Profinite group cohomology

chap | Topological groups
Profinite groups
Eq.

chap | Profinite cohomology.
Functoriality.

chap | Spectral - sequences
Hochschild - Serre Sp. seq.

References

① Matsushima : chap I groups Top.
② Shatz : Profinite groups.
③ Weibel : Introduct° to Hom Alg.
④ Serre : Galois Cohomology.

chap

§. Topological groups.

Def $G \in \underline{\text{TopGrp}}$:

$(G, e) \in \text{Grp}$; $(G, \mathcal{O}_G) \in \text{Top}$.

$\varphi : G \times G \longrightarrow G$ continuous.
$(x , y) \longmapsto xy^{-1}$

Prop $G \in \text{TopGrp}$, $g \in G$.

$L_g : G \longrightarrow G$; $R_g : G \longrightarrow G$ continuous.
$x \longmapsto g \cdot x$; $x \longmapsto x \cdot g$

Prop A open $< G$, $B < G$ $\Rightarrow A.B$, A^{-1} are open.

Def $G \supset U \in \mathcal{V}(g)$ if. U open and $g \in U$.

Prop G Hausdorff $\Longleftrightarrow \bigcap_{U \in \mathcal{V}(e)} U = \{e\}$.

Prop - Def $G \in \mathrm{GrpTop}$, $H < G$.

- $G/H \in \mathrm{Top}$ with O open in G/H iff $\pi^{-1}(O)$ open in G $\quad (\pi : G \longrightarrow G/H)$.

- $H \triangleleft G \Rightarrow G/H \in \mathrm{GrpTop}$

Def $X \in \mathrm{finTop}$ if $\left| \begin{array}{l} X \text{ finite} \\ X \in \mathrm{Top}. \end{array} \right.$

Rem $X \in \mathrm{finTop}$.

① Topologies on $X \longleftrightarrow$ preorders on X.
(reflex. + transitivity)

② X compact.

③ X Hausdorff $\Rightarrow$ X discrete
i.e each point is open.

§. Profinite groups

Def $(I, \leq)$ partially ordered index set.

$(G_i)_{i \in I}$ family of grps.

- $i \leq j \Rightarrow$ $\left| \begin{array}{l} \exists k \in I \mid i \leq k,\ j \leq k. \\ \exists f_j^i : G_j \longrightarrow G_i. \quad (*) \end{array} \right.$

- consistency. $i \leq j \leq k$.

$$\Longrightarrow \quad \begin{tikzcd} G_i & G_j \arrow[l, "f_j^i"'] & G_k \arrow[l, "f_k^j"'] \arrow[ll, bend left, "f_k^i"] \end{tikzcd}$$

① $(G_i, f_j^i)_I$ projective mapping family.

② Instead of $(*)$ $i \leq j \Rightarrow \exists f_i^j : G_i \longrightarrow G_j$

$(G_i, f_i^j)_I$ direct mapping family

Def

① Projective limit of $(G_i, f_j^i)_I$:

* $\lim G_i \in \underline{\underline{Grp}}$.

* UMP.

$$\begin{tikzcd}
 & & G_i \\
X \arrow[r, dashed] \arrow[urr, bend left, "g_i"] \arrow[drr, bend right, "g_j"'] & \lim G_i \arrow[ur, "f_i"] \arrow[dr, "f_j"'] & \\
 & & G_j \arrow[uu, "f_j^i"']
\end{tikzcd} \qquad i \leq j$$

② injective limits of $(G_i, f_i^j)_I$.

* $\operatorname{colim} G_i \in Grp$

* UMP.

$$i \leq j \qquad \begin{tikzcd}
G_i \arrow[dd, "f_i^j"'] \arrow[dr, "f_i"] \arrow[drr, bend left, "g_i"] & & \\
 & \operatorname{colim} G_i \arrow[r, dashed] & X \\
G_j \arrow[ur, "f_j"'] \arrow[urr, bend right, "g_j"'] & &
\end{tikzcd}$$

Thm lim and colim exists in Set, Grp...

$$* \lim G_i = \left\{ (x_i)_i \in \prod_I G_i \;\middle|\; i \leq j \Rightarrow f_j^i(x_j) = x_i \right\}.$$

$$* \operatorname{colim} G_i = \bigsqcup G_i /\sim.$$

$$G_i \ni x \sim y \in G_j \iff \left| \begin{array}{l} \exists k \in I \mid i \leq k,\ j \leq k \\ f_i^k(x) = f_j^k(y). \end{array} \right.$$

Ex $I = \mathbb{N}$; $\leq \, = \, |$; $G_n = \mathbb{Z}/n\mathbb{Z}$.

$$n \mid m \Rightarrow f_n^m : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z}.$$
$$\bar{x}^n \longmapsto \frac{m}{n}\bar{x}^m.$$

$$\begin{tikzcd}
\bar{x}^n \arrow[rrr, mapsto, bend left] & & & {[\frac{x}{n}]} \\
\mathbb{Z}/n\mathbb{Z} \arrow[dd, "f_n^m"', "n \mid m"'] \arrow[dr, "f_n"] \arrow[rrr, bend left, "\alpha_n"] & & & \\
& \operatorname{colim} \mathbb{Z}/n\mathbb{Z} \arrow[rr, dashed, "\varphi"] & & \mathbb{Q}/\mathbb{Z} \\
\mathbb{Z}/m\mathbb{Z} \arrow[ur, "f_m"'] \arrow[urrr, bend right, "\alpha_m"] & & & {[\frac{x}{m}]} \\
\bar{x}^m \arrow[urrr, mapsto, bend right] & & &
\end{tikzcd}$$

Show that φ bijective.

Def G is a profinite group.

$G = \lim G_i$, G_i finite.

Rem

① group finite $\Rightarrow$
- $\in$ GrpTop.
- compact.
- Hausdorff.

② G profinite $\Longrightarrow$ G compact Hausdorff.

THM

① G profinite group

$\Updownarrow$

② G compact, Hausdorff and its open normal subgroups form a fund. syst of neigh. of e.

$\Updownarrow$

③ G compact, totally disconnected. Hausdorff.

§. $\mathcal{C}_G$.

Def G profinite group.

① A <u>discrete G-module</u> : $(A \in \mathcal{C}_G)$.

- A discrete Ab grp.
- $\exists * : G \times A \longrightarrow A$ continuous.
  $(g, a) \longmapsto g*a$

② Morphism in $\mathcal{C}_G$.

$$\begin{tikzcd} G \times A \arrow[r] \arrow[d, "id_G \times f"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, "f"] \\ G \times B \arrow[r] & B \end{tikzcd}$$

i.e $f(g*a) = g*f(a)$

Lemma $A \in \mathcal{C}_G$, $a \in A$

① $G_a = \{g \in G \mid g*a = a\}$ open $< G$.

② $a \in A^{G_a} = \{x \in A \mid \forall g \in G_a,\ g*x = x\}$.

③ $A = \bigcup\limits_{U \text{ open} < G} A^U$.

Prop $\mathcal{C}_G$ abelian category.

Proof $\ker f$, $\operatorname{im} f$, $\operatorname{coker} f \in \mathcal{C}_G$.

Def $A \in \mathcal{C}_G$. $G^0 = \{e\}$; $G^n = G \times \cdots \times G$

- $C^n(G, A) = \{ f : G^n \longrightarrow A \mid \text{continuous} \}$.
- $C^0(G, A) = \{ f : \{e\} \longrightarrow A \} \simeq A$.

Prop $(C^n(G,A), +) \in \text{Abgp}$.

Prop $C^n(G, -) : \mathcal{C}_G \longrightarrow \text{Abgp}$.

$$\begin{array}{ccc} A & & C^n(G,A) \ni f \\ \varphi \big\downarrow & \longmapsto & \big\downarrow \varphi^* \qquad \big\downarrow \\ B & & C^n(G,B) \ni \varphi^*(f) = \varphi \circ f \end{array}$$

exact functor. i.e.

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0 \quad \text{ES}.$$

$$0 \longrightarrow C^n(G,A) \xrightarrow{\varphi^*} C^n(G,B) \xrightarrow{\psi^*} C^n(G,C) \longrightarrow 0 \quad \text{ES}$$

Proof.

① $\varphi^*(f)=0 \Rightarrow \varphi(f(g_1,\dots,g_n))=0 \quad \forall (g_i)$

$\Rightarrow f(g_1,\dots,g_n)=0_A \quad \forall (g_i)$

$(\varphi$ inject.$)$

$\Rightarrow f=0$. i.e φ^* inject.

② • $\psi^*\circ\varphi^*(f)=\psi\circ\varphi\circ f=0\circ f=0 \quad (\psi\circ\varphi=0)$

i.e $\operatorname{im}\varphi^*\subset\ker\psi^*$.

• $f\in\ker\psi^* \Rightarrow \psi^*(f)=0$.

$\Rightarrow \psi\circ f(g_1,\dots,g_n)=0 \quad \forall (g_i)$

$\Rightarrow f(g_1,\dots,g_n)\in\ker\psi=\operatorname{im}\varphi$.

$\Rightarrow \exists a\in A \mid f(g_1,\dots,g_n)=\varphi(a)$

setth: $G^n\longrightarrow A$

$(g_1,\dots,g_n)\mapsto a$.

$\Rightarrow f=\varphi\circ h=\varphi^*(h) \Rightarrow f\in\operatorname{im}\varphi^*$

③ $C^n(G,B)\xrightarrow{\psi^*}C^n(G,C)$

$f\in C^n(G,C)$

ψ surjective $\Rightarrow \exists$ section $s: C\longrightarrow B \mid \psi\circ s=\mathrm{id}_C$.

$\Rightarrow G^n\xrightarrow{f}C\xrightarrow{s}B$ continuous (discrete)

and $\psi^*(s\circ f)=\psi\circ s\circ f=\mathrm{id}_C\circ f=f$. i.e ψ^* surjective chap

§. Profinite cohom

Prop-Def $(C^n(G,A), d^n) \in \mathrm{coch}^+(A\mathcal{S})$ with

- $d^0 : C^0(G,A) \longrightarrow C^1(G,A)$

$$f \longmapsto d^0 f : G \longrightarrow A$$
$$g \mapsto d^0 f(g)$$

- $\boxed{(d^0 f)(g) = g * f(e) - f(e)}$

$d^1 : C^1(G,A) \longrightarrow C^2(G,A)$

$$f \longmapsto d^1 f : G^2 \longrightarrow A$$
$$(g_1, g_2) \mapsto d^1 f(g_1, g_2)$$

$\boxed{d^1 f(g_1, g_2) = g_1 * f(g_2) - f(g_1 g_2) + f(g_1)}$

$d^n : C^n(G,A) \longrightarrow C^{n+1}(G,A)$

$$f \longmapsto d^n f : G^{n+1} \longrightarrow A.$$
$$(g_1, \dots, g_{n+1}) \longmapsto d^n f(g_1, \dots, g_{n+1})$$

$$d^n f(g_1, \dots, g_{n+1}) =$$
$$g_1 * f(g_2, \dots, g_{n+1}) + \sum_{1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1})$$
$$+ (-1)^{n+1} f(g_1, \dots, g_n).$$

Proof show $d^n \circ d^{n-1} = 0$ (i.e. $\operatorname{im} d^{n-1} \subset \ker d^n$)

for instance

$$d^1 \circ d^0 (f)(g_1, g_2)$$

$$= g_1 * d^0(f)(g_2) - d^0 f(g_1 g_2) + d^0 f(g_1)$$

$$= g_1 * [g_2 * f(e) - f(e)] - [(g_1 g_2) * f(e) - f(e)]$$
$$+ [g_1 * f(e) - f(e)]$$

$= 0$ since $g_1 * g_2 * f(e) = (g_1 g_2) * f(e)$

Def n-cohom. group of G with coeff in A

$$H^n(G, A) = \frac{\ker(d^n : C^n(G,A) \longrightarrow C^{n+1}(G,A))}{\operatorname{im}(d^{n-1} : C^{n-1}(G,A) \longrightarrow C^n(G,A))}$$

Prop $H^0(G,A) = A^G$.

Proof $0 \to C^0(G,A) \xrightarrow{d^0} C^1(G,A) \to \dots$

. $H^0(G,A) = \dfrac{\ker d^0}{\operatorname{im} 0} = \ker d^0$.

. $\ker d^0 = \{ f \in C^0(G,A) \mid d^0 f = 0 \}$.

$(d^0 f)(g) = 0 \ \forall g$.

$f(e) * g - f(e) = 0 \ \forall g$.

$C^0(G,A) \cong A$.
$f(e) \mapsto a \in A$. $\Rightarrow \ker d^0 = \{ a \in A \mid \forall g \in G,\ a * g = g \}$

$= A^G$.

Prop $H^1(G,A) =$ group of $[f] = f + \operatorname{im} d^0$.
$f : G \to A$ cont. crossed-morph.

Proof $C^0(G,A) \xrightarrow{d^0} C^1(G,A) \xrightarrow{d^1} C^2(G,A) \to \dots$

. $H^1(G,A) = \dfrac{\ker d^1}{\operatorname{im} d^0}$.

$\ker d^1 = \{ f \in C^1(G, A) \mid d^1 f = 0 \}$.

$$(d^1 f)(g_1, g_2) = 0 \quad , \forall (g_1, g_2)$$

$$g_1 * f(g_2) - f(g_1 g_2) + f(g_1) = 0 .$$

$$f(g_1 g_2) = g_1 * f(g_2) + f(g_1)$$

(i.e f crossed-morph).

Rem $G \times A \longrightarrow A$. (trivial action)

$(g, a) \longmapsto g * a = a$

$\Rightarrow f(g_1 g_2) = g_1 * f(g_2) + f(g_1)$

$= f(g_2) + f(g_1)$

(i.e f usual morphism).

Prop $H^2(G, A)$ = group of classes of continuous factor systems from G to A.

Rem interpretation of higher cohom. grps.
i.e $H^n(G, A)$, $n \geq 3$. ?

§. Functoriality.

Def $A \in \mathcal{C}_G$; $\phi : G' \longrightarrow G$. // continuous

$A' \in \mathcal{C}_{G'}$; $\psi : A \longrightarrow A'$

$\langle \phi, \psi \rangle$ is a compatible pair if.

$\forall\, a \in A,\ g' \in G'$:

$$\begin{tikzcd} G \times \overset{\overset{a}{\cap}}{A} \arrow[r, "*"] \arrow[d, phantom] & A \arrow[d, "\psi"] \\ \underset{\underset{g'}{\cup}}{G'} \times A' \arrow[u, "\phi"] \arrow[r, "*"'] & A' \end{tikzcd}$$

i.e $\phi(g') * a = g' * \psi(a)$

Prop $\langle \phi, \psi \rangle$ compatible pair

$\Longrightarrow \alpha^n : C^n(G, A) \longrightarrow C^n(G', A')$

$f \longmapsto$

$$\begin{tikzcd} G'^n \arrow[r, "\alpha^n(f)"] \arrow[d, "\phi^n"'] & A' \\ G^n \arrow[r, "f"'] & A \arrow[u, "\psi"'] \end{tikzcd}$$

define $\alpha : C^*(G, A) \longrightarrow C^*(G', A')$

and $\alpha^* : H^*(G, A) \longrightarrow H^*(G', A')$

Def restriction.

$$\left| \begin{array}{l} S < G \\ A \in \mathcal{C}_G \end{array} \right. \Longrightarrow \left| \begin{array}{l} A \in \mathcal{C}_S \\ \langle i : S \hookrightarrow G ;\ id_A : A \to A \rangle \\ \text{compatible pair.} \end{array} \right.$$

$$\Longrightarrow \left| \ res : H^*(G, A) \longrightarrow H^*(S, A) \right.$$

Prop S closed $\triangleleft G$; $\pi : G \to G/S$ can. comorp.

① $A \in \mathcal{C}_G \Longrightarrow A^S \in \mathcal{C}_{G/S}$.

② $\langle i : A^S \hookrightarrow A ,\ \pi : G \to G/S \rangle$ compatible pair

Proof $G/S \times A^S \longrightarrow A^S$

$(\bar{g}, a) \longmapsto \bar{g} * a = g \cdot a$

$g \cdot a \in A^S$ since $s \cdot (g \cdot a) = (sg) \cdot a = (gs') \cdot a = g \cdot (s' \cdot a) = g \cdot a$

$(S \triangleleft G$ i.e. $gS = Sg)$ $\quad (a \in A^S)$

Def inflation with $\langle i, \pi \rangle$.

$$\inf : H^*(G/S, A^S) \longrightarrow H^*(G, A)$$

THM Assume:

① projective syst $(G_\alpha)_\alpha \in$ ProGrps.

$$\begin{tikzcd} & G_\alpha \\ \lim G_\alpha \arrow[ur, "\phi_\alpha"] \arrow[r, "\phi_\beta"'] & G_\beta \arrow[u, "\phi^\alpha_\beta"'] \end{tikzcd} \qquad \alpha \le \beta.$$

② direct system $(A_\alpha)_\alpha \in \mathcal{C}_{G_\alpha}$.

$$\begin{tikzcd} A_\alpha \arrow[dr, "\psi_\alpha"] \arrow[d, "\psi^\beta_\alpha"'] & \\ A_\beta \arrow[r, "\psi_\beta"'] & \operatorname{colim} A_\alpha \end{tikzcd} \qquad \alpha \le \beta$$

③
$$\begin{tikzcd} G_\alpha & \times & A_\alpha \arrow[d, "\psi^\beta_\alpha"] \arrow[r, "*"] & A_\alpha \arrow[d, "\psi^\beta_\alpha"] \\ G_\beta \arrow[u, "\phi^\alpha_\beta"] & \times & A_\beta \arrow[r, "*"] & A_\beta \end{tikzcd} \qquad \langle \phi^\alpha_\beta, \psi^\beta_\alpha \rangle \text{ compatible}$$

Then $\exists!$ action

ⓐ $\lim G_\alpha \times \operatorname{colim} A_\alpha \xrightarrow{*} \operatorname{colim} A_\alpha$

$$\Big(g = (g_\alpha)_\alpha, \ \underset{\psi_\beta(a_\beta)}{\overset{a}{\shortparallel}} \Big) \longmapsto \psi_\beta(g_\beta * a_\beta).$$

ⓑ
$$\begin{tikzcd} G_\alpha & \times & A_\alpha \arrow[d, "\psi_\alpha"] \arrow[r, "*"] & A_\alpha \arrow[d, "\psi_\alpha"] \\ \lim G_\alpha \arrow[u, "\phi_\alpha"] & \times & \operatorname{colim} A_\alpha \arrow[r, "*"] & \operatorname{colim} A_\alpha \end{tikzcd} \qquad \langle \phi_\alpha, \psi_\alpha \rangle \text{ compatible}$$

(c) $H^*(\lim_\alpha G_\alpha, \operatorname{colim} A_\alpha) = \operatorname{colim}_\alpha H^*(G_\alpha, A_\alpha)$.

<u>Proof</u> (sketch).

* check (c) well-defined action.
* $\langle \varphi_\alpha, \psi_\alpha \rangle$ compatibility in indices.

$$i_\alpha : C^n(G_\alpha, A_\alpha) \longrightarrow C^n(\lim G_\alpha, \operatorname{colim} A_\alpha)$$

*

$$\begin{tikzcd}
C^n(G_\alpha, A_\alpha) \arrow[d] \arrow[dr] \arrow[drr, bend left, "i_\alpha"] & & \\
{} \arrow[r, phantom, "\circlearrowleft"] & \operatorname{colim} C^n(G_\alpha, A_\alpha) \arrow[r, dashed, "\exists f"] & C^n(\lim_\alpha G_\alpha, \operatorname{colim}_\alpha A_\alpha) \\
C^n(G_\beta, A_\beta) \arrow[ur] \arrow[urr, bend right, "i_\beta"'] & &
\end{tikzcd}$$

check f <u>bijective</u>.

Def H closed $< G$.
$A \in \mathcal{C}_H$.
<u>induced G-module</u>

$$M_G^H(A) = \{ \varphi : G \to A \text{ cont} \mid \varphi(hx) = h * \varphi(x), \ \forall h \in H, x \in G \}.$$

Prop

① $M_G^H(A) \in \text{Ab Grp}$.

② $M_G^H(A) \in \mathcal{C}_G$ via the action

$$* : G \times M_G^H(A) \longrightarrow M_G^H(A)$$
$$(g, \varphi) \longmapsto g * \varphi : G \longrightarrow A, \quad x \longmapsto \varphi(gx).$$

③ $M_G^H : \mathcal{C}_H \longrightarrow \mathcal{C}_G$, $\quad A \longmapsto M_G^H(A)$

functor (exact, preserves injectives)

④ $\langle M_G^H(A) \to A,\ \varphi \mapsto \varphi(1) \ ; \ H \hookrightarrow G \rangle$ compatible pair

<u>Hence</u> $\exists$ $H^*(G, M_G^H(A)) \longrightarrow H^*(H, A)$.

Homological Algebra Reminder.

Def $\mathcal{A}$ abelian category.

(1) $I \in \mathcal{A}$ injective if $\mathcal{A}(-, I)$ exact functor

(2) $\mathcal{A}$ has enough injectives if.
$\forall A \in \mathcal{A}$, $\exists I$ injective and injection $A \rightarrowtail I$

(3) Injective resolution of $A \in \mathcal{A}$:
* $(I^\bullet, d^\bullet) \in \mathrm{coch}(\mathcal{A})$ such that
* $0 \to M \to I^0 \to I^1 \to \cdots$ exact.
* I^i injective $\forall i \geq 0$.

Lem $\mathcal{A}$ has enough injectives
$\Rightarrow \forall A \in \mathcal{A}$ has an injective resolut$^=$.

Def $\mathcal{A}, \mathcal{B}$ abelian categories
$F : \mathcal{A} \to \mathcal{B}$ left exact functor
$\mathcal{A}$ has enough injectives.
Right derived functors of F: R^iF $(i \geq 0$.
* $A \in \mathcal{A}$, choose an injective resolut$^=$
$A \to I^\bullet$
* $\boxed{R^iF(A) = H^i(F(I^\bullet))}$

Lemma G profinite group.

① $\mathbb{Z} \in \mathcal{C}_G$ via the trivial action.

$$G \times \mathbb{Z} \xrightarrow{*} \mathbb{Z}$$
$$(g, n) \longmapsto g * n = g.$$

② $\mathcal{C}_G(\mathbb{Z}, A) \cong A^G$.

Proof $\alpha \in \mathcal{C}_G(\mathbb{Z}, A) \Leftrightarrow \left| \begin{array}{l} \alpha(1) \in A. \\ g * \alpha(1) = \alpha(1) \quad \forall g \in G \end{array} \right.$

$\Leftrightarrow \alpha(1) \in A^G.$

Prop (Shapiro lemma)

$$H^*(G, M_G^H(A)) \xrightarrow{\cong} H^*(H, A)$$

Proof

$* = 0$: $H^0(H, A) \cong A^H \cong \mathcal{C}_H(\mathbb{Z}, A)$

$\cong \mathcal{C}_G(\mathbb{Z}, M_G^H(A))$

$\cong (M_G^H(A))^G.$

$\cong H^0(G, M_G^H(A))$

- Choose an injective resolution $A \to I^\bullet$
- $M_G^H(A) \to M_G^H(I^\bullet)$ injective resolution of $M_G^H(A)$ since $M_G^H(-)$ exact and preserves injectives

• $H^*(G, M_G^H(A)) \cong H^*(M_G^H(I^\bullet)^G)$

$\cong H^*((I^\bullet)^H)$ } by the case $*=0$

$= H^*(H, A)$.

Rem $H^*(H,A) \overset{\text{def}}{=} H^*((I^\bullet)^H)$ with.

$A \hookrightarrow I^\bullet$ injective resolut° of A.

Prop H open $< G$, $A \in \mathcal{C}_G$, $[G:H]$ finite.

$\Rightarrow$ $\pi : M_G^H(A) \longrightarrow A$ surjective in $\mathcal{C}_G$

$\varphi \longmapsto \pi(\varphi) = \sum_{\bar{x} \in G/H} x \cdot \varphi(x^{-1})$

Proof

① π is well-defined.

$y \in \bar{x} \Rightarrow x^{-1} y \in H$

$\Rightarrow \exists h \in H \mid x^{-1} = h y^{-1}$

$\Rightarrow \varphi(x^{-1}) = \varphi(h y^{-1}) = h \varphi(y^{-1})$

$\Rightarrow x \cdot \varphi(x^{-1}) = x h \varphi(y^{-1}) = y \varphi(y^{-1})$.

② π surjective.

•

Prop-Def

① $\pi : M_G^H(A) \longrightarrow A$ induces.

and
$$C^*(G, M_G^H(A)) \xrightarrow{\pi^*} C^*(G, A).$$
$$H^*(G, M_G^H(A)) \longrightarrow H^*(G, A).$$

② corestriction : composition

$$\begin{tikzcd} H^*(G,A) \arrow[r, "sh", "\cong"'] \arrow[rr, bend right] & H^*(G, M_G^H(A)) \arrow[r, "\pi^*"] & H^*(G,A) \end{tikzcd}$$

$$\mathrm{cor} = \pi^* \circ sh.$$

Prop H open $< G$; $[G:H]$ finite.

$$\begin{tikzcd} H^*(G,A) \arrow[r, "Res"] \arrow[rr, bend right] & H^*(H,A) \arrow[r, "Cor"] & H^*(G,A) \end{tikzcd}$$

$\mathrm{Cor} \circ \mathrm{Res}$

is a multiplication by $[G:H]$.

Proof $\mathrm{cor} \circ \mathrm{Res}$ comes from:

$$A \longrightarrow M_G^H(A) \longrightarrow A.$$
$$a \longmapsto \left(\varphi_a : \begin{array}{l} G \to A \\ x \mapsto x.a \end{array}\right) \longmapsto \sum_{\bar{x} \in G/H} a.\varphi_a(x^{-1})$$

and
$$\sum_{\bar{x} \in G/H} x.\varphi_a(x^{-1}) = \sum_{\bar{x} \in G/H} x.x^{-1}.a$$
$$= \sum_{\bar{x} \in G/H} a = |G/H|.a = [G:H].a.$$

chap Spectral sequences

Def First-quadrant cohomological spec. seq

① collection of $E_r^{p,q}$ groups, $p,q \in \mathbb{N}$.
$r \geq r_0$ fixed.

② collection of differentials

$$\cdots \longrightarrow E_r^{p-r,q+r-1} \xrightarrow{d_r^{p-r,q+r-1}} E_r^{p,q} \xrightarrow{d^{p,q}} E^{p+r,q-r+1} \longrightarrow \cdots$$

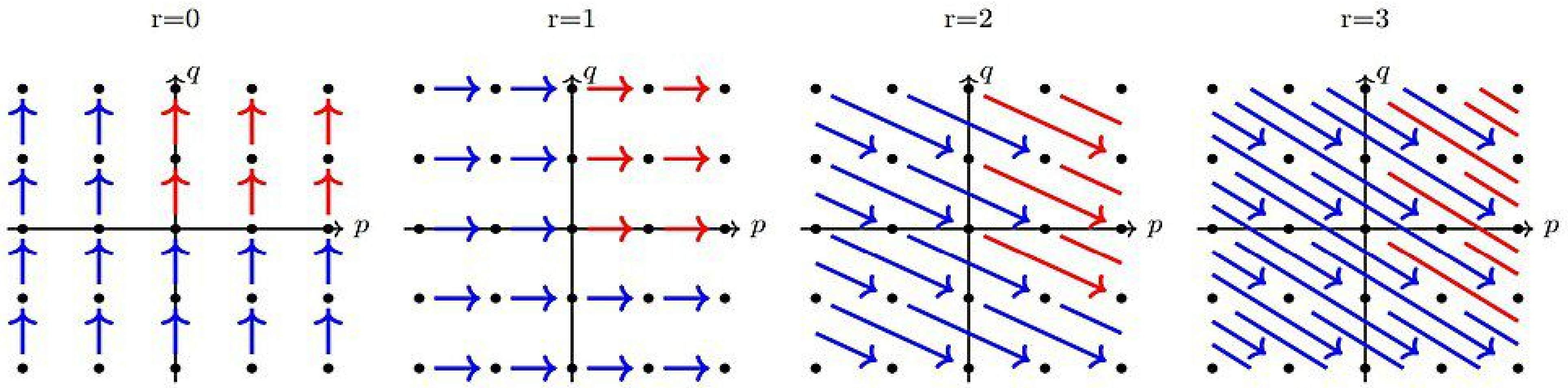
such that $d_r^{\cdot} \circ d_r^{\cdot} = 0$.

③ $E_{r+1}^{p,q}$ = cohomology at $E_r^{p,q}$ with respect to the above diff.

⊛ $E_r = \left(E_r^{p,q}, d_r^{p,q}\right)_{p,q}$: r^{th}-page.

⊛ the spectral sequence E converge if (stationary).
$\exists s \mid E_s = E_{s+1} = \cdots$

such page E_s is denoted E_∞.


r=0
r=1
r=2
r=3
q
p

Def A cochain $(C^i, \delta^i)_i$ is filtered if $\forall i$, C^i has a diff-preserving filtration.

i.e
$$F_0 C^i \subset F_1 C^i \subset \cdots \subset F_p C^i \subset F_{p+1} C^i \subset \cdots$$
&
$$\bigcup_p F_p C^i = C^i \,;\quad \bigcap_p F_p C^i = 0 \,;\quad \delta^i(F_p C^i) \subset F_p C^{i+1}.$$

Rem
① A filtration on (C^i, δ^i) induces a filtration $F_p H^*(C)$ on cohomology grps.

② Associated graded pieces
$$Gr_p H^n(C) = F_p H^n(C) \Big/ F_{p+1} H^n(C)$$

Def Spect. seq E converges to $H^*(C)$ $\left(E_\infty^{p,q} \Longrightarrow H^{p+q}(C) \right)$ if.

E converge. (Stationary at E_s)

$$E_s^{p,q} = E_\infty^{p,q} = Gr_p H^{p+q}(C).$$

Rem Convergent spect. seq does not compute $H^*(C)$, rather a filtration of it.

Def Double-complex

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & C^{p-1,q+1} \arrow[r] \arrow[u] & C^{p,q+1} \arrow[r] \arrow[u] & C^{p+1,q+1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & C^{p-1,q} \arrow[r] \arrow[u] \arrow[ur, phantom, "\circlearrowleft"] & C^{p,q} \arrow[r] \arrow[u] \arrow[ur, phantom, "\circlearrowleft"] & C^{p+1,q} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & C^{p-1,q-1} \arrow[r] \arrow[u] \arrow[ur, phantom, "\circlearrowleft"] & C^{p,q-1} \arrow[r] \arrow[u] \arrow[ur, phantom, "\circlearrowleft"] & C^{p+1,q-1} \arrow[r] \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

THM Double-complex spect. seq

$C = (C^{p,q}, d^{p,q})_{p,q}$ double-complex.

$\Longrightarrow$ $\exists$ 2 spectral sequences

$E_0^{p,q} = C^{p,q}$; $E_0^{p,q} = C^{q,p}$.

both converging to $H^{p+q}(\mathrm{Tot}\, C^{\bullet\bullet})$

(Diff in the resp. $E_0^{p,q}$ pages are induces from the maps in $(C^{p,q}, d^{p,q})$).

Lem (snake)

$$\begin{tikzcd}
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h & \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] \arrow[u, two heads] & B' \arrow[r, "\beta'"] \arrow[u, two heads] & C' \arrow[r] \arrow[u, two heads] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[u, "f"] & B \arrow[r, "\beta"] \arrow[u, "g"] & C \arrow[r] \arrow[u, "h"] & 0 \\
 & \ker f \arrow[r] \arrow[u, hook] & \ker g \arrow[r] \arrow[u, hook] & \ker h \arrow[u, hook] \arrow[lluuu, bend right, "\delta"'] &
\end{tikzcd}$$

with exact rows.

$\Rightarrow \exists$ connecting δ. such that:

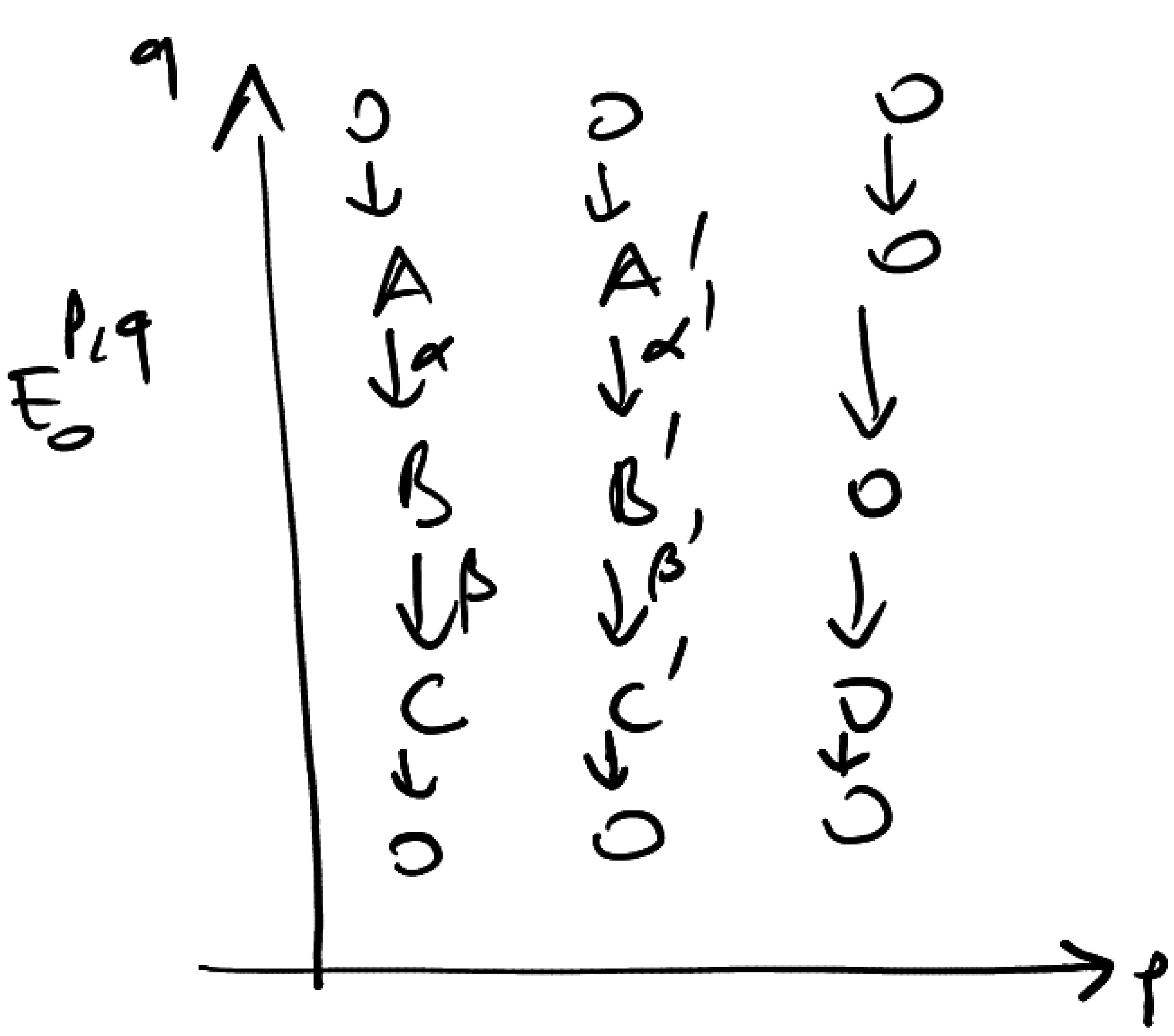
$$0 \to \ker f \to \ker g \to \ker h \xrightarrow{\delta} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0$$

exact.

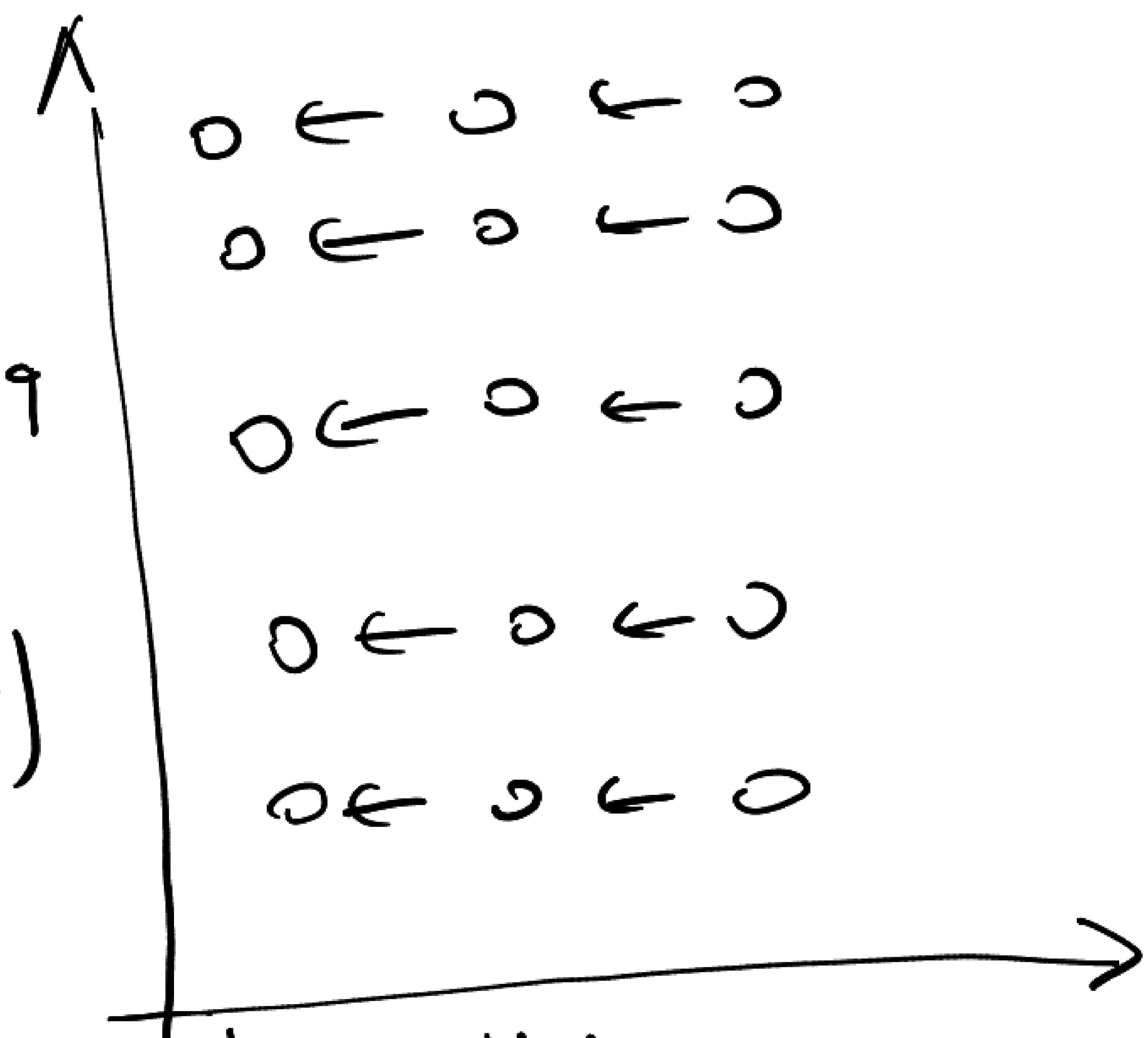
Proof ① By hand (very long: movie "it's my turn")

② alternative proof

⊛ $E_0^{p,q} = C^{q,p}$



$\Rightarrow E_1^{p,q} = H^*(E_0^{p,q})$



By exactness this converges to 0.

(*) $E_0^{p,q} = C^{p,q}$

$E_0^{p,q}$

$$\begin{tikzcd}
0 & 0 & 0 & 0 & 0 \\
0 \arrow[u] & A' \arrow[u] & B' \arrow[u] & C' \arrow[u] & 0 \arrow[u] \\
0 \arrow[u] & A \arrow[u] & B \arrow[u] & C \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

$\Downarrow$

$E_1^{p,q}$

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{coker} f \arrow[r, "\bar{\alpha}"] & \operatorname{coker} g \arrow[r, "\bar{\beta}"] & \operatorname{coker} h \\
0 \arrow[r] & \ker f \arrow[r, "\alpha"] & \ker g \arrow[r, "\beta"] & \ker h
\end{tikzcd}$$

$\Downarrow$

$E_2^{p,q}$

$$\begin{tikzcd}
0 & 0 & 0 & 0 & 0 \\
0 & \ker(\operatorname{coker} f \to \operatorname{coker} g) \arrow[drr, "d^2"] & 0 & 0 & 0 \\
0 & 0 & 0 & \ker h / \ker g & 0
\end{tikzcd}$$

E converges to $0 \Rightarrow d_2$ is an isom.

$\Rightarrow \delta = d_2^{-1}$ induces the connecting homomorphism of the snake lem.

<u>THM</u> (C, ∂) cochain having a regular filtration compatible with grading and diff.

$\Rightarrow (\exists$ spectral seq. $E_2^{p,q} \Rightarrow H^*(C))$..

hyp $E_2^{p,q} \Rightarrow H^*(C)$

$\Rightarrow \exists$ exact sequence of terms of low degrees

$$0 \to E_2^{1,0} \to H^1(C) \to E_2^{0,1} \to E_2^{2,0} \to H^2(C)$$

<u>THM</u> (sp seq of composed functors).

① $\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{C}$ abelian categories + left exact functors.

$\Gamma = F \circ G$

② $\mathcal{A}, \mathcal{B}, \mathcal{C}$ have enough injectives so that $R^n\Gamma, R^pG, R^qG$ exist.

③ Q injective $\Rightarrow R^pF(G(Q)) = 0 \quad \forall p > 0$
(i.e G takes injectives into F-acyclic objects)

$\Rightarrow \exists$ spect seq : $\forall A \in \mathcal{A}$.

$$R^pF(R^qG(A)) \Longrightarrow R^{p+q}\Gamma(A).$$

(Grothendieck spect seq).

Application Hochschild-Serre spect. seq.

① $G \in$ Prof. Grp., N closed $\triangleleft G$.

② $\mathcal{C}_G \xrightarrow{P} \mathcal{C}_{G/N} \xrightarrow{F}$

$A \longmapsto A^N$

$B \longmapsto B^{G/N}$.

③ $\Gamma = F \circ P$ i.e $\Gamma(A) = A^G$.

$\Longrightarrow$ $\left| \begin{array}{l} R^q P(A) = H^q(N, A) \\ R^p F(B) = H^p(G/N, A) \end{array}\right.$

$\exists$ spect. seq $H^p(G/N, H^q(N,A)) \Rightarrow H^{p+q}(G, A)$.

<u>Prop</u> Applicate exact seq of lower degree

$$0 \longrightarrow H^1(G/N, A^N) \xrightarrow{\theta_1} H^1(G, A) \xrightarrow{\theta_2} (H^1(N, A))^{G/N}$$

$$\downarrow d_2^{0,1}$$

$$H^2(G/N, A^N)$$

$$\downarrow \theta_3$$

$$H^2(G, A)$$

<u>with</u> $\left| \begin{array}{l} \theta_1, \theta_3 \text{ inflations} \\ \theta_2 \text{ restriction.} \\ d_2^{0,1} \text{ called } \underline{\text{transgression}}. \end{array}\right.$