A Solution to the Zariski-Closure Conjecture for Exponential Lie Groups: A Longstanding Program with D. Manchon II

University of Sfax

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Algèbres de Hopf combinatoires et Analyse
En l’honneur de Dominique Manchon
Aubière, France

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- Since 2004, our discussions took concrete shape within a CNRS–DGRST project, during which a conjecture, later known as the Zariski-Closure Conjecture, was formulated and supported. This result generates a dequantization procedure for coadjoint orbits of exponential groups, so what may be viewed as Pedersen’s perspective.

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- The eighth Tunisian-Japanese Conference TJC 8, was organized in his honor (October 27-30, 2025). https://sites.google.com/view/tunisian-japanese-conference/home/2025-8th/

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- Hence G is connected and simply connected.

- Let $\mathrm{ad}_{\mathfrak{g}}$ be the adjoint endomorphism defined by:

$$\mathrm{ad}_{\mathfrak{g}}(X)(Y) = \mathrm{ad}_X(Y) = [X, Y],\ X,\ Y \in \mathfrak{g}.$$

- The group G acts on $\mathfrak{g}$ by the adjoint representation Ad_G, i.e

$$\begin{aligned}\mathrm{Ad}_G(g)(Y) &= \mathrm{Ad}(g)(Y)\\ &= e^{\mathrm{ad}_X}Y,\ \ g=\exp X \in G,\end{aligned}$$

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- The unitary dual $\widehat{G}$ (the set of classes of unitary and irreducible representations of G) is parameterized via the Kirillov-Bernat orbit method.

The orbit method for exponential Lie groups

❍ A subspace $\mathfrak{h}[\ell]$ of the Lie algebra $\mathfrak{g}$ is called a polarization for $\ell \in \mathfrak{g}^*$ if $\mathfrak{h}[\ell]$ is a maximal dimensional isotropic subalgebra with respect to the skew-symmetric bilinear form B_ℓ defined by

$$B_\ell(X, Y) = \ell([X, Y]),\ X,\ Y \in \mathfrak{g}.$$

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- Let $\mathfrak{g}(\ell) = \{X \in \mathfrak{g},\ \ell([X, \mathfrak{g}]) = \{0\}\}$ be the radical of B_ℓ, which is the stabilizer of $\ell \in \mathfrak{g}^*$ in $\mathfrak{g}$. This is actually the Lie algebra of the Lie subgroup $G(\ell) = \{g \in G,\ g \cdot \ell = \ell\}$.

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- So, we can consider the unitary character

$$\chi_\ell(\exp X) = e^{-i\ell(X)},\ X \in \mathfrak{h}[\ell].$$

Any linear form $\ell \in \mathfrak{g}^*$ admits a real polarization $\mathfrak{h} = \mathfrak{h}[\ell]$. Define then

$$\pi_\ell = \pi_{\ell,\mathfrak{h}} = \mathrm{Ind}_H^G \chi_\ell, \ H = \exp \mathfrak{h}.$$

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- The Hilbert space $\mathcal{H}_\ell$ of the representation π_ℓ is the space $L^2(G/H, \chi_\ell)$ consisting of all measurable functions $\xi : G \longrightarrow \mathbb{C}$ such that

$$\xi(gh) = \Delta_{H,G}^{-1}(h)\chi_\ell(h^{-1})\xi(g), \ \Delta_{H,G} = \frac{\Delta_H}{\Delta_G}$$

for all $h \in H$, $g \in G$ and such that the function $|\xi|$ is contained in $L^2(G/H)$. π_ℓ acts on $\mathcal{H}_\ell$ by left translations.

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- Assume in addition that the polarization $\mathfrak{h}$ at ℓ satisfies Pukanszky's condition: $H \cdot \ell = \ell + \mathfrak{h}^\perp$, where $H := \exp \mathfrak{h}$.

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- Moreover, the Kirillov-Bernat-Vergne mapping

$$\begin{aligned} \bar{\theta}_G : \ \mathfrak{g}^*/G &\longrightarrow \widehat{G} \\ \Omega_\ell &\longmapsto [\pi_{\ell,\mathfrak{h}}] =: \pi_{\Omega_\ell} \end{aligned}$$

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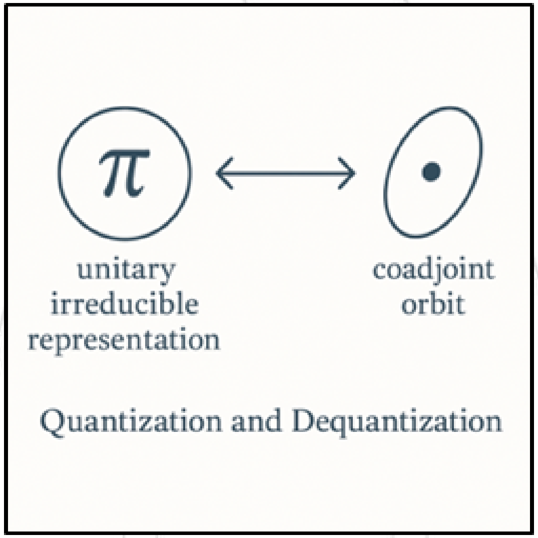
❍ Let $\mathcal{U}(\mathfrak{g}^{\mathbb{C}}) = T(\mathfrak{g}^{\mathbb{C}})/ < x \otimes y - y \otimes x - [x, y] >$ be the enveloping algebra of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$. We shall note $d\pi_{\ell,\mathfrak{h}}$ (and sometimes $\pi_{\ell,\mathfrak{h}}$) the derivative of the representation $\pi_{\ell,\mathfrak{h}}$ defined on $\mathcal{U}(\mathfrak{g}^{\mathbb{C}})$. A primitive ideal is by definition the annihilator of some $d\pi_{\ell,\mathfrak{h}}$ for a given $\ell \in \mathfrak{g}^*$.

Dequantization

❍ The dequantization refers to the inverse process: recovering the orbit from its associated representation.



- Theory of **Moment Maps**: (N. Wildberger, 1989).

Dequantization via Poisson characteristic varieties

❍ Let $d = \dim \Omega_\ell$. There exists a global Darboux coordinate system for Ω_ℓ:

$$\Theta : \mathbb{R}^d \longrightarrow \Omega, (p, q) = (p_1, \dots, p_{\frac{d}{2}}, q_1, \dots, q_{\frac{d}{2}}) \longmapsto \Theta(p, q)$$

with

$$\langle \Theta(p, q), X \rangle = X(p, q) = \sum_{u=1}^{d/2} a_{X,u}(q) p_u + a_{X,0}(q)$$

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❍ The functions $a_{X,u}, u = 0, \ldots, d/2$ are in $C^\infty(\mathbb{R}^{d/2})$, and are entire with at most an exponential growth.

❍ By Pedersen results, there exists a unique unitary strongly continuous representation ρ of G in $L^2(\mathbb{R}^{d/2})$ such that the space $\mathcal{H}_\rho^\infty$ of its C^∞-vectors contains $C_c^\infty(\mathbb{R}^{d/2})$, and such that for all $\xi \in C^\infty(\mathbb{R}^{d/2})$ and $X \in \mathfrak{g}$, we have :

$$d\rho(X)\xi(t) = \sum_{u=1}^{d/2} a_{X,u}(t)\frac{\partial \xi(t)}{\partial t_u} - i a_{X,0}(t)\xi(t) + \frac{1}{2}\Big(\sum_{u=1}^{d/2} \frac{\partial a_{X,u}}{\partial t_u}(t)\Big)\xi(t).$$

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- This representation is equivalent to the representation $\pi_{\ell,\mathfrak{h}}$.

Deforming unitary representations

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- Let $f \in \mathfrak{g}^*$, the coadjoint orbit $\Omega_{f,\hbar} = G_\hbar \cdot f \subset \mathfrak{g}^*$ is the same for all $\hbar \neq 0$.

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- We use the above results to construct a simultaneous realization of all the induced $\mathrm{Ind}_{H_\hbar}^{G_\hbar} \chi_{f,\hbar}$ in the same $L^2(\mathbb{R}^{d/2})$.

- Let $A = S(\mathfrak{g}^{\mathbb{C}})$ be the symmetric algebra of $\mathfrak{g}^{\mathbb{C}}$ endowed with the Poisson bracket. Let $\mathcal{A} = A[[\nu]]$ be the associated deformed algebra, for an indeterminate ν.

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❍ The Poisson bracket of Kirillov-Kostant-Souriau is given for $\varphi,\ \psi \in A$ and $\ell \in \mathfrak{g}^*$ by the formula:

$$\{\varphi,\ \psi\}(\ell) := < \ell,\ [d\varphi(\ell), d\psi(\ell)] >,$$

but the Poisson structure depends on ν : for all $\varphi, \psi \in C^\infty(\mathfrak{g}^*)$ we have :

$$\{\varphi, \psi\}_\nu(\ell) = \nu\{\varphi, \psi\}_1(\ell) = \nu < \ell,\ [d\varphi(\ell), d\psi(\ell)] > .$$

❍ Let $\tau : \mathcal{A} \to \mathcal{U}_\nu(\mathfrak{g}^{\mathbb{C}})$ be the Duflo isomorphism :

$$\tau = \sigma \circ J(D)^{1/2},$$

where σ denotes the symmetrization map and $J(D)$ is the differential operator of infinite order with constant coefficients corresponding to the formal series :

$$\left(\frac{sh\,\mathrm{ad}\,\frac{\nu}{2}x}{\mathrm{ad}\,\frac{\nu}{2}x}\right).$$

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$$\Big(\frac{sh \,\mathrm{ad}\, \frac{\nu}{2}x}{\mathrm{ad}\, \frac{\nu}{2}x}\Big).$$

❍ Then the algebra $(\mathcal{A}, *)$ is isomorphic to the enveloping formal complexified algebra:

$$\mathcal{U}_\nu(\mathfrak{g}^{\mathbb{C}}) = T(\mathfrak{g}^{\mathbb{C}})[[\nu]]/ < x \otimes y - y \otimes x - \nu[x, y] >,$$

and we have precisely:

$$f * g = \tau^{-1}(\tau f \cdot \tau g).$$

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- This star-product thus generates a family of non-commutative associative laws $(*_\nu)$ on $\mathcal{A}$, the parameter ν varying through the field of complex numbers.

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- For a purely imaginary complex number ν, we now consider the representation:

$$\pi_\nu(X) = i\rho_{-i\nu}(X),$$

which allows to obtain a family (π_ν) of representations on $(\mathcal{A}, *_\nu)$.

❍ The expression of the representation π_ν of the deformed algebra $\mathcal{A}$ is obtained as follows : for all $X \in \mathfrak{g}$ and $\xi \in C_c^{+\infty}(\mathbb{R}^{d/2})$ one has:

$$d\pi_\nu(X)\xi(t) = \sum_{u=1}^{\frac{d}{2}} i a_{X,u}(-i\nu t)\frac{\partial \xi}{\partial t_u}(t) + a_{X,0}(-i\nu t)\xi(t)$$

$$+\frac{\nu}{2}\sum_{u=1}^{\frac{d}{2}} \frac{\partial}{\partial t_u} a_{X,u}(-i\nu t)\xi(t),$$

and the action of an element of $\mathcal{A}$ is obtained via the identification $\mathcal{A} = \mathcal{U}_\nu(\mathfrak{g}^{\mathbb{C}})$ given by Duflo's isomorphism.

Characteristic manifolds

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- The set $V(J)$ of common zeroes of the elements of J is called the Poisson characteristic variety and denoted by $VA(\pi)$.

The Zariski closure Conjecture

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The Zariski closure Conjecture

- We (Bak.-Dhieb-Manchon) posed the following:

Conjecture 1: The Zariski closure Conjecture, 2005

Let G be an exponential solvable Lie group, with Lie algebra $\mathfrak{g}$, and let π be an irreducible unitary representation of G, associated to a coadjoint orbit Ω via the Kirillov orbit method. Then the Poisson characteristic variety $VA(\pi)$ coincides with the Zariski closure in $\mathfrak{g}^*$ of the orbit Ω.
In other words, the ideal J is rational (according to Dixmier's terminology).

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- The case where $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{g}(\ell)$, the proof is inspired from the nilpotent case.

- The case when π is realized with a normal Pukanszky polarization.

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This yields a more general question since the Hilbert space $\mathcal{H}$ of π is regarded as a quantization of the symplectic manifold Ω.

Question 1

Is it raisonnable to expect a close relation between the ideal

$$I(\Omega) := \{F \in S(\mathfrak{g}) : F|_{\Omega} = 0\}$$

and the annihilator

$$\ker d\pi := \{W \in \mathcal{U}(\mathfrak{g}) : d\pi(W) = 0\}?$$

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❍ One can expect the following:

Claim 1

There exists a generating family of $(Q_j)_j$ of $I(\Omega)$, such that the annihilator of π in $\mathcal{U}(\mathfrak{g})$ is the ideal generated by the elements u_j's with $\tau^{-1}(u_j)(\ell) = Q_j(i\ell)$ for $\ell \in \mathfrak{g}^*$.

❍ Claim 1 holds in the nilpotent setting (Pedersen, Godfrey). It is very complicated to construct generators of primitive ideals of the envelopping algebra of an exponential sovable Lie groups.

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Theorem 2: Bak-Ishi, 2024

Let G be a simply connected real exponential solvable Lie group. Let π_ℓ be the irreducible unitary representation of G corresponding to the coadjoint orbit Ω_ℓ through a linear form $\ell \in \mathfrak{g}^*$. Assume that $\mathfrak{g}(\ell)$ is an ideal of $\mathfrak{g}$. Then one has

$$\ker d\pi_\ell = \mathcal{U}(\mathfrak{g})\text{-span}\{T - i\ell(T), T \in \mathfrak{g}(\ell)\}.$$

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Theorem 2: Bak-Ishi, 2024

Let G be a simply connected real exponential solvable Lie group. Let π_ℓ be the irreducible unitary representation of G corresponding to the coadjoint orbit Ω_ℓ through a linear form $\ell \in \mathfrak{g}^*$. Assume that $\mathfrak{g}(\ell)$ is an ideal of $\mathfrak{g}$. Then one has

$$\ker d\pi_\ell = \mathcal{U}(\mathfrak{g})\text{-span}\{T - i\ell(T), T \in \mathfrak{g}(\ell)\}.$$

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Question 2

Is it possible to characterize Frobenuis exponential solvable Lie algebras by means of primitive ideals of $\mathcal{U}(\mathfrak{g})$?

Frobenius exponential Lie algebras

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Let $\mathfrak{g}$ be an exponential solvable Lie algebra. The following are equivalent:

1. $\mathfrak{g}$ is Frobenius Lie algebra.
2. There exists $\ell \in \mathfrak{g}^*$ such that $\mathfrak{g}(\ell)$ is an ideal of $\mathfrak{g}$ and $\ker d\pi_\ell$ is trivial.
3. There exists $\ell \in \mathfrak{g}^*$ such that the coadjoint orbit Ω_ℓ through ℓ is relatively open in an affine subspace of $\mathfrak{g}^*$ and $\ker d\pi_\ell$ is trivial.

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❍ The following is then immediate:

Corollary

Under the same circumstances, we have $\ker d\pi_\ell = \{0\}$ if Ω_ℓ is open. That is, if $\mathfrak{g}$ is Frobenuis, then $G = \exp(\mathfrak{g})$ admits a $\pi_\ell \in \hat{G}$ such that $\ker d\pi_\ell = \{0\}$ for some $\ell \in \mathfrak{g}^*$.

Two counterexamples

- We now produce two counterexamples of an exponential solvable Lie group with a Lie algebra $\mathfrak{g}$ admitting a linear form $\ell \in \mathfrak{g}^*$ such that $\ker d\pi_\ell$ is trivial but the corresponding orbit fails to be open in $\mathfrak{g}^*$.

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Example 1: First, we produce an example of 3-dimensional $\mathfrak{g}$ with a primitive zero ideal, which is automatically non-Frobenius because of its odd dimension.

- Let $\mathfrak{g} := \mathfrak{g}_3(1)$ be the Lie algebra generated by the three vectors $\{A, X, Y\}$ whose Lie brackets are given by : $[A, X] = X - Y$, $[A, Y] = X + Y$ and let $G = \exp \mathfrak{g}$. Hence, G is an exponential non-completely solvable Lie group.

Let $f = xX^* + yY^* + aA^* \in \mathfrak{g}^*$. If $x^2 + y^2 = 0$, then, the orbit of f is reduced to the unit set $\{f\}$.

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❍ Let $f = xX^* + yY^* + aA^* \in \mathfrak{g}^*$. If $x^2 + y^2 = 0$, then, the orbit of f is reduced to the unit set $\{f\}$.

❍ In the case where $x^2 + y^2 \neq 0$, the subalgebra $\mathfrak{b}$=$\mathbb{R}$-span $\{X, Y\}$ is a polarization of f satisfying the Pukanszky condition. Then, let χ_f be the character defined on $B = \exp \mathfrak{b}$ by $\chi_f(\exp U) = e^{-if(U)}$ and $\rho_f = \mathrm{Ind}_B^G \chi_f$. We know that there exists a unique $\theta \in [0, 2\pi[$ such that $\rho = \rho_\theta = \rho_{f_\theta}$ where $f_\theta = \cos\theta X^* + \sin\theta Y^*$. The orbit Ω_θ associated to ρ is:

$$\Omega_\theta = \{sA^* + e^{-t}\cos(t+\theta)X^* + e^{-t}\sin(t+\theta)Y^*, \quad s, t \in \mathbb{R}\}.$$

On the other hand, we have that :

$$\begin{cases} d\rho(A) = -\frac{d}{dt} \\ d\rho(X) = -ie^{t}\cos(\theta + t) \\ d\rho(Y) = -ie^{-t}\sin(\theta + t) \end{cases}$$

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- It follows that the annihilator of ρ is trivial.

Example 2:

❍ Let $\mathfrak{g} := \mathfrak{g}_4 = \mathbb{R}$-span $\{X_1, X_2, X_3, X_4\}$ be a real Lie algebra of dimension 4 defined by the following bracket relations:

$$[X_4, X_1] = X_1, \quad [X_4, X_2] = \sqrt{2}X_2, \quad [X_4, X_3] = \sqrt{3}X_3.$$

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- Let $\ell := X_1^* + X_2^* + X_3^* \in \mathfrak{g}^*$.

Then $\mathfrak{g}(\ell)$ is a two-dimensional subalgebra spanned by $-\sqrt{2}X_1 + X_2$ and $-\sqrt{3}X_1 + X_3$, hence $\mathfrak{g}(\ell)$ is not an ideal of $\mathfrak{g}$.

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- We take a polarization $\mathfrak{b} := \mathbb{R}$-span$\{X_1, X_2, X_3\}$ satisfying the Pukanszky condition at ℓ and its unitary character χ_ℓ.

- Since $G = \exp \mathbb{R}X_4 \ltimes B$, we realize $\pi_\ell = \mathrm{Ind}_B^G \chi_\ell$ on $L^2(\mathbb{R})$ as

$$\pi_\ell(\exp(t_1X_1 + t_2X_2 + t_3X_3))\phi(t) = \chi_\ell((\exp tX_4)^{-1} \exp(t_1X_1 + t_2X_2 + t_3X_3) \exp tX_4)\phi(t)$$
$$= e^{i(e^{-t}t_1 + e^{-\sqrt{2}t}t_2 + e^{-\sqrt{3}t}t_3)}\phi(t),$$

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$$\pi_\ell(\exp t_4X_4)\phi(t) = \phi(t - t_4) \qquad (\phi \in L^2(\mathbb{R})).$$

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❍ We thus obtain the following description of the infinitesimal representation $d\pi_\ell$ as

$$d\pi_\ell(X_1)\phi(t) = ie^{-t}\phi(t), \quad d\pi_\ell(X_2)\phi(t) = ie^{-\sqrt{2}t}\phi(t),$$
$$d\pi_\ell(X_3)\phi(t) = ie^{-\sqrt{3}t}\phi(t), \quad d\pi_\ell(X_4)\phi(t) = -\frac{d}{dt}\phi(t).$$

❍ Namely, $\ker d\pi_\ell = \{0\}$, whereas Ω_ℓ is not open.

Quasi-Frobenius Lie algebras

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Question 3

How to characterize Quasi-Frobenuis exponential solvable Lie algebras? (by means of primitive ideals of $\mathcal{U}(\mathfrak{g})$, for instance).

The Zariski closure conjecture: A new approach

- Let

$$\mathcal{S} : \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{m-1} \subset \mathfrak{g}_m = \mathfrak{g}$$

be a composition series of ideals of $\mathfrak{g}$ which passes the nilpotent radical $\mathfrak{n}$ of $\mathfrak{g}$, say $\mathfrak{g}_{j_0} = \mathfrak{n}$. Hence, $1 \leq \dim(\mathfrak{g}_j/\mathfrak{g}_{j-1}) \leq 2$.

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- At the passage from $\mathfrak{g}_{j-1}$ to $\mathfrak{g}_j$, how to decide if there is a new comer $A_j \in \mathcal{U}(\mathfrak{g}_j)$, which is a Casimir element? There exists a non-empty G-invariant Zariski open set $\mathcal{Z}$ of $\mathfrak{g}^*$ such that $\pi_\ell(A_j)$ is a scalar operator for $\ell \in \mathcal{Z}$.

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❍ We write $\pi_\ell(A_j) = \varphi_{A_j}(\ell)I$, the Casimir function on $\mathcal{Z}$ corresponding to the Casimir element A_j. Then obviously $A_j - \varphi_{A_j}(\ell)$ turns out to be a generator of the primitive ideal $\ker(\pi_\ell)$.

Primitive ideals: The general case

Theorem 4: Bak-Fujiwara, 2026

Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group. Let $\pi = \pi_\ell \in \widehat{G}$ and let $\Omega(\pi)$ be the coadjoint orbit of G corresponding to π. Then there exists an index set $\mathcal{J}_0 \subset \{1, ..., n\}$ such that:

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2. The polynomial $\sigma^{-1}(A_j)$ of $S(\mathfrak{g})$ vanishes on the orbit $\Omega(\pi)$ for any $j \in \mathcal{J}_0$.
3. If $\mathscr{A}$ designates the family of generators of $\ker \pi$ obtained in (1), then $I(\Omega(\pi))$ is generated by the family $\sigma^{-1}(\mathscr{A})$.

Proof of the Zariski Closure Conjecture

Corollary 1

The Zariski Closure Conjecture holds for exponential solvable Lie groups.

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Corollary 2

An exponential Lie algebra $Log(G)$ is quasi-Frobenius if and only if, $\hat{G}$ contains a representation of trivial annihilator.

The setting of solvable Lie algebras unsolved

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- Moreover, the Diximier map $\mathcal{I}$ induces a bijection from the orbit space $\mathcal{G}\backslash\mathfrak{s}^*$ onto $\mathrm{Prim}(\mathcal{U}(\mathfrak{s}))$.

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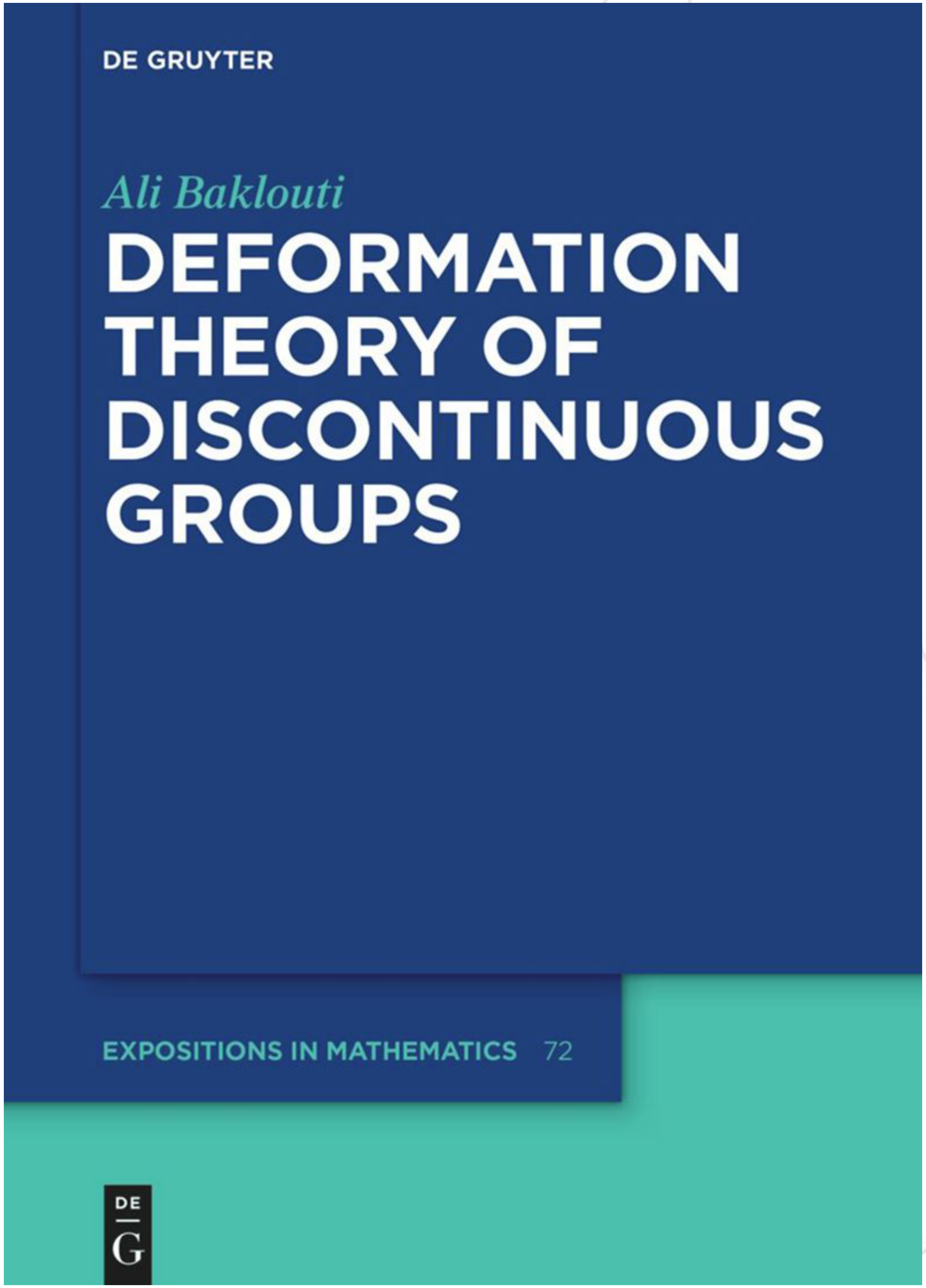
- The process of dequantization consists in how to realize J_I, the associate graduate $Gr(I)$ does not help much for the moment, but at least we know that J_I is the ideal generated by $\sigma^{-1}(\mathscr{A})$ (cf. Theorem 4) in the exponential setting.



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Thank you for your attention

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