Operad Theory for singular SPDEs

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Conférence "Algèbres de Hopf combinatoires et Analyse"
en l'honneur de Dominique Manchon.
27th November 2025, Clermont-Ferrand.

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European Research Council
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Singular SPDEs

Let consider the following SPDE:

$$(\partial_t - \mathcal{L})\, u = f(u, \nabla u, \ldots) + g\,(u, \nabla u, \ldots)\, \xi.$$

Regularisation of the noises with a mollifier ϱ: $\xi_\varepsilon = \varrho_\varepsilon * \xi$.

Find an ansatz for the solution u_ε such that

- the solution map $\Phi : \xi_\varepsilon, \text{Ansatz} \longmapsto u_\varepsilon$ is continuous.
- $\xi_\varepsilon \to \xi$ when $\varepsilon \to 0$.

Various ansatz

One wants to decompose u_ε into

$$u_\varepsilon = \text{Finite sum of stochastic Iterated Integrals} + \text{Remainder.}$$

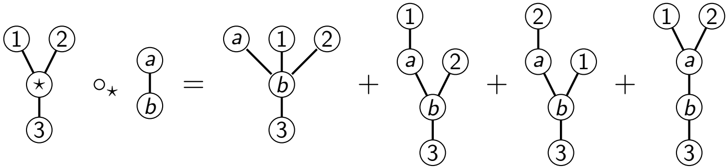
Various ansatz:

- Regularity Structures: Local expansion.
  [Hairer ; 14]
- Paracontrolled Calculus: Harmonic Analysis.
  [Gubinelli-Imkeller-Perkowski ; 15]
- Renormalisation Group: Discrete scales. [Kupiainen ; 16]
  Polchinski Flow: Continuous scales. [Duch ; 25]

Same combinatorial sets: Decorated trees and Multi-indices.

Operad theory

One considers trees with labels on the nodes and defines partial composition operations. As an example



One has for τ_1, τ_2, τ_3 labelled trees

$$(\tau_1 \circ_i \tau_2) \circ_j \tau_3 = (\tau_1 \circ_j \tau_3) \circ_i \tau_2, \quad (\tau_1 \circ_i \tau_2) \circ_j \tau_3 = \tau_1 \circ_i (\tau_2 \circ_j \tau_3),$$

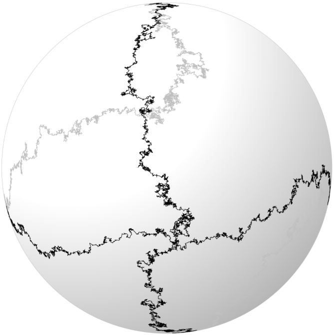
$$\tau_1 \circ_i \mathbf{1} = \mathbf{1} \circ_j \tau_1 = \tau_1.$$

Used in [B.-Chandra-Chevyrev-Hairer ; 21] extending a proof from [Chapoton-Livernet ; 01] .

Geometric stochastic heat equations

Stochastic dynamics in the space of loops in a compact Riemannian manifold:

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\,\partial_x u^\beta \partial_x u^\gamma + \sigma_i^\alpha(u)\xi_i, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{T}$$



Renormalised equation

Theorem (B.-Chandra-Chevyrev-Hairer 2021)

There exist renormalisation constants $C_\varepsilon(\tau)$ *such that the renormalised equation is given by:*

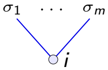
$$\partial_t u_\varepsilon^\alpha = \partial_x^2 u_\varepsilon^\alpha + \Gamma^\alpha_{\beta\gamma}(u_\varepsilon)\,\partial_x u_\varepsilon^\beta \partial_x u_\varepsilon^\gamma + \sigma_i^\alpha(u_\varepsilon)\,\xi_i^\varepsilon + \sum_{\tau\in\mathfrak{S}_\xi} C_\varepsilon(\tau)\Upsilon^\alpha_{\Gamma,\sigma}[\tau](u_\varepsilon).$$

Renormalised equation for any subcritical (non)-Gaussian noise ξ ($\xi \in \mathcal{C}^\alpha$, $\alpha \in (-2, -\frac{3}{2}]$).

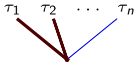
Chain rule and **Itô Isometry** studied for space-time white noise ($\alpha = -\frac{3}{2}$) [B.-Gabriel-Hairer-Zambotti ; 22].

Christoffel trees $\mathfrak{S}_\xi$

Trees generated by some rules coming from the non-linearities



$\equiv u^m \xi_i,$



$\equiv u^{n-2}(\partial_x u)^2.$

One puts an equivalence class on the $\circ_i$. One has

$C_\varepsilon($  $) = \mathbb{E}\left(\xi_i^\varepsilon G * \xi_i^\varepsilon\right),\quad C_\varepsilon($  $) = \mathbb{E}\left((\partial_x G * \xi_i^\varepsilon)^2\right),$

$\Upsilon^\alpha_{\Gamma,\sigma}[$  $](u) = 2\partial_\zeta \sigma_i^\alpha(u)\partial_\eta \Gamma^\zeta_{\beta\gamma}(u)\,\sigma_j^\eta(u)\,\sigma_i^\gamma(u)\,\sigma_j^\beta(u)\,.$

Given ξ, keep only the decorated trees with negative degree.

Main result for Chain rule

Theorem (B.-Dotsenko 2024)

There exist $C_\varepsilon(\tau)$ such that the renormalised equation is

$$\partial_t u_\varepsilon^\alpha = \partial_x^2 u_\varepsilon^\alpha + \Gamma^\alpha_{\beta\gamma}(u_\varepsilon)\,\partial_x u_\varepsilon^\beta \partial_x u_\varepsilon^\gamma + \sigma_i^\alpha(u_\varepsilon)\,\xi_i^\varepsilon + \sum_{\tau\in\mathfrak{B}_\xi} C_\varepsilon(\tau)\Upsilon^\alpha_{\Gamma,\sigma}[\tau](u_\varepsilon)\,,$$

where $\mathfrak{B}_\xi$ depends on the ξ_i and the $\Upsilon_{\Gamma,\sigma}[\tau](u_\varepsilon)$ are computed with the σ_i and the covariant derivative $\nabla_X Y$ defined by

$$(\nabla_X Y)^\alpha(u) = X^\beta(u)\,\partial_\beta Y^\alpha(u) + \Gamma^\alpha_{\beta\gamma}(u)\,X^\beta(u)\,Y^\gamma(u)\,.$$

The renormalised equation transforms according to the chain rule under composition with diffeomorphisms.

Quasilinear Equations

Theorem (B.-Gerencser-Nadeem 2024)

Let a, taking values in $[\lambda, \lambda^{-1}]$ for some $\lambda > 0$. Let $u_0 \in \mathcal{C}^\alpha(\mathbb{T})$ for some $\alpha > 0$. There is a choice of the $C_\varepsilon^{a(u_\varepsilon)}(\tau)$ such that the solution u_ε

$$\partial_t u_\varepsilon - a(u_\varepsilon)\partial_x^2 u_\varepsilon = f(u_\varepsilon)(\partial_x u_\varepsilon)^2 + g(u_\varepsilon)\xi_\varepsilon + \sum_{\tau \in \mathfrak{B}_\xi} C_\varepsilon^{a(u_\varepsilon)}(\tau) \frac{\Upsilon_F[\tau](u_\varepsilon)}{S(\tau)}$$

converges locally in time. For space-time white noise, the solution converges globally in time for $t \in [0,1]$.

- Locality of counter-terms obtained from the chain rule.
- Global existence in time coming from the stochastic quasilinear multiplicative heat equation.

Geometric counter-terms

Given a diffeomorphism φ, one has

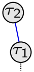
$$(\varphi\cdot\Gamma)^{\alpha}_{\eta\zeta}(\varphi(u))\,\partial_\beta\varphi^\eta(u)\,\partial_\gamma\varphi^\zeta(u) = \partial_\mu\varphi^\alpha(u)\,\Gamma^{\mu}_{\beta\gamma}(u) - \partial^2_{\beta\gamma}\varphi^\alpha(u)\;,$$
$$(\varphi\cdot\sigma)^\alpha(\varphi(u)) = \partial_\beta\varphi^\alpha(u)\,\sigma^\beta(u)\;.$$

The space $V_{\mathrm{geo}} \subset \langle\mathfrak{S}_\xi\rangle$ consists of those elements τ such that

$$\varphi\cdot\Upsilon_{\Gamma,\sigma}[\tau] = \Upsilon_{\varphi\cdot\Gamma,\varphi\cdot\sigma}[\tau].$$

Generating set

Covariant derivative on decorated trees

$$\nabla_{\tau_1}\tau_2 = $$ 

$$+ \frac{1}{2}$$ 

and

$$\Upsilon^{\alpha}_{\Gamma,\sigma}[\nabla_{\tau_1}\tau_2] = \Upsilon^{\beta}_{\Gamma,\sigma}[\tau_1]\,\partial_\beta \Upsilon^{\alpha}_{\Gamma,\sigma}[\tau_2] + \Gamma^{\alpha}_{\beta\gamma}\,\Upsilon^{\beta}_{\Gamma,\sigma}[\tau_1]\,\Upsilon^{\gamma}_{\Gamma,\sigma}[\tau_2].$$

For space-time white noise ξ, one has

$$\begin{aligned}\mathfrak{B}_\xi = \{&\nabla_{\circ}\circ,\ \nabla_{\bullet}\nabla_{\circ}\nabla_{\bullet}\circ,\ \nabla_{\circ}\nabla_{\bullet}\nabla_{\bullet}\circ,\ \nabla_{\bullet}\nabla_{\bullet}\nabla_{\circ}\circ,\ \nabla_{\bullet}\nabla_{\nabla_{\bullet}\circ}\circ,\ \nabla_{\nabla_{\bullet}\circ}\nabla_{\bullet}\circ,\\ &\nabla_{\nabla_{\circ}\bullet}\nabla_{\bullet}\circ,\ \nabla_{\nabla_{\bullet}\bullet}\nabla_{\circ}\circ,\ \nabla_{\nabla_{\bullet}\nabla_{\bullet}\circ}\circ,\ \nabla_{\nabla_{\bullet}\nabla_{\circ}\circ}\bullet,\ \nabla_{\nabla_{\nabla_{\bullet}\circ}\circ}\bullet,\\ &\nabla_{\nabla_{\nabla_{\bullet}\circ}\bullet}\circ,\ \nabla_{\circ}\nabla_{\nabla_{\bullet}\circ}\bullet,\ \nabla_{\nabla_{\circ}\nabla_{\bullet}\circ}\bullet,\ \nabla_{\nabla_{\nabla_{\circ}\circ}\bullet}\bullet,\ \nabla_{\bullet}\nabla_{\nabla_{\circ}\circ}\bullet\ \}.\end{aligned}$$

Dimension

Theorem (B.-Dotsenko 2024)

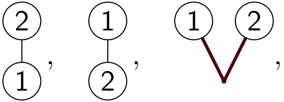
For all subcritical noise ξ, sufficiently high dimension $d(\xi)$, one can compute the dimension of $(\Upsilon_{\Gamma,\sigma}[\tau])_{\tau\in\mathfrak{B}_\xi}$.

Geometric subspace	$\mathcal{V}^{g}_{\mathrm{geo},2}$	$\mathcal{V}^{g}_{\mathrm{geo},4}$	$\mathcal{V}^{c}_{\mathrm{geo},4}$	$\mathcal{V}^{g}_{\mathrm{geo},6}$	$\mathcal{V}^{c}_{\mathrm{geo},7}$	$\mathcal{V}^{g}_{\mathrm{geo},8}$
Dimension	1	15	22	507	17646	29880

Theorem (Bellingeri-B. 2024)

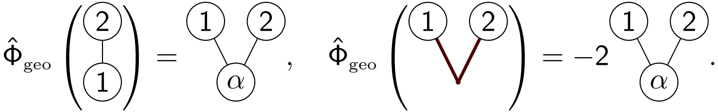
For all subcritical noise ξ, $d=1$ and $m=1$, the dimension of $(\Upsilon_{\Gamma,\sigma}[\tau])_{\tau\in\mathfrak{B}_\xi}$ is the dimension of the free Novikov algebra.

The binary operations of our operad are



$$\nabla\mathrm{Trees} = \mathrm{PreLie} \vee \mathrm{ComMag}\,.$$

We set $\hat{\Phi}_{\mathrm{geo}} : \nabla\mathrm{Trees} \to \nabla\mathrm{Trees} \vee \mathbb{K}\alpha$



Geometric counter-terms given by $\ker \hat{\Phi}_{\mathrm{geo}}$.

Homological arguments for showing $\ker \hat{\varphi}_{\mathrm{geo}} = \langle \mathfrak{B}_{\xi} \rangle$.

Multi-indices

Generalised KPZ equation

$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + f(u_\varepsilon)\,(\partial_x u_\varepsilon)^2 + g(u_\varepsilon)\xi_\varepsilon .$$

Local expansion indexed by decorated trees $\mathcal{T}$

$$u_\varepsilon = \sum_{\tau \in \mathcal{T}} c^\varepsilon_{\tau,z} u^\varepsilon_{\tau,z} + R^\varepsilon_{\mathcal{T},z}.$$

Observation:

$\tau_1 =$ , $\tau_2 =$ , $c^\varepsilon_{\tau_1,z} = c^\varepsilon_{\tau_2,z} = 2f(u_\varepsilon(z))^2 g(u_\varepsilon(z))^3 g'(u_\varepsilon(z)).$

Grouping terms

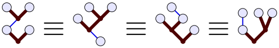
$$c^\varepsilon_{\tau_1,z} u^\varepsilon_{\tau_1,z} + c^\varepsilon_{\tau_2,z} u^\varepsilon_{\tau_2,z} = c^\varepsilon_{\tau_1,z} \left(u^\varepsilon_{\tau_1,z} + u^\varepsilon_{\tau_2,z} \right).$$

New coding

One fixes variables corresponding to nodes with their fertility

$$z_{(\circ,k)}, \quad z_{(\mathbf{V},k)}, \quad k \in \mathbb{N}.$$

One has



$$\equiv \left(z_{(\mathbf{V},0)}\right)^2 \left(z_{(\circ,0)}\right)^3 z_{(\circ,1)}.$$

Then, one has the existence of a surjection

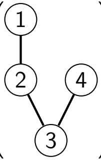
$$\Psi : \nabla\text{Trees} \to \nabla\text{Multi-indices}$$

Can one find C such that:

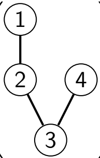
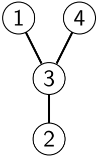
$$\text{Trees} \twoheadrightarrow \text{C} \twoheadrightarrow \text{Multi-indices}.$$

Elementary differentials of arity n

We denote by $\mathcal{W}_d(n)$, the elementary differentials of arity n. Let $f, g, h, p \in \mathcal{C}^\infty(\mathbb{R}^d)$.

F () $(f, g, h, p) = \sum_{i,j,k=1}^{d} f_i \partial_i g_j p_k \partial_j \partial_k h \in \mathcal{W}_d(4).$

One defines an action of $\mathfrak{S}(n)$ on Trees with n nodes. For example,

$(1\ 3\ 2).$ () $=$  .

Main negative results

Theorem (B.-Laubie 2025)

In dimension $d \neq 1$,

1. *Trees are the only combinatorial description of $\mathcal{W}_d$ compatible with the trees.*
2. *$\mathcal{W}_d$ does not admit any faithful combinatorial description compatible with Multi-indices.*

Ideas of the proof

1. Let τ_1, τ_2 with the same mutli-index. There exists σ such that $\sigma.\tau_1 = \tau_2$. Then

$$F(\tau_2) = F(\tau_1) \iff F(\sigma.\tau_1) = F(\tau_1) \iff \tau_2 = \sigma.\tau_1 = \tau_1.$$

2. Polynomial identities in Witt algebras (Study of operadic ideals).

Perspectives

- Itô Isometry in the full subcritical regime: dimension of a properad.
- Another symmetry is needed for selecting a natural solution in the full subcritical regime.
- Regularity Structures via Arborical Species.
  [B.-Laubie ; in progress 25].