum_{n_1 > \cdots > n_\ell \geq 1} \frac{\xi_1^{n_1} \cdots \xi_\ell^{n_\ell}}{n_1^{s_1} \cdots n_\ell^{s_\ell}},$$

These numbers also have an integral representation, as iterated integrals of *dlog* of t, and $t - \xi$, $\xi \in \mu_N$.

Consider the moduli $\mathcal{C}_{n,p+1+q}$ of marked disks. These are given by inclusions

$$\{1, \ldots, n\} \hookrightarrow \mathrm{int}(D)$$

$$\{-\bar{p}, \ldots, -\bar{1}, 0, \bar{1}, \ldots, \bar{q}, \infty\} \hookrightarrow \partial D\,.$$

Consider the moduli $\mathcal{C}_{n,p+1+q}$ of marked disks. These are given by inclusions

$$\{1, \ldots, n\} \hookrightarrow \mathrm{int}(D)$$

$$\{-\bar{p}, \ldots, -\bar{1}, 0, \bar{1}, \ldots, \bar{q}, \infty\} \hookrightarrow \partial D\,.$$

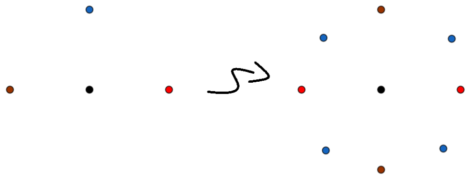
We have a map $\iota : \mathcal{C}_{n,p+1+q} \hookrightarrow \mathcal{M}_{0,N(2n+p+q)+2}$ sending all coloured (blue, brown and red) points, that we see as points in the upper half-plane, to their N-th roots and complex conjugates.

Consider the moduli $\mathcal{C}_{n,p+1+q}$ of marked disks. These are given by inclusions

$$\{1, \ldots, n\} \hookrightarrow \mathrm{int}(D)$$

$$\{-\bar{p}, \ldots, -\bar{1}, 0, \bar{1}, \ldots, \bar{q}, \infty\} \hookrightarrow \partial D\,.$$

We have a map $\iota : \mathcal{C}_{n,p+1+q} \hookrightarrow \mathcal{M}_{0,N(2n+p+q)+2}$ sending all coloured (blue, brown and red) points, that we see as points in the upper half-plane, to their N-th roots and complex conjugates.



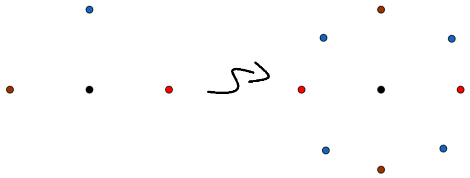
An illustration of the map ι for $N = 2$

Consider the moduli $\mathcal{C}_{n,p+1+q}$ of marked disks. These are given by inclusions

$$\{1, \dots, n\} \hookrightarrow \mathrm{int}(D)$$

$$\{-\bar{p}, \dots, -\bar{1}, 0, \bar{1}, \dots, \bar{q}, \infty\} \hookrightarrow \partial D\,.$$

We have a map $\iota : \mathcal{C}_{n,p+1+q} \hookrightarrow \mathcal{M}_{0,N(2n+p+q)+2}$ sending all coloured (blue, brown and red) points, that we see as points in the upper half-plane, to their N-th roots and complex conjugates.



An illustration of the map ι for $N = 2$

One then defines the sheaf $\mathcal{U}_N^\bullet := \iota^*\mathcal{U}^\bullet$ of N-coloured polylogarithmic forms.

Theorem [C]

Fiber-integrating along "forgetting-a-point" maps sends N-coloured polylogarithmic forms to N-coloured polylogarithmic forms.

Theorem [C]

Fiber-integrating along “forgetting-a-point” maps sends N-coloured polylogarithmic forms to N-coloured polylogarithmic forms.

$(N = 2) \Rightarrow$ **Weights (a-k-a Feynman amplitudes) appearing in [C–Felder-Ferrario–Rossi] for the deformation quantization in the presence of two branes are $\mathbb{Q}[(2\pi\mathrm{i})^{-1}]$-linear combinations of alternating multiple zeta values.**

Theorem [C]

Fiber-integrating along "forgetting-a-point" maps sends N-coloured polylogarithmic forms to N-coloured polylogarithmic forms.

$(N = 2) \Rightarrow$ **Weights (a-k-a Feynman amplitudes) appearing in [C–Felder-Ferrario–Rossi] for the deformation quantization in the presence of two branes are $\mathbb{Q}[(2\pi \mathrm{i})^{-1}]$-linear combinations of alternating multiple zeta values.**

Questions:

- Occurences of N-coloured MZVs in the Poisson σ-model for $N \notin \{1, 2\}$?
- Nature of the weights when there are more branes?
- Higher genus version? E.g.: do eMZVs appear if one replaces the source with a genus one curve in the Poisson σ-model?

Thank you Dominique !!!