Homological properties of Braided Hopf Algebras

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Cohomological dimension

Let A be an algebra and let M be a (left) A-module.

- The **projective dimension** of the A-module M is the smallest possible length for a projective resolution of M:

$$\text{pd}_{\,_A\mathcal{M}}(M) = \sup\{d \in \mathbb{N} \mid \exists N \in {}_A\mathcal{M}, \mathrm{Ext}^d_{\,_A\mathcal{M}}(M, N) \neq \{0\}\} \in \mathbb{N} \cup \{\infty\}$$

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$$\mathrm{l.gldim}(A) = \sup\{\mathrm{pd}_{{}_A\mathcal{M}}(M),\ M \in {}_A\mathcal{M}\}$$

We always have

$$\mathrm{l.gldim}(A) \leq \mathrm{cd}(A).$$

Motivation

Let k be a field.

- **Example**: The first Weyl algebra $A_1(k) = k\langle x, y \mid xy - yx = 1\rangle$,
  If k has characteristic zero, then $\mathrm{gldim}(A_1(k)) = 1$ (Rinehart, 1962).
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  So, for an algebra A, when do we have $\mathrm{cd}(A) = \mathrm{gldim}(A)$?

- **Classical cases:**
  - If A is a graded connected algebra, $\mathrm{cd}(A) = \mathrm{gldim}(A) = \mathrm{pd}_A({}_\varepsilon k) = \mathrm{pd}_{A^{\mathrm{op}}}(k_\varepsilon)$ (Berger, 2005),
  - If A is a Hopf algebra, $\mathrm{cd}(A) = \mathrm{gldim}(A) = \mathrm{pd}_A({}_\varepsilon k) = \mathrm{pd}_{A^{\mathrm{op}}}(k_\varepsilon)$ (follows from Ginzburg - Kumar, 1993)

Objective I - The main result:

Theorem (Bichon - N, 2024)

Let A be a Hopf algebra in the braided category $\mathcal{M}^H$ of comodules over a coquasitriangular cosemisimple Hopf algebra H. Then we have

$$\mathrm{cd}(A) = \mathrm{l.gldim}(A) = \mathrm{r.gldim}(A) = \mathrm{pd}_A({}_\varepsilon k) = \mathrm{pd}_{A^{\mathrm{op}}}(k_\varepsilon)$$

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Braided categories

A *braided category* is a monoidal category $\mathcal{C}$ endowed with a *braiding*, i.e a family of natural isomorphisms

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

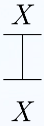
such that for all objects X, Y, Z in $\mathcal{C}$, the following diagrams

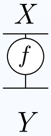
$$\begin{array}{ccc} X \otimes Y \otimes Z & \xrightarrow{c_{X,Y\otimes Z}} & Y \otimes X \otimes Z \\ \downarrow{\scriptstyle c_{X,Y}\otimes \mathrm{id}_Z} & \nearrow{\scriptstyle \mathrm{id}_Y \otimes c_{X,Z}} & \\ Y \otimes X \otimes Z & & \end{array} \qquad \begin{array}{ccc} X \otimes Y \otimes Z & \xrightarrow{c_{X\otimes Y,Z}} & Z \otimes X \otimes Y \\ \downarrow{\scriptstyle \mathrm{id}_X \otimes c_{Y,Z}} & \nearrow{\scriptstyle c_{X,Z}\otimes \mathrm{id}_Y} & \\ X \otimes Z \otimes Y & & \end{array}$$

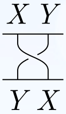
commute.

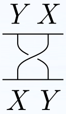
Notations

For objects X and Y in $\mathcal{C}$, as usual we have some diagrammatic notations:

$\mathrm{id}_X =$ 

and $f =$ 

$c_{X,Y} =$ 

and $c^{-1}_{X,Y} =$ 

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Examples

- The category ${}_k\mathcal{M}$ of k-vector spaces is a braided category with the braiding given by the flip operators.

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- A **coquasitriangular Hopf algebra** is a Hopf algebra H equipped with a convolution invertible linear form $r : H \otimes H \to k$ (called a universal r-form) such that, for any $x, y, z \in H$,

$$yx = r(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}r^{-1}(x_{(3)}, y_{(3)})$$

$$r(xy, z) = r(x, z_{(1)})r(y, z_{(2)}), \quad r(x, yz) = r(x_{(1)}, z)r(x_{(2)}, y)$$

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Then, the category of H-comodules $\mathcal{M}^H$ is a braided category with the braiding

$$\begin{aligned} r_{V,W} : V \otimes W &\longrightarrow W \otimes V \\ v \otimes w &\longmapsto r(v_{(1)}, w_{(1)})w_{(0)} \otimes v_{(0)} \end{aligned}$$

A concrete example

- Let Γ be an abelian group. Then the universal r-forms on the group algebra $k\Gamma$ correspond to the bicharacters $\Gamma \times \Gamma \to k^*$, i.e the maps ψ such that

$$\psi(xy, z) = \psi(x, z)\psi(y, z); \; \psi(x, yz) = \psi(x, y)\psi(x, z) \quad \text{for} \quad x, y, z \in \Gamma.$$

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- $\mathcal{M}^{k\Gamma}$ identifies with the category of Γ-graded vector spaces as follows:
  If $V = (V, \alpha)$ is a right $k\Gamma$-comodule, put, for $g \in \Gamma$, $V_g = \{v \in V \mid \alpha(v) = v \otimes g\}$.
  Then $V = \bigoplus_{g\in\Gamma} V_g$ defines a Γ-grading on V.
  Conversely, if $V = \bigoplus_{g\in\Gamma} V_g$ is Γ-graded, putting $\alpha(v) = v \otimes g$ for $v \in V_g$, defines a structure of $k\Gamma$-comodule on V.

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Given a bicharacter ψ, the category $\mathcal{M}^{k\Gamma}$ is braided with braiding:

$$\begin{aligned} c_{V,W} : \quad & V \otimes W \longrightarrow W \otimes V \\ v \otimes w \in V_g \otimes W_h & \longmapsto \psi(g, h) w \otimes v \end{aligned}$$

Hopf algebras in a monoidal category

Let $\mathcal{C}$ be a monoidal category.

- An **algebra** in $\mathcal{C}$ is an object A of $\mathcal{C}$ endowed with morphisms $m : A \otimes A \to A$ (the product) and $\eta : I \to A$ (the unit) such that

$$m(m \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes m) \quad \text{and} \quad m(\mathrm{id}_A \otimes \eta) = \mathrm{id}_A = m(\eta \otimes \mathrm{id}_A).$$

- A **coalgebra** in $\mathcal{C}$ is an object C of $\mathcal{C}$ endowed with morphisms $\Delta : C \to C \otimes C$ (the coproduct) and $\varepsilon : C \to I$ (the counit) such that

$$(\Delta \otimes \mathrm{id}_C)\Delta = (\mathrm{id}_C \otimes \Delta)\Delta \quad \text{and} \quad (\mathrm{id}_C \otimes \varepsilon)\Delta = \mathrm{id}_C = (\varepsilon \otimes \mathrm{id}_C)\Delta.$$

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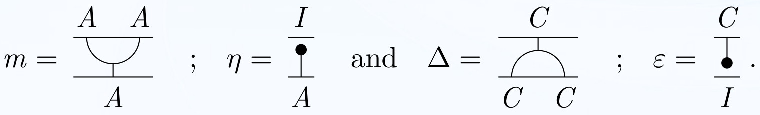
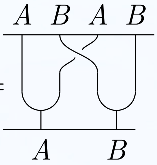
In diagrammatic notation, we denote



Braided tensor product algebras

Let $\mathcal{C}$ be a braided category. Let A and B be two algebras in $\mathcal{C}$. The braiding of $\mathcal{C}$ gives rise to an algebra structure on the object $A \otimes B$ with multiplication given by

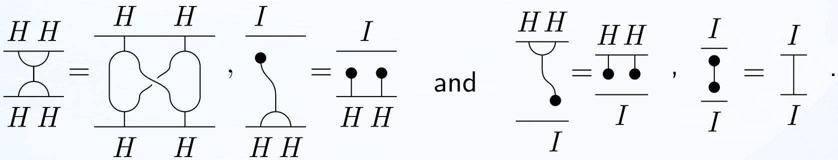
$m_{A\otimes_c B} =$ 

and unit $\eta_A \otimes \eta_B$.
The resulting algebra in $\mathcal{C}$ is denoted by $A \otimes_c B$ and is called the **braided tensor product algebra** of A and B.

Braided Hopf algebras

Let $\mathcal{C}$ be a braided category, with a braiding c.

- A **bialgebra** in $\mathcal{C}$ is an object H of $\mathcal{C}$ endowed with an algebra structure and a coalgebra structure in $\mathcal{C}$ such that its coproduct Δ and its counit ε are algebra morphisms, that is,



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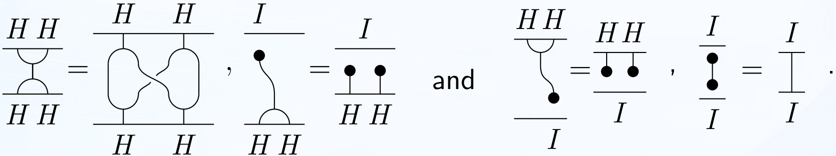
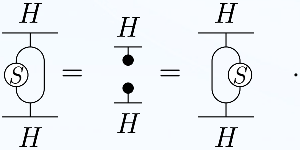


- A **braided Hopf algebra** is a bialgebra H in $\mathcal{C}$ such that there exists a morphism $S : H \to H$ in $\mathcal{C}$ (the antipode of H) with $m(S \otimes \mathrm{id}_H)\Delta = \eta\varepsilon = m(\mathrm{id}_H \otimes S)\Delta$, i.e



Example : two-parameter braided quantum SL_2

Definition

Let $p, q \in k^*$. The algebra $\mathscr{O}_{p,q}(\mathsf{SL}_2(k))$ is the algebra presented by generators a, b, c, d with the relations

$$ba = qab,\, ca = pac,\, db = qbd,\, dc = pcd,\, bc = cb$$

$$ad - p^{-1}bc = 1 = da - qbc$$

Proposition

The algebra $\mathscr{O}_{p,q}(\mathsf{SL}_2(k))$ has a $k\mathbb{Z}$-comodule algebra structure whose coaction is defined by the algebra map

$$\delta : \mathscr{O}_{p,q}(\mathsf{SL}_2(k)) \longrightarrow \mathscr{O}_{p,q}(\mathsf{SL}_2(k)) \otimes k\mathbb{Z}$$
$$a,\ b,\ c,\ d \longmapsto a \otimes 1,\ b \otimes z^{-1},\ c \otimes z,\ d \otimes 1$$

where z is a fixed generator of the infinite cyclic group $\mathbb{Z}$.

Example : two-parameter braided quantum SL_2

Let $A = \mathscr{O}_{p,q}(\mathrm{SL}_2(k))$.
Consider the bicharacter $\psi : \mathbb{Z} \times \mathbb{Z} \to k^*$, $\psi(z,z) = p^{-1}q$. Recall that ψ induces a braiding on $\mathcal{M}^{k\mathbb{Z}}$, with, for instance, $c_{A,A}(b \otimes c) = \psi(z^{-1}, z) c \otimes b = pq^{-1} c \otimes b$.

Example : two-parameter braided quantum SL_2

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Proposition

A is a Hopf algebra in the braided category $\mathcal{M}^{k\mathbb{Z}}$ with the structure

$$\begin{aligned} \Delta : A &\longrightarrow A \otimes_c A \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix} \end{aligned}$$

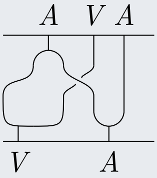
$$\begin{aligned} \varepsilon : \quad A &\longrightarrow \quad k, \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \qquad \begin{aligned} S : \quad A &\longrightarrow \quad A^{op,c} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} d & -qb \\ -p^{-1}c & a \end{pmatrix} \end{aligned}$$

Modules and bimodules over a braided Hopf algebra

Proposition

Let $\mathcal{C}$ be a braided category and let A be a bialgebra in $\mathcal{C}$. Let V be a left A-module in $\mathcal{C}$. Endow $V \otimes A$ with the right A-module structure defined by right multiplication. Then the morphism

$\mu^l_{V\otimes A} =$



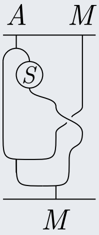
provides $V \otimes A$ with a left A-module structure, hence with an A-bimodule structure in $\mathcal{C}$. Denoting the resulting A-bimodule by $V \boxtimes A$, this construction yields a functor

$$L = - \boxtimes A : {}_A\mathcal{C} \longrightarrow {}_A\mathcal{C}_A$$
$$V \longmapsto V \boxtimes A.$$

Proposition

Let $\mathcal{C}$ be a braided category and A be a Hopf algebra in $\mathcal{C}$. Let M be an A-bimodule in $\mathcal{C}$, the morphism

$$\mu^l_{\widetilde{M}} =$$



endows M with a left A-module structure in $\mathcal{C}$. We then denote by $\widetilde{M}$ the resulting left A-module. This construction gives us a functor

$$R : {}_A\mathcal{C}_A \longrightarrow {}_A\mathcal{C}$$

$$M \mapsto \widetilde{M}$$

Proposition

Let $\mathcal{C}$ be a braided category and A be a Hopf algebra in $\mathcal{C}$. Then the functor $R : {}_A\mathcal{C}_A \longrightarrow {}_A\mathcal{C}$ is right adjoint to the functor $L = - \boxtimes A : {}_A\mathcal{C} \longrightarrow {}_A\mathcal{C}_A$.

This will enable us to use the following classical result.

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This will enable us to use the following classical result.

Proposition

Let $\mathcal{C}$ and $\mathcal{D}$ be k-linear abelian categories, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be some k-linear functors with G right adjoint to F. Suppose that $\mathcal{C}$ has enough projectives and that F is exact. Then we have natural isomorphisms

$$\mathrm{Ext}^*_{\mathcal{D}}(F(X), V) \cong \mathrm{Ext}^*_{\mathcal{C}}(X, G(V))$$

for any $X \in \mathrm{Ob}(\mathcal{C})$ and $V \in \mathrm{Ob}(\mathcal{D})$.

Corollary

Let $\mathcal{C}$ be an abelian k-linear braided category with enough projectives and let A be a Hopf algebra in $\mathcal{C}$. There exists natural isomorphisms

$$\mathrm{Ext}^*_{{}_A\mathcal{C}_A}(A, M) \cong \mathrm{Ext}^*_{{}_A\mathcal{C}}({}_\varepsilon I, \widetilde{M}).$$

and we have $\mathrm{pd}_{{}_A\mathcal{C}_A}(A) = \mathrm{pd}_{{}_A\mathcal{C}}({}_\varepsilon I) = \mathrm{pd}_{\mathcal{C}_A}(I_\varepsilon)$.

(The functor $- \otimes -$ is assumed to be exact and k-linear, and it follows that the categories ${}_A\mathcal{C}_A$ and ${}_A\mathcal{C}$ are abelian k-linear). Indeed, we have, for M in ${}_A\mathcal{C}_A$,

$$\mathrm{Ext}^*_{{}_A\mathcal{C}_A}({}_\varepsilon I \boxtimes A, M) \cong \mathrm{Ext}^*_{{}_A\mathcal{C}}({}_\varepsilon I, \widetilde{M})$$

and we get

$$\mathrm{pd}_{{}_A\mathcal{C}_A}(A) \leq \mathrm{pd}_{{}_A\mathcal{C}}({}_\varepsilon I).$$

Corollary

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and we get

$$\mathrm{pd}_{{}_A\mathcal{C}_A}(A) \leq \mathrm{pd}_{{}_A\mathcal{C}}({}_\varepsilon I).$$

Then, we also have $\widetilde{M_\varepsilon} \cong M$ in ${}_A\mathcal{C}$ and we obtain

$$\mathrm{Ext}^*_{{}_A\mathcal{C}}({}_\varepsilon I, M) \cong \mathrm{Ext}^*_{{}_A\mathcal{C}}({}_\varepsilon I, \widetilde{M_\varepsilon}) \cong \mathrm{Ext}^*_{{}_A\mathcal{C}_A}(A, M_\varepsilon)$$

Hence, $\mathrm{pd}_{{}_A\mathcal{C}}({}_\varepsilon I) \leq \mathrm{pd}_{{}_A\mathcal{C}_A}(A)$.

Cohomological dimension of braided Hopf algebras

Assume now that $\mathcal{C} = \mathcal{M}^H$ for H a coquasitriangular Hopf algebra. Since $\mathrm{pd}_{{}_A\mathcal{C}_A}(A) = \mathrm{pd}_{{}_A\mathcal{C}}({}_\varepsilon I)$, we have

$$\mathrm{l.\,gldim}(A) \leq \mathrm{cd}(A) = \mathrm{pd}_{{}_A\mathcal{M}_A}(A) \leq \mathrm{pd}_{{}_A\mathcal{M}_A^H}(A) = \mathrm{pd}_{{}_A\mathcal{M}^H}({}_\varepsilon k)$$
$$\stackrel{?}{=} \mathrm{pd}_{{}_A\mathcal{M}}({}_\varepsilon k) \leq \mathrm{l.\,gldim}(A).$$

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Definition (Nastasescu,Van den Bergh,Van Oystaeyen, 1989)

Let $\mathcal{C}, \mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F induces a natural transformation $\mathcal{P}_{-,-} : \mathrm{Hom}_{\mathcal{C}}(-,-) \longrightarrow \mathrm{Hom}_{\mathcal{D}}\big(F(-), F(-)\big)$. We say that F is a **separable functor** if there is a natural transformation

$$\mathbf{M}_{-,-} : \mathrm{Hom}_{\mathcal{D}}\big(F(-), F(-)\big) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(-,-)$$

such that $\mathbf{M}_{-,-} \circ \mathcal{P}_{-,-} = \mathbf{1}_{\mathrm{Hom}_{\mathcal{C}}(-,-)}$.

Cohomological dimension of braided Hopf algebras

Proposition

Let $\mathcal{C}$ and $\mathcal{D}$ be k-linear abelian categories with enough projective objects, and let $F : \mathcal{C} \to \mathcal{D}$ be a k-linear functor. Assume that F is exact, preserves projective objects and is separable. Then for any object X in $\mathcal{C}$, we have $\mathrm{pd}_{\mathcal{C}}(X) = \mathrm{pd}_{\mathcal{D}}(F(X))$.

Cohomological dimension of braided Hopf algebras

Proposition

Let $\mathcal{C}$ and $\mathcal{D}$ be k-linear abelian categories with enough projective objects, and let $F : \mathcal{C} \to \mathcal{D}$ be a k-linear functor. Assume that F is exact, preserves projective objects and is separable. Then for any object X in $\mathcal{C}$, we have $\mathrm{pd}_{\mathcal{C}}(X) = \mathrm{pd}_{\mathcal{D}}(F(X))$.

We recall some of the main examples of separable functors:

Proposition (Caenepeel-Militaru-Ion-Zhu 1999)

Let H be a cosemisimple Hopf algebra and let A be a right H-comodule algebra. The forgetful functors ${}_A\mathcal{M}^H \to {}_A\mathcal{M}$ and $\mathcal{M}_A^H \to \mathcal{M}_A$ are separable.

This result can be proven directly by using the Haar integral.

Cohomological dimension of braided Hopf algebras

Assuming moreover that H is cosemisimple, we thus have

$$\begin{aligned} \mathrm{l.gldim}(A) \leq \mathrm{cd}(A) = \mathrm{pd}_{{}_A\mathcal{M}_A}(A) \leq \mathrm{pd}_{{}_A\mathcal{M}_A^H}(A) = \mathrm{pd}_{{}_A\mathcal{M}^H}({}_\varepsilon k) \\ = \mathrm{pd}_{{}_A\mathcal{M}}({}_\varepsilon k) \leq \mathrm{l.gldim}(A) \end{aligned}$$

Cohomological dimension of braided Hopf algebras

Assuming moreover that H is cosemisimple, we thus have

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and we obtain

Theorem

Let A be a Hopf algebra in the braided category $\mathcal{M}^H$ of comodules over a coquasitriangular cosemisimple Hopf algebra H. Then we have

$$\mathrm{cd}(A) = \mathrm{l.\,gldim}(A) = \mathrm{r.\,gldim}(A) = \mathrm{pd}_A({}_\varepsilon k) = \mathrm{pd}_{A^{\mathrm{op}}}(k_\varepsilon)$$

A free resolution of ${}_\varepsilon k$

Let $A = \mathscr{O}_{p,q}(\mathrm{SL}_2(k))$.
The following generalizes a construction of Hadfield-Krämer (2005) in the $p = q$ case:

Proposition

The following is a resolution of ${}_\varepsilon k$ by free left A-modules:

$$(P_*) : \quad 0 \longrightarrow A \xrightarrow{\phi_3} A^3 \xrightarrow{\phi_2} A^3 \xrightarrow{\phi_1} A \xrightarrow{\varepsilon} k \longrightarrow 0.$$

where $\phi_1(x, y, z) = x(a-1) + yb + zc$, $\phi_3(x) = x(c, -b, pqa - 1)$ *and*

$$\phi_2(x, y, z) = (x, y, z) \begin{pmatrix} b & 1 - qa & 0 \\ c & 0 & 1 - pa \\ 0 & c & -b \end{pmatrix}.$$

$$\Longrightarrow \mathrm{pd}_{{}_A\mathcal{M}}({}_\varepsilon k) \le 3.$$

Ext-space

For $t \in k^*$, there exists an algebra map

$$\varepsilon_t : A \longrightarrow k$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

Proposition

For $p, q \in k^*$, put $t = (pq)^{-1}$. We have

$$\mathrm{Ext}^3_A({}_{\varepsilon}k, {}_{\varepsilon_t}k) \cong k.$$

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For $t \in k^*$, there exists an algebra map

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Proposition

For $p, q \in k^*$, put $t = (pq)^{-1}$. We have

$$\mathrm{Ext}^3_A({}_\varepsilon k, {}_{\varepsilon_t} k) \cong k.$$

$$\Longrightarrow \mathrm{pd}_{{}_A\mathcal{M}}({}_\varepsilon k) \geq 3.$$

Corollary

We have $\mathrm{cd}(\mathscr{O}_{p,q}(\mathsf{SL}_2(k))) = 3$ for any $p, q \in k^*$.

Table of Contents

Twisted Calabi-Yau algebras

Let A be a k-algebra;

- A is said to be *smooth* if A is of type FP as an A-bimodule, that is, A has a finite resolution by finitely generated projective A^e-modules.

Twisted Calabi-Yau algebras

Let A be a k-algebra;

- A is said to be *smooth* if A is of type FP as an A-bimodule, that is, A has a finite resolution by finitely generated projective A^e-modules.
- A is said to be *twisted Calabi-Yau* of dimension $n \geq 0$ if A is smooth and

$$H^i(A, {}_A A \otimes A_A) \simeq \begin{cases} \{0\} & \text{if} \quad i \neq n \\ A_\mu & \text{if} \quad i = n \end{cases}$$

as A-bimodules, for an algebra automorphism $\mu \in \mathrm{Aut}(A)$, called the *Nakayama* automorphism of A.

Homological duality

The motivation for the concept of twisted Calabi-Yau algebra comes from the following result:

Theorem (Van Den Bergh)

If A is a twisted Calabi-Yau algebra of dimension n with Nakayama automorphism μ, then necessarily $n = \mathrm{cd}(A)$, and if M is an A-bimodule, then we have for any $i \geq 0$

$$H^i(A, M) \simeq H_{n-i}(A, {}_{\mu^{-1}}M)$$

Objective II - The main result:

This result provides examples of twisted Calabi-Yau algebras in the setting of braided Hopf algebras; it generalizes a previous result of Brown-Zhang (2008) for ordinary Hopf algebras.

Theorem (Bichon - N, 2024)

Let A be a Hopf algebra with bijective antipode in the braided category $\mathcal{M}^H$ of comodules over a coquasitriangular Hopf algebra H (with the r-form $\mathbf{r}$). Assume that the A-module ${}_\varepsilon k$ is of type FP in ${}_A\mathcal{M}^H$ and that there is an integer $n \geq 0$ such that $\mathrm{Ext}^i_A({}_\varepsilon k, A) = \{0\}$ for $i \neq n$ and $\mathrm{Ext}^n_A({}_\varepsilon k, A)$ is one-dimensional. Then A is twisted Calabi-Yau of dimension n, with Nakayama automorphism defined by

$$\mu(a) = \psi(a_{[1]})\,\mathbf{r}\big(a_{[2](1)}, S_H(a_{[2](2)})g^{-1}\big)\,S_A^2(a_{[2](0)})$$

where $\psi : A \to k$ is the algebra map corresponding to the A-module structure on $\mathrm{Ext}^n_A({}_\varepsilon k, A)$ and satisfies $\psi(a_{(0)})a_{(1)} = \psi(a)1$ for any $a \in A$, and $g \in H$ is the group-like element corresponding to an appropriate H-comodule structure on $\mathrm{Ext}^n_A({}_\varepsilon k, A)$.

Finiteness conditions

Let $\mathcal{C}$ be an abelian k-linear monoidal category (this always mean that $-\otimes-$ is exact in each variable) and let A be an algebra in $\mathcal{C}$.

- An object V in $\mathcal{C}$ is said to have a left dual if there exists an object V^* together with morphisms $e: V^* \otimes V \to I$ and $\delta: I \to V \otimes V^*$ such that

$$(\mathrm{id}_V \otimes e) \circ (\delta \otimes \mathrm{id}_V) = \mathrm{id}_V, \quad (e \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes \delta) = \mathrm{id}_{V^*}$$

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- A left A-module M in $\mathcal{C}$ is said to be finite relative projective if M is isomorphic, as an A-module, to a direct summand of a free A-module $A\otimes V$, with V an object of $\mathcal{C}$ having a left dual.

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- A left A-module M in $\mathcal{C}$ is said to be finite relative projective if M is isomorphic, as an A-module, to a direct summand of a free A-module $A \otimes V$, with V an object of $\mathcal{C}$ having a left dual.
- A left A-module M in $\mathcal{C}$ is said to be of type FP if it has a finite resolution by finite relative projectives, in the sense that there exists an exact sequence of A-modules

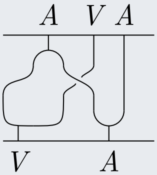
$$0 \to P_n \to P_{n-1} \to \cdots P_2 \to P_1 \to P_0 \to M \to 0$$

where for each i, the A-module P_i is finite relative projective.

Smoothness

Proposition

Let $\mathcal{C}$ be a braided category and let A be a bialgebra in $\mathcal{C}$. Let V be a left A-module in $\mathcal{C}$. Endow $V \otimes A$ with the right A-module structure defined by right multiplication. Then the morphism

$\mu^l_{V\otimes A} =$ 

provides $V \otimes A$ with a left A-module structure, hence with an A-bimodule structure in $\mathcal{C}$. Denoting the resulting A-bimodule by $V \boxtimes A$, this construction yields a functor

$$\begin{aligned} L = - \boxtimes A : {}_A\mathcal{C} &\longrightarrow {}_A\mathcal{C}_A \\ V &\longmapsto V \boxtimes A. \end{aligned}$$

Smoothness

Proposition

Let $\mathcal{C}$ *be a braided category and let* A *be an algebra in* $\mathcal{C}$. *The functor* $L = - \boxtimes A : {}_A\mathcal{C} \longrightarrow {}_A\mathcal{C}_A$ *transforms free* A*-modules into free* A*-bimodules. If moreover* $\mathcal{C}$ *is an abelian* k*-linear braided category, then the functor* L *transforms objects that are of type* FP *in* ${}_A\mathcal{C}$ *into objects that are of type* FP *in* ${}_A\mathcal{C}_A$.

Smoothness

Proposition

Let $\mathcal{C}$ be a braided category and let A be an algebra in $\mathcal{C}$. The functor $L = - \boxtimes A : {}_A\mathcal{C} \longrightarrow {}_A\mathcal{C}_A$ transforms free A-modules into free A-bimodules. If moreover $\mathcal{C}$ is an abelian k-linear braided category, then the functor L transforms objects that are of type FP in ${}_A\mathcal{C}$ into objects that are of type FP in ${}_A\mathcal{C}_A$.

Since $\mathcal{C}$ is an abelian k-linear braided category, the functor $- \otimes -$ is assumed to be exact, hence it suffices to prove that $(A \otimes V) \boxtimes A \simeq A \otimes V \otimes A$ as A-bimodules.

Smoothness

Proposition

Let $\mathcal{C}$ be a braided category and let A be an algebra in $\mathcal{C}$. The functor $L = - \boxtimes A : {}_A\mathcal{C} \longrightarrow {}_A\mathcal{C}_A$ transforms free A-modules into free A-bimodules. If moreover $\mathcal{C}$ is an abelian k-linear braided category, then the functor L transforms objects that are of type FP in ${}_A\mathcal{C}$ into objects that are of type FP in ${}_A\mathcal{C}_A$.

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Theorem

Let A be a Hopf algebra in the braided category $\mathcal{M}^H$ of comodules over a coquasitriangular Hopf algebra H. If ${}_\varepsilon k$ is of type FP in ${}_A\mathcal{M}^H$, then A is a smooth algebra.

Take $\mathcal{C} = \mathcal{M}^H$. We have $L({}_\varepsilon k) \simeq A$, hence A is of type FP in ${}_A\mathcal{M}^H_A$ and thus, of type FP in ${}_A\mathcal{M}_A$.

Sweedler's Notation

We fix a coquasitriangular Hopf algebra H and a Hopf algebra A in the braided category $\mathcal{M}^H$. We denote by, for $a \in A, x \in H$,

- $\Delta_A(a) = a_{[1]} \otimes a_{[2]}$, the comultiplication of A;
- $\Delta_H(x) = x_{(1)} \otimes x_{(2)}$, the comultiplication of H;
- $\alpha(a) = a_{(0)} \otimes a_{(1)}$, the H-coaction on A.

Theorem (Bichon - N, 2024)

Let A be a Hopf algebra in the braided category $\mathcal{M}^H$ of comodules over a coquasitriangular Hopf algebra H. If ${}_{\varepsilon}k$ is of type FP in ${}_A\mathcal{M}^H$, then there is an isomorphism of right A^e-modules

$$H^*(A, {}_AA \otimes A_A) \simeq \mathrm{Ext}^*_A({}_{\varepsilon}k, {}_AA) \otimes A$$

Theorem (Bichon - N, 2024)

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$$H^*(A, {}_AA \otimes A_A) \simeq \mathrm{Ext}^*_A({}_\varepsilon k, {}_AA) \otimes A$$

where the right A^e-action on $\mathrm{Ext}^*_A({}_\varepsilon k, {}_AA) \otimes A$ is defined by

$$([f] \otimes a') \cdot (a \otimes b) = ([f] \cdot a_{[1]})_{(0)} \otimes ba' S_A^2(a_{[2](0)}) \mathbf{r}\left[a_{[2](1)}, S_H\left(a_{[2](2)}\right) S_H\left(([f] \cdot a_{[1]})_{(1)}\right)\right]$$

with the right A-structure on $\mathrm{Ext}^*_A({}_\varepsilon k, {}_AA)$ induced by right multiplication in A and the right H-comodule structure is given by (see next slide)

$$\begin{aligned} \bar{\delta} : \mathrm{Ext}^*_A({}_\varepsilon k, {}_AA) &\longrightarrow \mathrm{Ext}^*_A({}_\varepsilon k, {}_AA) \otimes H \\ [f] &\longmapsto [f]_{(0)} \otimes [f]_{(1)} = [f_{(0)}] \otimes f_{(1)} \end{aligned}$$

Lemma

Let P be a finite relative projective object in ${}_A\mathcal{M}^H$. Then there is a map

$$\delta : \mathrm{Hom}_A(P, A) \longrightarrow \mathrm{Hom}_A(P, A) \otimes H$$
$$f \longmapsto f_{(0)} \otimes f_{(1)}$$

such that for all $x \in P$, $f_{(0)}(x) \otimes f_{(1)} = f(x_{(0)})_{(0)} \otimes S_H^{-1}(x_{(1)})f(x_{(0)})_{(1)}$ that endows $\mathrm{Hom}_A(P, A)$ with an H-comodule structure, and makes it into an object in $\mathcal{M}_A^H$.

Lemma

Let P be a finite relative projective object in ${}_A\mathcal{M}^H$. Then there is a map

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Lemma

Let M be an A-module of type FP in ${}_A\mathcal{M}^H$. For $n \in \mathbb{N}$, the map

$$\begin{aligned}\bar{\delta} : \mathrm{Ext}_A^n(M, A) &\longrightarrow \mathrm{Ext}_A^n(M, A) \otimes H \\ [f] &\longmapsto [f]_{(0)} \otimes [f]_{(1)} = [f_{(0)}] \otimes f_{(1)}\end{aligned}$$

making $\mathrm{Ext}_A^n(M, A)$ into an H-comodule, and an object in $\mathcal{M}_A^H$.

The main result

Theorem (Bichon - N, 2024)

Let A be a Hopf algebra with bijective antipode in the braided category $\mathcal{M}^H$ of comodules over a coquasitriangular Hopf algebra H. Assume that the A-module ${}_\varepsilon k$ is of type FP in ${}_A\mathcal{M}^H$ and that there is an integer $n \geq 0$ such that $\mathrm{Ext}^i_A({}_\varepsilon k, A) = \{0\}$ for $i \neq n$ and $\mathrm{Ext}^n_A({}_\varepsilon k, A)$ is one-dimensional. Then A is twisted Calabi-Yau of dimension n, with Nakayama automorphism defined by

$$\mu(a) = \psi(a_{[1]})\,\mathbf{r}\big(a_{[2](1)}, S_H(a_{[2](2)})g^{-1}\big)\,S_A^2(a_{[2](0)})$$

where $\psi : A \to k$ is the algebra map corresponding to the A-module structure on $\mathrm{Ext}^n_A({}_\varepsilon k, A)$ and satisfies $\psi(a_{(0)})a_{(1)} = \psi(a)1$ for any $a \in A$, and $g \in H$ is the group-like element corresponding to the H-comodule structure on $\mathrm{Ext}^n_A({}_\varepsilon k, A)$.

Proof:

Since $H^*(A, {}_AA \otimes A_A) \simeq \mathrm{Ext}^*_A({}_\varepsilon k, {}_AA) \otimes A$,

- Assuming that $\mathrm{Ext}^i_A({}_\varepsilon k, A) = \{0\}$ for $i \neq n$, we obtain that $H^i(A, {}_AA \otimes A_A) = \{0\}$ for $i \neq n$.

Proof:

Since $H^*(A, {}_AA \otimes A_A) \simeq \mathrm{Ext}_A^*({}_\varepsilon k, {}_AA) \otimes A$,

- Assuming that $\mathrm{Ext}_A^i({}_\varepsilon k, A) = \{0\}$ for $i \neq n$, we obtain that $H^i(A, {}_AA \otimes A_A) = \{0\}$ for $i \neq n$.
- Assuming moreover $\mathrm{Ext}_A^n({}_\varepsilon k, A)$ is one dimensional. The H-comodule structure on $\mathrm{Ext}_A^n({}_\varepsilon k, A)$ corresponds to a group-like element $g \in H$. Let

$$\psi : A \to k$$

be the algebra map associated with the A-module structure on $\mathrm{Ext}_A^n({}_\varepsilon k, A)$. It follows from the fact that $\mathrm{Ext}_A^n({}_\varepsilon k, A)$ is an object in $\mathcal{M}_A^H$ that ψ satisfies $\psi(a_{(0)})a_{(1)} = \psi(a)1$ for any $a \in A$. Then the right A^e-action on $\mathrm{Ext}_A^*({}_\varepsilon k, {}_AA) \otimes A$ is

$$([f] \otimes a') \cdot (a \otimes b) = [f] \otimes ba'\psi(a_{[1]})S_A^2(a_{[2](0)})\mathbf{r}\left[a_{[2](1)}, S_H\left(a_{[2](2)}\right)g^{-1}\right].$$

and this gives the announced formula for μ.

Two-parameter braided quantum SL_2

We recall the example of $\mathscr{O}_{p,q}(\mathsf{SL}_2(k))$:

Definition

Let $p, q \in k^*$. The algebra $\mathscr{O}_{p,q}(\mathsf{SL}_2(k))$ is the algebra presented by generators a, b, c, d with the relations

$$ba = qab, ca = pac, db = qbd, dc = pcd, bc = cb$$

$$ad - p^{-1}bc = 1 = da - qbc$$

Recall that $\mathcal{M}^{k\mathbb{Z},\xi}$ is an abelian k-linear braided category, where $\mathbb{Z}$ is the infite cyclic group with generator z, and the bicharacter

$$\psi : \mathbb{Z} \times \mathbb{Z} \longrightarrow k^*$$
$$(z, z) \longmapsto \xi.$$

A free resolution of ${}_\varepsilon k$ in $\mathcal{M}^{k\mathbb{Z}}$

Let $A = \mathscr{O}_{p,q}(\mathsf{SL}_2(k))$.

Proposition

Let V, W be the 3-dimensional $k\mathbb{Z}$-comodules with respective bases (e_1, e_2, e_3) and (e_1', e_2', e_3'), and coactions defined by

$$\delta_V : V \longrightarrow V \otimes k\mathbb{Z} \qquad\qquad \delta_W : W \longrightarrow W \otimes k\mathbb{Z}$$

$$e_1, e_2, e_3 \longmapsto e_1 \otimes 1, e_2 \otimes z^{-1}, e_3 \otimes z \qquad\qquad e_1', e_2', e_3' \longmapsto e_1' \otimes z^{-1}, e_2' \otimes z, e_3' \otimes 1.$$

Then we have a resolution of ${}_\varepsilon k$ by free A-modules in $\mathcal{M}^{k\mathbb{Z}}$

$$0 \to A \longrightarrow A \otimes W \longrightarrow A \otimes V \longrightarrow A \xrightarrow{\varepsilon} k \to 0$$

In particular ${}_\varepsilon k$ is of type FP in ${}_A\mathcal{M}^{k\mathbb{Z}}$.

Thus $\mathscr{O}_{p,q}(\mathsf{SL}_2(k))$ is smooth.

Ext-space

For $t \in k^*$, there exists an algebra map

$$\begin{aligned} \varepsilon_t : A &\longrightarrow k \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \end{aligned}$$

Proposition

For $p, q \in k^*$, put $t = (pq)^{-1}$. We have $\mathrm{Ext}^n_A({}_\varepsilon k, A) = 0$ if $n \neq 3$, and $\mathrm{Ext}^3_A({}_\varepsilon k, A) \simeq k_{\varepsilon_{(pq)^{-1}}}$ as right A-modules.

Theorem

The algebra $\mathscr{O}_{p,q}(\mathsf{SL}_2(k))$ *is twisted Calabi-Yau of dimension* 3*, with Nakayama automorphism defined by*

$$\begin{aligned}\mu: \quad \mathscr{O}_{p,q}(\mathsf{SL}_2(k)) &\longrightarrow \mathscr{O}_{p,q}(\mathsf{SL}_2(k)) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} (pq)^{-1}a & b \\ c & (pq)\,d \end{pmatrix}.\end{aligned}$$

- THE END -

Thank you for your attention

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