# Magnus Expansions ... old and new ...

Kurusch Ebrahimi-Fard

NTNU Trondheim

Lie–Størmer Center

Conférence

"Algèbres de Hopf combinatoires et Analyse"

en l'honneur de Dominique Manchon

27-28 November 2025

# Why the Magnus Expansion... ?

2006: ◇ *Birkhoff type decompositions and the Baker–Campbell–Hausdorff recursion* (with L. Guo).

2009: ◇ *A Magnus- and Fer-type formula in dendriform algebras*

◇ *Dendriform Equations*

◇ *A noncommutative Bohnenblust–Spitzer identity for Rota–Baxter algebras solves Bogoliubov's counterterm recursion* (with F. Patras)

2014: ◇ *The tridendriform structure of a discrete Magnus expansion*

◇ *The Magnus expansion, trees and Knuth's rotation correspondence*

2022: ◇ *Post-Lie-Magnus expansion and BCH-recursion* (with M. Al-Kaabi)

2024: ◇ *Free post-groups, post-groups from group actions, and post-Lie algebras* (with M. Al-Kaabi)

2025: ◇ *Algebraic aspects of connections: from torsion, curvature, and post-Lie algebras to Gavrilov's double exponential and special polynomials* (with M. Al-Kaabi and H. Munthe-Kaas)

# What is the Magnus expansion?

# ~~What is the~~ Who is Magnus[1] ~~expansion~~ ?

**Wilhelm Magnus** was a German-American mathematician who worked in combinatorial group theory, Lie theory, mathematical physics, elliptic functions, and the study of tessellations.

61 PhD students

Born 1907 Berlin, Germany
Died 1990 New Rochelle, New York, USA

1931 PhD, Univ. of Frankfurt, Germany
1930-1932 Univ. of Göttingen, Germany
1933-1938 Univ. of Frankfurt, Germany
1934/35 Princeton Univ., USA
1939 Univ. of Königsberg, Germany
1939 Telefunken (industry), Germany
1944 Univ. of Königsberg, Germany
1946 Univ. of Göttingen, Germany
1948-1950 Caltech, USA
1950 Courant Inst., USA
1973 Polytechnic Inst. of New York, USA

[1] MacTutor Biographies: https://mathshistory.st-andrews.ac.uk/Biographies/Magnus/

# What is the Magnus expansion?

$$\dot{Y}(t) = \lambda A(t)Y(t), \quad Y(0) = \mathbf{1}$$

**W. Magnus** (1954)[2]: exponential solution

$$\begin{aligned} Y(t) &= \mathbf{1} + \sum_{n>0} \lambda^n \int_0^t \int_0^{s_1} \int_0^{s_2} \cdots \int_0^{s_{n-1}} A(s_1)A(s_2)\cdots A(s_n)ds_1 ds_2 \cdots ds_n \\ &= \exp\big(\Omega(\lambda A)(t)\big) \end{aligned}$$

Ansatz:

$$\Omega(\lambda A)(t) = \lambda \int_0^t A(s)ds$$

Recall:

$$\int_0^t A(s)ds \int_0^t A(u)du = \int_0^t A(s)\Big(\int_0^s A(u)du\Big)ds + \int_0^t \Big(\int_0^s A(u)du\Big)A(s)ds$$

[2] On the exponential solution of differential equations for a linear operator, Comm. Pure Appl. Math., 7 (1954) 649.

# Magnus expansion up to 2nd order

$$\exp\left(\lambda\int_0^t A(s)ds\right) = \mathbf{1} + \lambda\int_0^t A(s)ds + \frac{\lambda^2}{2}\left(\int_0^t A(s)\left(\int_0^s A(u)du\right)ds + \int_0^t \left(\int_0^s A(u)du\right)A(s)ds\right) + \cdots$$

Observe:

$$\mathbf{1} + \lambda\int_0^t A(s)ds - \frac{\lambda^2}{2}\int_0^t \left[\int_0^s A(u)du, A(s)\right]ds + \frac{\lambda^2}{2}\left(\int_0^t A(s)\left(\int_0^s A(u)du\right)ds + \int_0^t \left(\int_0^s A(u)du\right)A(s)ds\right)$$

$$= \mathbf{1} + \lambda\int_0^t A(s)ds + \lambda^2\int_0^t A(s)\left(\int_0^s A(u)du\right)ds$$

... up to order $\lambda^2$

$$Y = \exp\left(\lambda\int_0^t A(s)ds - \frac{\lambda^2}{2}\int_0^t \left[\int_0^s A(u)du, A(s)\right]ds\right)$$

# Magnus expansion up to 3rd order

$$
\begin{aligned}
\Omega(\lambda A)(t) = \lambda \int_0^t A(s)ds \\
&- \frac{\lambda^2}{2} \int_0^t \Big[ \int_0^{s_1} A(s_2)ds_2, A(s_1)\Big] ds_1 \\
&+ \frac{\lambda^3}{4} \int_0^t \Big[ \int_0^{s_1} \Big[ \int_0^{s_2} A(s_3)ds_3, A(s_2)\Big] ds_2, A(s_1)\Big] ds_1 \\
&+ \frac{\lambda^3}{12} \int_0^t \Big[ \int_0^{s_1} A(s_2)ds_2, \Big[ \int_0^{s_1} A(s_2)ds_2, A(s_1)\Big]\Big] ds_1 + \cdots
\end{aligned}
$$

# Solution

## Magnus ODE

$$\dot{\Omega}(A) = A + \sum_{n>0} \frac{B_n}{n!} \mathsf{ad}_{\Omega}^{(n)}(A)$$

Bernoulli numbers: $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_{2k+1} = 0$, $k > 0$

## Magnus recursion

$$\Omega_1(A)(t) = B_0 \int_0^t ds\, A(s),$$

$$\Omega_k(A)(t) = \int_0^t ds \sum_{m>0} \frac{B_m}{m!} \sum_{\substack{r_1+\cdots+r_m=k-1 \\ r_i>0}} \Big[\Omega_{r_1}(A)(s), \big[\Omega_{r_2}(A)(s), \cdots [\Omega_{r_m}(A)(s), A(s)]\big] \cdots \Big]$$

# Magnus expansion up to 3rd order

$$\Omega(\lambda A)(t) = \lambda \int_0^t A(s)ds$$

$$- \frac{\lambda^2}{2} \int_0^t \Big[ \int_0^{s_1} A(s_2)ds_2, A(s_1) \Big] ds_1$$

$$+ \frac{\lambda^3}{4} \int_0^t \Big[ \int_0^{s_1} \Big[ \int_0^{s_2} A(s_3)ds_3, A(s_2) \Big] ds_2, A(s_1) \Big] ds_1$$

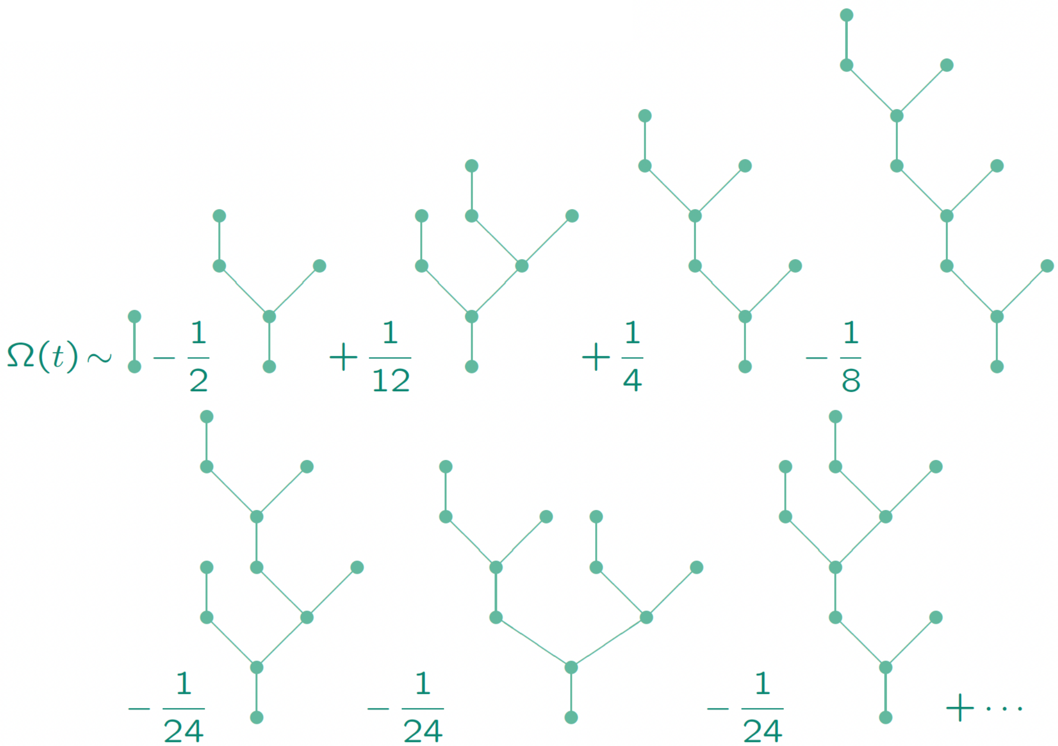
$$+ \frac{\lambda^3}{12} \int_0^t \Big[ \int_0^{s_1} A(s_2)ds_2, \Big[ \int_0^{s_1} A(s_2)ds_2, A(s_1) \Big] \Big] ds_1 + \cdots$$

Building block:

$$\Big[ \int_0^{s_1} A(s_2)ds_2, A(s_1) \Big]$$

# Magnus expansion as tree series

Iserles–Nørsett (1999[3]): tree expansion




[3] On the solution of linear differential equations in Lie groups, Philos. Trans. R. Soc. A, 357, (1999), 983

# The building block in Magnus expansion

The bilinear operation

$$(A \rhd B)(t) := \Big[ \int_0^t A(s)ds, B(t) \Big]$$

defines a left pre-Lie product.

**Definition**: $(\mathcal{P}, \rhd)$ is a (left) *pre-Lie algebra* if for all $x, y, z \in \mathcal{P}$

$$x \rhd (y \rhd z) - (x \rhd y) \rhd z = y \rhd (x \rhd z) - (y \rhd x) \rhd z.$$

**Theorem**: Let $(\mathcal{P}, \rhd)$ be a pre-Lie algebra. Then the commutator bracket

$$[x, y] := x \rhd y - y \rhd x$$

satisfies the Jacobi identity, i.e., $(\mathcal{P}, \rhd)$ is Lie admissible.

# Pre-Lie Magnus expansion

$$\dot{\Omega}(A) = A + \sum_{n>0} \frac{B_n}{n!} L^{(n)}_{\dot{\Omega}\rhd}(A) \qquad L^{(n)}_{B\rhd}(A) := L^{(n-1)}_{B\rhd}(B \rhd A)$$

$$\begin{aligned}\dot{\Omega}(A) = A &- \frac{1}{2}A \rhd A + \frac{1}{4}(A \rhd A) \rhd A + \frac{1}{12}A \rhd (A \rhd A) \\ &- \frac{1}{8}((A \rhd A) \rhd A) \rhd A - \frac{1}{24}(A \rhd A) \rhd (A \rhd A) \\ &- \frac{1}{24}(A \rhd (A \rhd A)) \rhd A - \frac{1}{24}A \rhd ((A \rhd A) \rhd A) + \cdots\end{aligned}$$

Observation: pre-Lie relation implies reduction in the number of terms

$$(A \rhd A) \rhd (A \rhd A) - ((A \rhd A) \rhd A) \rhd A = A \rhd ((A \rhd A) \rhd A) - (A \rhd (A \rhd A)) \rhd A$$

$$\dot{\Omega}_4(A) = -\frac{1}{6}((A \rhd A) \rhd A) \rhd A - \frac{1}{12}A \rhd ((A \rhd A) \rhd A)$$

# Pre-Lie Magnus expansion

$$\dot{\Omega}(A) = A + \sum_{n>0} \frac{B_n}{n!} L^{(n)}_{\dot{\Omega}\rhd}(A) \qquad L^{(n)}_{B\rhd}(A) := L^{(n-1)}_{B\rhd}(B \rhd A)$$

$$\begin{aligned}\dot{\Omega}(A) = A - \frac{1}{2}A \rhd A + \frac{1}{4}(A \rhd A) \rhd A + \frac{1}{12}A \rhd (A \rhd A)\\ -\frac{1}{8}((A \rhd A) \rhd A) \rhd A - \frac{1}{24}(A \rhd A) \rhd (A \rhd A)\\ -\frac{1}{24}(A \rhd (A \rhd A)) \rhd A - \frac{1}{24}A \rhd ((A \rhd A) \rhd A) + \cdots\end{aligned}$$

Observation: pre-Lie relation implies reduction in the number of terms

$$(A \rhd A) \rhd (A \rhd A) - ((A \rhd A) \rhd A) \rhd A = A \rhd ((A \rhd A) \rhd A) - (A \rhd (A \rhd A)) \rhd A$$

$$\dot{\Omega}_4(A) = -\frac{1}{6}((A \rhd A) \rhd A) \rhd A - \frac{1}{12}A \rhd ((A \rhd A) \rhd A)$$

# Guin–Oudom construction

For any pre-Lie algebra $(\mathcal{P}, \triangleright)$, extend $\triangleright$ to the symmetric algebra $(S(\mathcal{P}), \cdot, \Delta_{\mathrm{sh}}, \epsilon, S)$

$$\Delta_{\mathrm{sh}}(a_1 \cdots a_n) = \sum_{\{i_1,\dots,i_k\} \coprod \{j_1,\dots,j_{n-k}\} = [n]} a_{i_1} \cdots a_{i_k} \otimes a_{j_1} \cdots a_{j_{n-k}}$$

Grossman–Larson product on $\mathcal{S}(\mathcal{P})$

$$A * B := A' \cdot (A'' \triangleright B)$$

$$(A * B) \triangleright C = A \triangleright (B \triangleright C)$$

$(\mathcal{S}(\mathcal{P}), *, \Delta_{\mathrm{sh}})$ is a graded connected cocommutative Hopf algebra.

Observe: the product in $(S(\mathcal{P}), \cdot, \Delta_{\mathrm{sh}}, \epsilon, S)$ can be expressed in terms of the Grossman–Larson product

$$A \cdot B = A' * (S_*(A'') \triangleright B)$$

# Grossman–Larson product

$$x * y = xy + x \rhd y$$

$$\begin{aligned} x * y * z &= x(yz + y \rhd z) + x \rhd (yz + y \rhd z) \\ &= xyz + x(y \rhd z) + (x \rhd y)z + y(x \rhd z) + x \rhd (y \rhd z) \end{aligned}$$

Observe: $\theta_1(x) := x$ and for $n > 1$, $\theta_n(x) := x \rhd \theta_{n-1}(x)$

$$x^{*n} = x^n + \sum_{\substack{\pi \in P_n \\ \pi \neq 0_n}} \prod_{b \in \pi} \theta_{|b|}(x)$$

Define

$$\theta(x) = \sum_{n>0} \frac{1}{n!}\theta_n(x) = x + \frac{1}{2} x \rhd x + \frac{1}{3!} x \rhd (x \rhd x) + \frac{1}{4!} x \rhd (x \rhd (x \rhd x)) + \cdots$$

We deduce in $(\mathcal{S}(\mathcal{P}), *, \Delta_{\mathrm{sh}})$

$$\exp^*(x) = \exp^{\cdot}(\theta(x))$$

# Comparing two exponentials

Note that:

$$\theta(x) = \sum_{n \geqslant 0} \frac{1}{(n+1)!} L^{(n)}_{x \rhd} x$$

and

$$\theta^{-1}(x) = \Omega(x) = \sum_{n \geqslant 0} \frac{B_n}{n!} L^{(n)}_{\Omega(x) \rhd}(x)$$

which implies

$$\exp^*(\Omega(x)) = \exp^{\cdot}(x).$$

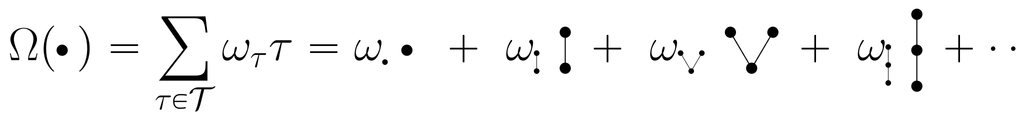
Observe (open problem):

$$\Omega_1(x) = \theta_1(x), \qquad \Omega_2(x) = \frac{1}{2}\theta_2(x), \qquad \Omega_3(x) = \frac{1}{3}\theta_3(x) + \frac{1}{12}[\theta_2(x), \theta_1(x)]$$

$$\Omega_4(x) = \frac{1}{4}\theta_4(x) + \frac{1}{12}[\theta_3(x), \theta_1(x)] = \frac{1}{6}((x \rhd x) \rhd x) \rhd x + \frac{1}{12} x \rhd ((x \rhd x) \rhd x)$$

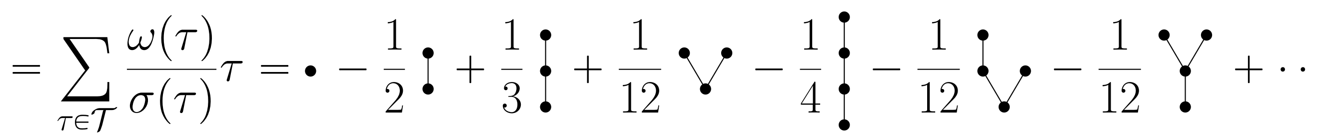
# Rooted tree expansion

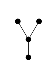
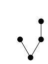
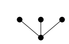
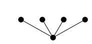
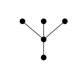
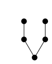
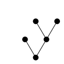
**Question:** What is the (pre-Lie) Magnus expansion in free pre-Lie $(\mathcal{T}, \triangleright)$?



**Ander Murua/Frédéric Chapoton** (2002):

$$\Omega(\bullet) = \bullet + \sum_{n>0} \frac{B_n}{n!} L^{(n)}_{\Omega\triangleright}(\bullet)$$



| $\tau$ |  |  |  |  |  |  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\omega$ | $1$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{4}$ | $\frac{1}{12}$ | $0$ | $-\frac{1}{30}$ | $\frac{1}{30}$ | $\frac{1}{30}$ | $\frac{1}{60}$ |
| $\sigma$ | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 3! | 4! | 3! | 2 | 2 |

# Pre-Lie Magnus expansion and BCH formula

$$\exp^*\big(\Omega(x)\big) * \exp^*\big(\Omega(y)\big) = \exp^*\big(\mathsf{BCH}^*(\Omega(x), \Omega(y))\big)$$

$$\begin{aligned}
\exp^*\big(\Omega(x)\big) * \exp^*\big(\Omega(y)\big) &= \exp^*\big(\Omega(x)\big)\Big(\exp^*\big(\Omega(x)\big) \rhd \exp^*\big(\Omega(y)\big)\Big) \\
&= \exp^*\big(\Omega(x)\big)\exp^*\Big(\Omega\big(\exp^*\big(\Omega(x)\big) \rhd y\big)\Big) \\
&= \exp^*\big(\Omega(x + \exp^*(\Omega(x)) \rhd y)\big)
\end{aligned}$$

$$x \# y := x + \exp^*(\Omega(x)) \rhd y$$

$$\exp^*\big(\Omega(x\#y)\big) = \exp^*\big(\mathsf{BCH}^*(\Omega(x), \Omega(y))\big)$$

# What is the Magnus expansion?

On the Lie algebra level we have two group laws:

$$U \bullet V := \mathsf{BCH}\big(U, V\big) \quad \text{and} \quad A\#B = A + e^{L_{\Omega(A)\rhd}} B$$

Pre-Lie BCH formula

$$\Omega(A\#B) = \mathsf{BCH}\big(\Omega(A), \Omega(B)\big)$$

and its inversion, i.e., for $A = \theta(U)$, $B = \theta(V)$

$$\theta(U)\#\theta(V) = \theta\big(\mathsf{BCH}\big(U, V)\big)\big)$$

$\Omega$ and its inverse $\theta$ are a relative Rota–Baxter operator respectively a crossed morphism:

$$\Omega(A) \bullet \Omega(B) = \Omega(A + e^{L_{\Omega(A)\rhd}} B)$$
$$\theta\big(U \bullet V\big) = \theta(U) + e^{L_{U\rhd}} \theta(V)$$

# Crossed morphism

Assume $G$ and $H$ to be groups and let

$$\phi : G \to \mathrm{Aut}(H)$$

be a group morphism. We will denote by $\phi_g$ the image of $g \in G$ via $\phi$, i.e. $\phi_g := \phi(g) \in \mathrm{Aut}(H)$ for all $g \in G$. We can say that that $G$ acts on $H$ on by automorphisms.

**Definition** [Crossed morphism of groups]
A map $f : G \to H$ is a crossed morphism relative to $\phi$ if for all elements $g_1, g_2 \in G$

$$f(g_1 g_2) = f(g_1)\phi_{g_1}(f(g_2)). \tag{1}$$

Moreover, suppose that $f : G \to H$ is an invertible crossed morphism. Then for $f^{-1} : H \to G$ and $h_1, h_2 \in H$

$$f^{-1}(h_1)f^{-1}(h_2) = f^{-1}(h_1 \phi_{f^{-1}(h_1)}(h_2)).$$

This follows at once from (1) and $h_i = f(g_i)$, $i = 1, 2$. Such an $f^{-1}$ is called a Rota–Baxter operator relative to $\phi$.

# Post-Lie algebra

A post-Lie algebra consists of a Lie algebra $(\mathfrak{g}, [\cdot\,,\cdot])$ and a product $\triangleright : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, compatible with the Lie bracket, in the following sense:

$$x \triangleright [y, z] = [x \triangleright y, z] + [y, x \triangleright z]$$

$$[x, y] \triangleright z = \mathrm{a}_{\triangleright}(x, y, z) - \mathrm{a}_{\triangleright}(y, x, z)$$

Recall:

$$a_{\triangleright}(x, y, z) := x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright z$$

Let $(\mathfrak{g}, [\cdot\,,\cdot], \triangleright)$ be a post-Lie algebra. Then the bracket

$$[\![x, y]\!] = x \triangleright y - y \triangleright x + [x, y]$$

satisfies the Jacobi identity.

# Guin–Oudom construction

Extend the post-Lie product $\rhd$ to $U(\mathfrak{g})$
Grossman–Larson product

$$A * B := A' \cdot (A'' \rhd B),$$

$$(A * B) \rhd C = A \rhd (B \rhd C)$$

$$A \cdot B = A' * (S_*(A'') \rhd B)$$

Post-Lie Magnus expansion:

$$\exp^* \big(\chi(x)\big) = \exp^\cdot(x)$$

$$\begin{aligned}\frac{d}{dt}\exp^*(\chi(tx)) &= \exp^\cdot(tx) \cdot y \\ &= \exp^\cdot(tx) * \Big(S_*\big(\exp^\cdot(tx)\big) \rhd x\Big)\end{aligned}$$

# Post-Lie flow equation

$$\alpha(x,t) := S_*\big(\exp^{\cdot}(tx)\big) \rhd x = \exp^*(-\chi(tx)) \rhd x$$

satisfies the differential equation

$$\begin{aligned}
\frac{d}{dt}\alpha(x,t) &= \frac{d}{dt}S_*\big(\exp^{\cdot}(tx)\big) \rhd x \\
&= S_*\Big(\frac{d}{dt}\exp^*(\chi(tx))\Big) \rhd x \\
&= S_*\Big(\exp^{\cdot}(tx) * \big(S_*\big(\exp^{\cdot}(tx)\big) \rhd x\big)\Big) \rhd x \\
&= \Big( - \big(S_*\big(\exp^{\cdot}(tx)\big) \rhd x\big) * S_*\big(\exp^{\cdot}(tx)\big)\Big) \rhd x = -\alpha(x,t) \rhd \alpha(x,t)
\end{aligned}$$

Hence, $\alpha(x,t)$ satisfies the *post-Lie flow equation*

$$\begin{cases} \alpha(x,0) = x \\ \dfrac{d}{dt}\alpha(x,t) = -\alpha(x,t) \rhd \alpha(x,t) \end{cases}$$

# Grossman–Larson ODE

The Grossman–Larson exponential $\exp^*(\chi(tx))$ is the solution

$$\frac{d}{dt}\exp^*(\chi(tx)) = \exp^*(\chi(tx)) * \alpha(x,t).$$

$$\chi(tx) = \tilde{\Omega}^*\big(\alpha(x,t)\big)$$

$$\alpha(x,t) = \sum_{j=0}^{\infty} \alpha_j t^j = S_*\big(\exp^{\cdot}(tx)\big) \rhd x = x + \sum_{i>0} \frac{t^i}{i!} S_*(x^i) \rhd x$$

$$\alpha_0 = x$$

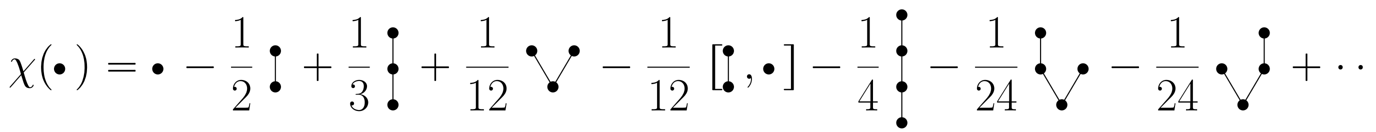
$$\alpha_1 = -x \rhd x$$

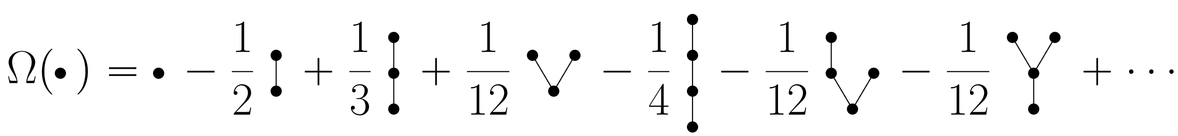
$$\alpha_2 = \frac{1}{2} S_*(x^2) \rhd x = S_*(x * x - x \rhd x) \rhd x = (x \rhd x) \rhd x + x \rhd (x \rhd x)$$

$$\alpha_n = \frac{1}{n!} S_*(x^n) \rhd x = S_*(x^{n-1})(x \rhd x) - S_*(x \rhd x^{n-1}) \rhd x$$

# Post-Lie Magnus expansion

$$\chi(x) \quad = \quad y - \frac{1}{2} x \rhd x + \frac{1}{12} x \rhd (x \rhd x) + \frac{1}{4}(x \rhd x) \rhd x + \frac{1}{12}[x \rhd x, x] + \cdots$$





# Post-Lie BCH relation

$$\exp^*\big(\chi(x)\big) * \exp^*\big(\chi(y)\big)$$

$$= \exp^*\big(\chi(x)\big)\Big(\exp^*\big(\chi(x)\big) \rhd \exp^*\big(\chi(y)\big)\Big)$$

$$= \exp^*\big(\chi(x)\big)\Big(\exp^*(\chi(x)) \rhd \exp(y)\Big)$$

$$= \exp^*\big(\chi(x)\big)\exp^*\big(\chi(\exp^*(\chi(x)) \rhd y)\big)$$

$$= \exp(x)\exp(\exp^*(\chi(x)) \rhd y) = \exp^*\Big(\chi\Big(\mathrm{BCH}(x, \exp^*(\chi(x)) \rhd y)\Big)\Big)$$

$$\chi\big(\mathsf{BCH}(x, \exp^*(\chi(x)) \rhd y)\big) = \mathsf{BCH}_*(\chi(x), \chi\big(y\big))$$

Thank you for your attention!