Outline

Doug Ulmer

Unlikely intersections on elliptic surfaces (w/ Urzúa)

(Notes, not slides)

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1. Context and basic question

$/\mathbb{C}$

$\mathcal{C}$ = smooth projective curve of genus g

$\pi : \mathcal{E} \to \mathcal{C}$ Jacobian elliptic surface with 0-section O (picture) (Can specify with a Weierstrass equation.)

Generic fiber is an elliptic curve E over $K = \mathbb{C}(\mathcal{C})$. Sections from a group iso to $E(K)$.

P_1, P_2 sections. Urzua: "What do the intersection between P_1 and P_2 look like?"

Standardize to O and P a non-torsion section

$nP.O \sim Cn^2$ ($C =$ height of P)

How many of these intersections are transverse?

Expect “usually” since a tangency is an unlikely intersection in $\mathbb{P}T_{\mathcal{E}}$.

2. Main Results

$$T := \bigcup_{n>0} \{t \in \mathcal{C} \mid nP \text{ tangent to } O \text{ at } t\}$$

(Here we identify O and $\mathcal{C}$.)

1. $|T|$ is finite
2. In fact, $|T| \leq 2g - 2 - d + \delta$ where $\delta =$ number of singular fibers of π and $d = \deg O^*(\Omega^1_{\mathcal{E}/\mathcal{C}}) =$ "height" of $\mathcal{E}$. ($p_g(\mathcal{E}) = d + g - 1$ if $d > 0$).
3. Over suitable moduli spaces of pairs $(\mathcal{E}, P)$, a very general point has no tangencies
4. There are examples of $(\mathcal{E}, P)$ (of every degree d) defined over $\mathbb{Q}$ with no tangencies
5. The bound on $|T|$ is sharp.

Finiteness is not so hard: Over smooth locus of π, in local coords nP and O are analytic, so they meet at isolated points (otherwise they would coincide). Have to rule out an accumulation of intersection points near bad locus of π. Use Kodaira's description of $\mathcal{E}$ near bad fibers to rule this out. Comes down to knowing the possible growth rates of an analytic function.

Specifically, near a point of I_1 (nodal) reduction, an accumulation woud lead to an analytic function f on the disc Δ and a sequence tending to zero such that

$$f'(z_i) = \frac{f(z_i)}{z_i} \frac{\log|f(z_i)|}{\log|z_i|}$$

and this leads to a contradiction to f being holomorphic.

3. Applications

(i) Geography

For surfaces of general type, distribution of K_X^2 is of interest.

Famously it has no strictly decreasing sequences for KSBA stable surfaces. (Kollar, Shepherd-Barron, Alekseev)

For smooth X, $K_X^2 \leq 12(1+p_g)$ by Noether. We show that: for every $g \geq 0$ and N, there exists a normal projective surface X with

1. $p_g(X) = g$
2. X has exactly one singular point, which is log-terminal
3. K_X is $\mathbb{Q}$-Cartier and ample
4. $K_X^2 > N$.

Idea: Start with $\pi : \mathcal{E} \to \mathbb{P}^1$ of height $d = g + 1$ and P such that for all n, nP meets O transversally in $d(n^2 - 1)$ points. (These are constructed in the proof of point 4 above.)

Blow up all but one of the points of intersection: $Y \to \mathcal{E}$, and let C_0, C_n and $\tilde{F}$ be the strict transforms of O, nP, and a fiber of π.

Check Artin's criterion to contract $C_0 \cup C_n$. This is X with the contraction being the singular point x. $p_g(X) = g$ (birational invariant).

x is a cyclic quotient singularity, so X is $\mathbb{Q}$-Gorenstein and K_X is $\mathbb{Q}$-Cartier.

Log-terminality of x, the formula $K_X^2 \sim dn^2$ as $n \to \infty$, and ampleness of K_X all follow from calculations of intersection numbers on Y. See the paper for details.

(ii) Height bounds for integral points

Take $\pi : \mathcal{E} \to \mathcal{C}$ with general fiber E over $K = \mathbb{C}(\mathcal{C})$. There is a canonical height function

$$\hat{h} : E(K) \to \mathbb{Q}$$

Let $S \subset \mathcal{C}$ be a finite set of points, and choose an "S-integral" model for E:

$$y^2 = x^3 + Ax + B$$

where A and B are regular outside S. Let S' be $S \cup$ the set of points of bad reduction of this model.

Conjectures of Lang suggest that the set of S-integral points (solutions with x, y regular outside S) should be bounded. Hindry-Silverman proved a strong form of this, namely a height bound (Inv. '88). We improve this slightly:

If $P = (x, y)$ is a non-torsion S-integral point on E, then

$$\hat{h}(P) \leq 4g - 4 + 2|S'|.$$

The height is an intersection number

$$-(P - O + D_P).(P - O)$$

which is essentially $2(P.O) + 2d$. The proof of our main theorem gives good control on $P.O$ and this yields the desired estimate.

We also give examples where the estimate is sharp.

4. Change of perspective and constant case

Let $\mathcal{C}^0$ be the open set where π has smooth fibers.

Let $\mathcal{E}[n]$ be the union of all n-torsion points in every fiber. This is quasi-finite over $\mathcal{C}$ of degree n^2, (in fact finite étale over $\mathcal{C}^0$) and usually not finite over $\mathcal{C}$.

nP is tangent to O over t if and only if P is tangent to $\mathcal{E}[n]$ over t. (Multiplication by n is étale on $\mathcal{E}^{sm} = \mathcal{E}$ minus the singular points in the bad fibers.)

So we redefine

$$T_{tor} := \cup_{n>0} \{t \in \mathcal{C} \mid P \text{ is tangent to } \mathcal{E}[n] \text{ over } t\}$$

Consider the constant case $\pi : \mathcal{E} = E_0 \times \mathcal{C} \to \mathcal{C}$. (picture)

Sections are graphs of morphisms: $P = Gr(f_P)$ where $f_P : \mathcal{C} \to E_0$.

Torsion sections $\leftrightarrow$ constant maps with image a torsion point of E_0.

There are other constant sections.

Define

$$T_{const} := \cup_{n>0} \{t \in \mathcal{C} \mid P \text{ is tangent to a constant section over } t\}$$

Obviously, $T_{tor} \subset T_{const}$.

$$I(P,t) := \text{int mult of } P \text{ and the constant section } P(t)\times\mathcal{C}$$

$I(P,t)\geq 1$ and ≥ 2 iff P is tangent to a constant section at t.

Let ω be a non-zero differential on E_0 and set

$$J(P,t) := \operatorname{ord}_t P^*(\omega)$$

Then one has the key formula

$$J(P,t) = e_t(f_P) - 1 = I(P,t) - 1$$

Finally,

$$|T_{tor}| \leq |T_{const}| \leq \sum_t (I(P,t)-1) = \sum_t J(P,t) = 2g-2.$$

This is the theorem, since $d=\delta=0$ in the constant case. To generalize, need a local trivialization of $\mathcal{E}\to\mathcal{C}$ and a version of the key formula.

The Betti foliation

Observation: A family of elliptic curves has a canonical "local parallel transport". Know how to move torsion sections, and they are dense. (picture)

To make more precise, use Kodaira:

$$\begin{tikzcd}
\frac{\mathbb{C}}{\mathbb{Z}+\mathbb{Z}\tau(z)} \arrow[r, hook] \arrow[d] & \frac{\Delta\times\mathbb{C}}{\mathbb{Z}^2} \arrow[r, hook] \arrow[d] & \mathcal{F} \arrow[r, hook] \arrow[d] & \mathcal{E}^0 \arrow[r, hook] \arrow[d] & \mathcal{E} \arrow[d, "\pi"] \\
\{z\} \arrow[r, hook] & \Delta \arrow[r, equal] & \Delta \arrow[r, hook] & \mathcal{C}^0 \arrow[r, hook] & \mathcal{C}
\end{tikzcd}$$

and there is a holomorphic (period) function $\tau : \Delta \to \mathfrak{h}$ such that $\pi^{-1}(z) = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau(z)$.

Trivialize the family of real Lie groups $\mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau(z) \cong (\mathbb{R}/\mathbb{Z})^2$ as follows:

$$(r, s) \mapsto r + s\tau(z) \pmod{\mathbb{Z} + \mathbb{Z}\tau(z)}$$

NB: This is not holomorphic: if w is the coordinate on $\mathbb{C}$, then $[w]$ corresponds to $s = (\operatorname{Im} w)/(\operatorname{Im} \tau) = (w - \overline{w})/(\tau - \overline{\tau})$ and $r = w - s\tau$.

A local section is torsion iff it is the image of $(r, s) \in (\mathbb{Q}/\mathbb{Z})^2$.

Globalize to a foliation of $\mathcal{E}^0$ by passing to universal cover $\tilde{\mathcal{C}}^0$ and $\mathcal{F}^0 \to \tilde{\mathcal{C}}^0$, then take quotient by global monodromy $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$.

The “Betti foliation” has these properties:

The torsion multsections are closed leaves, and if j is not constant, they are the only closed leaves. Other leaves are everywhere dense. (A fiber looks like an orbit of Γ on $(\mathbb{R}/\mathbb{Z})^2$.)

A section is a leaf if and only if it is torsion.

Near the bad fibers, one distinguishes between “vanishing leaves “ and “invariant leaves” using the local monodromy group.

Invariant leaves have limits that are in the closure of the torsion points of the bad fiber and each such point is the limit of a unique invariant leaf.

Define

$$T_{Betti} := \{t \in \mathcal{C} \mid P \text{ is tangent to a Betti leaf at } t\}$$

In the constant case, $T_{Betti} = T_{const}$ and in general

$$T_{tor} \subset T_{Betti}$$

6. Local intersection numbers

Define

$$I(P,t) := \text{intersection number of } P \text{ and the Betti leaf passing through } P(t)$$

(If $t \in \mathcal{C} \setminus \mathcal{C}^0$ and if $P(t)$ is not in the closure of the torsion points, we set $I = 0$.)

Then $I(P,t) \geq 1$ if $t \in \mathcal{C}^0$, and in general it is ≥ 2 iff P is tangent to the Betti foliation over t iff $t \in T_{Betti}$.

7. A real analytic 1-form and local indices

Define (in coordinates used for the local trivialization pp.13-14)

$$\eta := dw - \frac{\operatorname{Im} w}{\operatorname{Im} \tau} \, d\tau.$$

Note that $\ker \eta_{| T^0_{\mathcal{E},x}}$ is the tangent space to the leaf of the Betti foliation through x.

Calculating the action of monodromy, one finds that η descends to a *real analytic* section over $\mathcal{E}^0$ of

$$\Omega^1_{\mathcal{E}^0} \otimes \left(\Omega^1_{\mathcal{E}^0/\mathcal{C}^0} \right)^{-1}$$

Now given a non-torsion section P, define

$$\eta_P := P^*(\eta)$$

which is a real-analytic section of $\Omega^1_{\mathcal{C}} \otimes \omega^{-1}$ over $\mathcal{C}^0$. Here $\omega = O^*\Omega^1_{\mathcal{E}/\mathcal{C}}$ and $d = \deg(\omega)$.

We define local indices (analogous to order of vanishing) for a smooth section of a line bundle using winding numbers. If s_0 is a generating section on a nbhd of x, and if $s = fs_0$ for some smooth f, then the index is $(1/2\pi i) \int d \log f$. *Note that s does not need to be defined at x.* When s is meromorphic, this is the usual ord.

Then for all $t \in \mathcal{C}$, we define

$$J(P, t) := \text{index of } \eta_P \text{ at } t$$

The degree of a line bundle is a topological invariant! In particular, the sum of the indices of a smooth section defined and non-vanishing off a finite set is the sum of the indices at the missing points. (The indices where the section is defined and non-zero are zero.)

In our context:

$$\begin{aligned}\sum_{t\in\mathcal{C}} J(P,t) &= \deg \Omega^1_{\mathcal{C}} \otimes \omega^{-1} \\ &= 2g-2-d\end{aligned}$$

8. Key equality and end of proof

Finally, one checks (cases according to the reduction type) that for every $t \in \mathcal{C}$,

$$J(P, t) = I(P, t) - 1.$$

Thus we have

$$2g - 2 - d = \sum_t J(P, t) = \sum_t (I(P, T) - 1) \geq |T_{Betti}| - \delta$$

and

$$|T_{tor}| \leq |T_{Betti}| \leq 2g - 2 - d + \delta.$$

Examples show that this bound is sharp! (Start with $\mathcal{E} \to \mathbb{P}^1$ with degree 1 and 3 singular fibers. Then pull back to $\mathcal{C} \to \mathbb{P}^1$ where the tangencies are controlled by the ramification of the map.)

Questions:

1) Char p? Finteness surely holds if we restrict to $p \not| n$ but I have no idea how to produce a bound.

2) Other locally trivial situations, such as abelian varieties.

3) EDS and primality

Elliptic divisibilty sequences

EDS: $n \mapsto a_n$ such that $m|n \Rightarrow a_m|a_n$.

E.g., $a_n = 2^n - 1$ or $c^n - 1$. $p|a_n$ iff c is n-torsion in $\mathbb{G}_m$.

E an elliptic curve over $\mathbb{Q}$ and $P \in E(\mathbb{Q})$ not torsion.
$nP = (a_n/c_n^2, b_n/c_n^3)$ and $n \mapsto c_n$ is a divisibility sequence, an EDS.
$p|c_n$ iff P is n-torsion.

Divisors $P.O$ are an analogue. Get polynomials if $\mathcal{C} = \mathbb{P}^1$ and we choose suitable coordinates.

Our result say they are usually square-free. (Finiteness, where Vojta conjectures suggest $o(n^2)$.

A potentially intersting problem is to do this over $\mathbb{Q}(t)$ getting rational polynomials which may (?) often be irreducible.

Families

The moduli of triples (E, P, ω) is $k[a_2, a_3, a_4]$.

Given $\mathcal{C}$, choose globally generated L. Then sections $L^2 \oplus L^3 \oplus L^4$ give rise to $\mathcal{E} \to \mathcal{C}$ and P.

Get families parameterized by open sets in RR spaces, and this is the context in which one can show that for very general points, there are no tangencies.