Profinite group cohomology

<u>chap</u> | Topological groups
Profinite groups
Eq.

<u>chap</u> | Profinite cohomology.
Functoriality.

<u>chap</u> | Spectral - sequences
Hochschild - Serre Sp. seq.

<u>References</u>

① Matsushima : chap I groups Top.
② Shatz : Profinite groups.
③ Weibel : Introduct° to Hom Alg.
④ Serre : Galois Cohomology.

chap.

§. Topological groups.

Def $G \in \underline{\text{TopGrp}}$:
$\left| \begin{array}{l} (G, e) \in \text{Grp}. \ ; \ (G, \mathcal{O}_G) \in \text{Top}. \\ \varphi : G \times G \longrightarrow G. \quad \text{continuous.} \\ \quad (x, y) \longmapsto xy^{-1} \end{array} \right.$

Prop $G \in \text{TopGrp}, \ g \in G.$

$L_g : G \longrightarrow G$; $R_g : G \longrightarrow G$ | continuous.
$x \longmapsto g \cdot x$; $x \longmapsto x \cdot g$

Prop $\left| \begin{array}{l} A \text{ open} < G \\ B < G \end{array} \right. \Rightarrow A.B, \ A^{-1}$ are open.

Def $G \supset U \in \mathcal{V}(g)$ if. U open and $g \in U$.

Prop G $\underline{\text{Hausdorff}} \Longleftrightarrow \bigcap_{U \in \mathcal{V}(e)} U = \{e\}.$

Prop - Def $G \in \mathrm{GrpTop}$, $H < G$.

- $G/H \in \mathrm{Top}$ with O open in G/H iff $\pi^{-1}(O)$ open in G $\quad (\pi : G \longrightarrow G/H)$.

- $H \triangleleft G \Rightarrow G/H \in \mathrm{GrpTop}$

Def $X \in \mathrm{finTop}$ if $\left| \begin{array}{l} X \text{ finite} \\ X \in \mathrm{Top}. \end{array} \right.$

Rem $X \in \mathrm{finTop}$.

① Topologies on $X \Longleftrightarrow$ preorders on X.
(reflex. + transitivity)

② X compact.

③ X Hausdorff $\Rightarrow X$ discrete
i.e each point is open.

§. Profinite groups

Def $(I, \leq)$ partially ordered index set.

$(G_i)_{i \in I}$ family of groups.

- $i \leq j \Rightarrow \left| \begin{array}{l} \exists k \in I \mid i \leq k,\ j \leq k. \\ \exists f^i_j : G_j \longrightarrow G_i. \quad (*) \end{array} \right.$

- consistency. $\quad i \leq j \leq k.$

$$\Longrightarrow \quad \begin{tikzcd} G_i & G_j \arrow[l, "f^i_j"'] & G_k \arrow[l, "f^j_k"'] \arrow[ll, bend left, "f^i_k"] \end{tikzcd}$$

① $(G_i, f^i_j)_I$ projective mapping family.

② Instead of $(*)$ $i \leq j \Rightarrow \exists f^j_i : G_i \longrightarrow G_j$

$(G_i, f^j_i)_I$ direct mapping family

Def

① Projective limit of $(G_i, f_j^i)_I$:

* $\lim G_i \in \underline{\underline{Grp}}$.

* UMP.

$$
\begin{tikzcd}
 & & G_i \\
X \arrow[r, dashed] \arrow[urr, bend left, "g_i"] \arrow[drr, bend right, "g_j"'] & \lim G_i \arrow[ur, "f_i"] \arrow[dr, "f_j"'] & \\
 & & G_j \arrow[uu, "f_j^i"']
\end{tikzcd}
\qquad i \leq j
$$

② Injective limits of $(G_i, f_i^j)_I$.

* $\mathrm{colim}\, G_i \in Grp$

* UMP.

$$
i \leq j \qquad
\begin{tikzcd}
G_i \arrow[d, "f_i^j"'] \arrow[dr, "f_i"] \arrow[drr, bend left, "g_i"] & & \\
 & \mathrm{colim}\, G_i \arrow[r, dashed] & X \\
G_j \arrow[ur, "f_j"'] \arrow[urr, bend right, "g_j"'] & &
\end{tikzcd}
$$

__Thm__ lim and colim exists in Set, Grp...

* $\lim G_i = \{ (x_i)_i \in \prod_I G_i \mid i \le j \Rightarrow f^i_j(x_j) = x_i \}$.

* $\operatorname{colim} G_i = \bigsqcup G_i /_{\sim}$.

$$G_i \ni x \sim y \in G_j \iff \left|\begin{array}{l} \exists k \in I \mid i \le k,\ j \le k \\ f^k_i(x) = f^k_j(y). \end{array}\right.$$

__Ex__ $I = \mathbb{N}$; $\le \; = \; |$; $G_n = \mathbb{Z}/n\mathbb{Z}$.

$$n \mid m \Rightarrow f^m_n : \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z}.$$
$$\bar{x}^n \longmapsto \tfrac{m}{n}\bar{x}^m.$$

$$\begin{tikzcd}
\bar{x}^n \arrow[rrr, mapsto, bend left] & & & {[\frac{x}{n}]} \\
\mathbb{Z}/n\mathbb{Z} \arrow[dr, "f_n"] \arrow[rrr, bend left, "\alpha_n"] \arrow[dd, "f^m_n"', "n|m"'] & & & \\
& \operatorname{colim} \mathbb{Z}/n\mathbb{Z} \arrow[rr, dashed, "\varphi"] & & \mathbb{Q}/\mathbb{Z} \\
\mathbb{Z}/m\mathbb{Z} \arrow[ur, "f_m"] \arrow[urrr, bend right, "\alpha_m"'] & & & {[\frac{x}{m}]} \\
\bar{x}^m \arrow[urrr, mapsto, bend right] & & &
\end{tikzcd}$$

Prove that φ bijective.

Def G is a profinite group.

$G = \lim G_i$, G_i finite.

Rem

① group finite $\Rightarrow$
- $\in$ GrpTop.
- compact.
- Hausdorff.

② G profinite $\Longrightarrow$ G compact Hausdorff.

THM ① G profinite group

$\Updownarrow$

② G compact, Hausdorff.
and its open normal subgroups
form a fund. syst of neigh. of e.

$\Updownarrow$

③ G compact, totally disconnected.
Hausdorff.

§. $\mathcal{C}_G$.

Def G profinite group.

(1) A <u>discrete G-module</u>: $(A \in \mathcal{C}_G)$.

- A discrete Ab grp.
- $\exists\, * : G \times A \longrightarrow A$. continuous.
  $(g, a) \longmapsto g * a$

(2) Morphism in $\mathcal{C}_G$.

$$\begin{tikzcd} G \times A \arrow[r] \arrow[d, "\mathrm{id}_G \times f"'] & A \arrow[d, "f"] \\ G \times B \arrow[r] & B \end{tikzcd}$$

i.e $f(g * a) = g * f(a)$

Rmk $A \in \mathcal{C}_G$, $a \in A$

(1) $G_a = \{g \in G \mid g * a = a\}$ open $< G$.

(2) $a \in A^{G_a} = \{x \in A \mid \forall g \in G_a,\ g * x = x\}$.

(3) $A = \bigcup_{U \text{ open} < G} A^U$.

Prop $\mathcal{C}_G$ abelian category.

Proof $\ker f$, $\operatorname{im} f$, $\operatorname{coker} f \in \mathcal{C}_G$.

Def $A \in \mathcal{C}_G$. $G^0 = \{e\}$; $G^n = G \times \dots \times G$

- $C^n(G, A) = \{ f : G^n \longrightarrow A \mid \text{continuous} \}$.
- $C^0(G, A) = \{ f : \{e\} \to A \} \simeq A$.

Prop $(C^n(G,A), +) \in \text{Ab grp}$.

Prop $C^n(G, -) : \mathcal{C}_G \longrightarrow \text{Ab grp}$.

$$\begin{array}{ccc} A & & C^n(G,A) \ni f \\ \varphi \downarrow & \longmapsto & \downarrow \varphi^* \qquad \downarrow \\ B & & C^n(G,B) \ni \varphi^*(f) = \varphi \circ f \end{array}$$

exact functor. i.e.

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0 \qquad \underline{\text{ES}}.$$

$$0 \longrightarrow C^n(G,A) \xrightarrow{\varphi^*} C^n(G,B) \xrightarrow{\psi^*} C^n(G,C) \longrightarrow 0 \qquad \underline{\text{ES}}$$

Proof.

① $\varphi^*(f)=0 \Rightarrow \varphi(f(g_1,\dots,g_n))=0 \quad \forall (g_i)$

$\Rightarrow f(g_1,\dots,g_n)=0_A . \quad \forall (g_i)$

(φ inject.)

$\Rightarrow f=0$. i.e φ^* inject.

② • $\psi^*\circ\varphi^*(f)=\psi\circ\varphi\circ f=0\circ f=0$. $(\psi\circ\varphi=0)$

i.e $\operatorname{im}\varphi^* \subset \ker\psi^*$.

• $f\in\ker\psi^* \Rightarrow \psi^*(f)=0$.

$\Rightarrow \psi\circ f(g_1,\dots,g_n)=0 \quad \forall (g_i)$

$\Rightarrow f(g_1,\dots,g_n)\in\ker\psi=\operatorname{im}\varphi$.

$\Rightarrow \exists a\in A \mid f(g_1,\dots,g_n)=\varphi(a)$

set $h: G^n \longrightarrow A$

$(g_1,\dots,g_n)\mapsto a$.

$\Rightarrow f=\varphi\circ h=\varphi^*(h) \Rightarrow f\in\operatorname{im}\varphi^*$

③ $C^n(G,B) \xrightarrow{\psi^*} C^n(G,C)$

$f\in C^n(G,C)$

ψ surjective $\Rightarrow \exists$ section $s: C\longrightarrow B \mid \psi\circ s=\mathrm{id}_C$.

$\Rightarrow G^n \xrightarrow{f} C \xrightarrow{s} B$ continuous (discrete)

and $\psi^*(s\circ f)=\psi\circ s\circ f=\mathrm{id}_C\circ f=f$. i.e ψ^* surjective chap

§. Profinite cohom

Prop-Def $(C^n(G,A), d^n) \in \mathrm{coch}^+(A\mathcal{S})$ with

- $d^0 : C^0(G,A) \longrightarrow C^1(G,A)$

$$f \longmapsto d^0 f : G \longrightarrow A$$
$$g \mapsto d^0 f(g)$$

- $\boxed{(d^0 f)(g) = g * f(e) - f(e)}$

$d^1 : C^1(G,A) \longrightarrow C^2(G,A)$

$$f \longmapsto d^1 f : G^2 \longrightarrow A$$
$$(g_1, g_2) \mapsto d^1 f(g_1, g_2)$$

$$\boxed{d^1 f(g_1, g_2) = g_1 * f(g_2) - f(g_1 g_2) + f(g_1)}$$

$d^n : C^n(G,A) \longrightarrow C^{n+1}(G,A)$

$$f \longmapsto d^n f : G^{n+1} \longrightarrow A.$$
$$(g_1, \dots, g_{n+1}) \longmapsto d^n f(g_1, \dots, g_{n+1})$$

$$d^n f(g_1, \dots, g_{n+1}) =$$
$$g_1 * f(g_2, \dots, g_{n+1}) + \sum_{1}^{n} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1})$$
$$+ (-1)^{n+1} f(g_1, \dots, g_n).$$

Proof show $d^n \circ d^{n-1} = 0$ (i.e. $\operatorname{im} d^{n-1} \subset \ker d^n$)

for instance

$d^1 \circ d^0 (f)(g_1, g_2)$

$= g_1 * d^0(f)(g_2) - d^0 f(g_1 g_2) + d^0 f(g_1)$

$= g_1 * [g_2 * f(e) - f(e)] - [(g_1 g_2) * f(e) - f(e)]$
$\quad + [g_1 * f(e) - f(e)]$

$= 0$ since $g_1 * g_2 * f(e) = (g_1 g_2) * f(e)$

Def n-cohom. group of G with coeff in A

$$H^n(G, A) = \frac{\ker\left(d^n : C^n(G,A) \to C^{n+1}(G,A)\right)}{\operatorname{im}\left(d^{n-1} : C^{n-1}(G,A) \to C^n(G,A)\right)}$$

<u>Prop</u> $H^0(G,A) = A^G$.

<u>Proof</u> $0 \longrightarrow C^0(G,A) \xrightarrow{\delta^0} C^1(G,A) \longrightarrow \cdots$

. $H^0(G,A) = \dfrac{\ker \delta^0}{\operatorname{im} 0} = \ker \delta^0$.

. $\ker \delta^0 = \{ f \in C^0(G,A) \mid \delta^0 f = 0 \}$.

$(\delta^0 f)(g) = 0 \quad \forall g$.

$f(e) * g - f(e) = 0 \quad \forall g$.

$C^0(G,A) \cong A$.
$f(e) \leadsto a \in A$. $\Big| \Rightarrow \ker \delta^0 = \{ a \in A \mid \forall g \in G,\ a * g = g \}$
$= A^G$.

<u>Prop</u> $H^1(G,A) =$ group of $[f] = f + \operatorname{im} \delta^0$.
$f : G \to A$ cont. <u>crossed-morph</u>.

<u>Proof</u> $C^0(G,A) \xrightarrow{\delta^0} C^1(G,A) \xrightarrow{\delta^1} C^2(G,A) \longrightarrow \cdots$

. $H^1(G,A) = \dfrac{\ker \delta^1}{\operatorname{im} \delta^0}$.

$\ker d^1 = \{ f \in C^1(G, A) \mid d^1 f = 0 \}$.

$$(d^1 f)(g_1, g_2) = 0 \quad , \forall (g_1, g_2)$$

$$g_1 * f(g_2) - f(g_1 g_2) + f(g_1) = 0.$$

$$f(g_1 g_2) = g_1 * f(g_2) + f(g_1)$$

(i.e f crossed-morph).

Rem $G \times A \longrightarrow A$. (trivial action)

$(g, a) \longmapsto g * a = a$

$$\Rightarrow f(g_1 g_2) = g_1 * f(g_2) + f(g_1)$$

$$= f(g_2) + f(g_1)$$

(i.e f usual morphism).

Prop $H^2(G, A) =$ group of classes of continuous factor systems from G to A.

Rem interpretation of higher cohom. grps. i.e $H^n(G, A)$, $n \geq 3$. ?

§. Functoriality.

Def $A \in \mathcal{C}_G$; $\phi : G' \longrightarrow G$. // continuous

$A' \in \mathcal{C}_{G'}$; $\psi : A \longrightarrow A'$

$\langle \phi, \psi \rangle$ is a compatible pair if.

$\forall a \in A , g' \in G'$:

$$\begin{tikzcd}
G \times \underset{a}{\overset{\in}{A}} \arrow[r, "*"] \arrow[d, "\psi"] & A \arrow[d, "\psi"] \arrow[dl, phantom, "\circlearrowleft"] \\
G' \times A' \arrow[u, "\phi"] \arrow[r, "*"'] & A'
\end{tikzcd}$$

$\underset{g'}{\overset{\in}{G'}}$

i.e $\underline{\phi(g')*a = g'*\psi(a)}$

Prop $\langle \phi, \psi \rangle$ compatible pair

$\Longrightarrow \alpha^n : C^n(G, A) \longrightarrow C^n(G', A')$

$f \longmapsto$

$$\begin{tikzcd}
G'^n \arrow[r, "\alpha^n(f)"] \arrow[d, "\phi^n"] & A' \\
G^n \arrow[r, "f"'] & A \arrow[u, "\psi"']
\end{tikzcd}$$

define $\alpha : C^*(G, A) \longrightarrow C^*(G', A')$

and $\alpha^* : H^*(G, A) \longrightarrow H^*(G', A')$

Def restriction.

$$\left| \begin{array}{l} S < G \\ A \in \mathcal{C}_G \end{array} \right. \Rightarrow \left| \begin{array}{l} A \in \mathcal{C}_S \\ \langle i : S \hookrightarrow G ; id_A : A \to A \rangle \\ \text{compatible pair.} \end{array} \right.$$

$$\Rightarrow \left| \ res : H^*(G, A) \longrightarrow H^*(S, A) \right.$$

Prop S closed $\triangleleft G$; $\pi : G \to G/S$ cay. canonical

① $A \in \mathcal{C}_G \Rightarrow A^S \in \mathcal{C}_{G/S}$.

② $\langle i : A^S \hookrightarrow A , \pi : G \to G/S \rangle$ compatible pair

Proof $G/S \times A^S \longrightarrow A^S$

$(\bar{g}, a) \mapsto \bar{g} * a = g \cdot a$

$g \cdot a \in A^S$ since $s \cdot (g \cdot a) = (sg) \cdot a = (gs') \cdot a = g \cdot (s' \cdot a) = g \cdot a$ *

$(S \triangleleft G$ i.e $gS = Sg)$ $\quad (a \in A^S)$

Def inflation with $\langle i, \pi \rangle$.

$$\inf : H^*(G/S, A^S) \longrightarrow H^*(G, A)$$

Thm Assume:

① projective syst $(G_\alpha)_\alpha \in$ ProGrp.

$$\begin{tikzcd}
 & G_\alpha \\
\lim G_\alpha \arrow[ur, "\phi_\alpha"] \arrow[dr, "\phi_\beta"'] & \\
 & G_\beta \arrow[uu, "\phi_\beta^\alpha"']
\end{tikzcd} \qquad \alpha \le \beta.$$

② direct system $(A_\alpha)_\alpha \in \mathcal{C}_{G_\alpha}$.

$$\begin{tikzcd}
A_\alpha \arrow[dr, "\psi_\alpha"] \arrow[dd, "\psi_\alpha^\beta"'] & \\
 & \operatorname{colim} A_\alpha \\
A_\beta \arrow[ur, "\psi_\beta"'] &
\end{tikzcd} \qquad \alpha \le \beta$$

③
$$\begin{tikzcd}
G_\alpha \times A_\alpha \arrow[r, "*"] \arrow[d, "\psi_\alpha^\beta"'] \arrow[dr, phantom, "\circlearrowleft"] & A_\alpha \arrow[d, "\psi_\alpha^\beta"] \\
G_\beta \times A_\beta \arrow[u, "\phi_\beta^\alpha"] \arrow[r, "*"] & A_\beta
\end{tikzcd} \qquad \langle \phi_\beta^\alpha, \psi_\alpha^\beta \rangle \text{ compatible}$$

Then $\exists!$ action

ⓐ $\lim G_\alpha \times \operatorname{colim} A_\alpha \xrightarrow{*} \operatorname{colim} A_\alpha$

$$\left(g = (g_\alpha)_\alpha,\ \underset{\psi_\beta(a_\beta)}{\overset{a}{\shortparallel}}\right) \longmapsto \psi_\beta(g_\beta * a_\beta).$$

ⓑ
$$\begin{tikzcd}
G_\alpha \times A_\alpha \arrow[r, "*"] \arrow[d, "\psi_\alpha"'] & A_\alpha \arrow[d, "\psi_\alpha"] \\
\lim G_\alpha \times \operatorname{colim} A_\alpha \arrow[u, "\phi_\alpha"] \arrow[r, "*"] & \operatorname{colim} A_\alpha
\end{tikzcd} \qquad \langle \phi_\alpha, \psi_\alpha \rangle \text{ compatible}$$

(c) $H^*(\lim_\alpha G_\alpha, \operatorname{colim} A_\alpha) = \operatorname{colim}_\alpha H^*(G_\alpha, A_\alpha)$.

<u>Proof</u> (sketch).

⊛ check (c) well-defined action.

⊛ $\langle \phi_\alpha, \psi_\alpha \rangle$ compatibility in indices.

$$i_\alpha : C^n(G_\alpha, A_\alpha) \longrightarrow C^n(\lim G_\alpha, \operatorname{colim} A_\alpha)$$

⊛
$$\begin{tikzcd}
C^n(G_\alpha, A_\alpha) \arrow[dr] \arrow[dd] \arrow[drr, bend left, "i_\alpha"] \arrow[drr, phantom, bend left=15, "\circlearrowleft"] & & \\
\arrow[r, phantom, "\circlearrowleft"] & \operatorname{colim} C^n(G_\alpha, A_\alpha) \arrow[r, dashed, "\exists f"] & C^n(\lim G_\alpha, \operatorname{colim} A_\alpha) \\
C^n(G_\beta, A_\beta) \arrow[ur] \arrow[urr, bend right, "i_\beta"'] \arrow[urr, phantom, bend right=15, "\circlearrowleft"] & &
\end{tikzcd}$$

check f <u>bijective</u>.

<u>Def</u> H closed $< G$.

$A \in \mathcal{C}_H$.

<u>induced G-module</u>

$$M_G^H(A) = \{\varphi : G \to A \text{ cont} \mid \varphi(hx) = h * \varphi(x), \ \forall h \in H, x \in G\}.$$

<u>Prop</u>

① $M_G^H(A) \in \text{AbGrp}$.

② $M_G^H(A) \in \mathcal{C}_G$ via the action

$$* : G \times M_G^H(A) \longrightarrow M_G^H(A)$$
$$(g, \varphi) \longmapsto g * \varphi : G \longrightarrow A, \quad x \longmapsto \varphi(gx).$$

③ $M_G^H : \mathcal{C}_H \longrightarrow \mathcal{C}_G$
$$A \longmapsto M_G^H(A)$$

functor (exact, preserves injectives)

④ $\langle M_G^H(A) \to A, \ \varphi \mapsto \varphi(1) \ ; \ H \hookrightarrow G \rangle$ compatible pair

<u>Hence</u> $\exists$ $H^*(G, M_G^H(A)) \longrightarrow H^*(H, A)$.

Homological Algebra Reminder.

Def $\mathcal{A}$ abelian category.

① $I \in \mathcal{A}$ injective if $\mathcal{A}(-, I)$ exact functor

② $\mathcal{A}$ has enough injectives if.
$\forall A \in \mathcal{A}$, $\exists I$ injective and injection $A \rightarrowtail I$

③ Injective resolution of $A \in \mathcal{A}$:
* $(I^\bullet, d^\bullet) \in \mathrm{coch}(\mathcal{A})$ such that
* $0 \to M \to I^0 \to I^1 \to \cdots$ exact.
* I^i injective $\forall i \geq 0$.

Lem $\mathcal{A}$ has enough injectives
$\Rightarrow \forall A \in \mathcal{A}$ has an injective resolut$^=$.

Def $\mathcal{A}, \mathcal{B}$ abelian categories
$F: \mathcal{A} \to \mathcal{B}$ left exact functor
$\mathcal{A}$ has enough injectives.
Right derived functors of F: R^iF $(i \geq 0$.
* $A \in \mathcal{A}$, choose an injective resolut$^=$

$$A \to I^\bullet$$

* $\boxed{R^iF(A) = H^i(F(I^\bullet))}$.

Lem G profinite group.

① $\mathbb{Z} \in \mathcal{C}_G$ via the trivial action.

$$G \times \mathbb{Z} \xrightarrow{*} \mathbb{Z}$$
$$(g, n) \longmapsto g * n = g.$$

② $\mathcal{C}_G(\mathbb{Z}, A) \cong A^G$.

Proof $\alpha \in \mathcal{C}_G(\mathbb{Z}, A) \Leftrightarrow \begin{cases} \alpha(1) \in A. \\ g * \alpha(1) = \alpha(1) \quad \forall g \in G \end{cases}$

$\Leftrightarrow \alpha(1) \in A^G$.

Prop (Shapiro lemma)

$$H^*(G, M_G^H(A)) \xrightarrow{\cong} H^*(H, A)$$

Proof

$* = 0$: $H^0(H, A) \cong A^H \cong \mathcal{C}_H(\mathbb{Z}, A)$

$\cong \mathcal{C}_G(\mathbb{Z}, M_G^H(A))$

$\cong (M_G^H(A))^G$.

$\cong H^0(G, M_G^H(A))$

- Choose an injective resolution $A \to I^\bullet$
- $M_G^H(A) \to M_G^H(I^\bullet)$ injective resolution of $M_G^H(A)$ since $M_G^H(-)$ exact and preserves injectives

- $H^*(G, M_G^H(A)) \cong H^*(M_G^H(I^\bullet)^G)$ } by the case $*=0$

$\cong H^*((I^\bullet)^H)$

$= H^*(H, A)$.

Rem $H^*(H, A) \overset{\text{def}}{=} H^*((I^\bullet)^H)$ with.

$A \hookrightarrow I^\bullet$ injective resolut° of A.

Prop H open $< G$, $A \in \mathcal{C}_G$, $[G:H]$ finite.

$\Rightarrow \pi : M_G^H(A) \longrightarrow A$ surjective in $\mathcal{C}_G$

$$\varphi \longmapsto \pi(\varphi) = \sum_{\bar{x} \in G/H} x \cdot \varphi(x^{-1})$$

Proof

① π is well-defined.

$y \in \bar{x} \Rightarrow x^{-1} y \in H$

$\Rightarrow \exists h \in H \mid x^{-1} = h y^{-1}$

$\Rightarrow \varphi(x^{-1}) = \varphi(h y^{-1}) = h \varphi(y^{-1})$

$\Rightarrow x \cdot \varphi(x^{-1}) = x h \varphi(y^{-1}) = y \varphi(y^{-1})$.

② π surjective.

•

Prop-Def

① $\pi : M_G^H(A) \longrightarrow A$ induces.

$C^*(G, M_G^H(A)) \xrightarrow{\pi^*} C^*(G, A)$.

and

$H^*(G, M_G^H(A)) \longrightarrow H^*(G, A)$.

② corestriction : composition

$$\begin{tikzcd} H^*(G,A) \arrow[r, "Sh", "\cong"'] \arrow[rr, bend right, "\mathrm{cor} = \pi^* \circ Sh"'] & H^*(G, M_G^H(A)) \arrow[r, "\pi^*"] & H^*(G,A) \end{tikzcd}$$

cor $= \pi^* \circ Sh$.

Prop H open $< G$; $[G:H]$ finite.

$$\begin{tikzcd} H^*(G,A) \arrow[r, "\mathrm{Res}"] \arrow[rr, bend right, "\mathrm{Cor} \circ \mathrm{Res}"'] & H^*(H,A) \arrow[r, "\mathrm{Cor}"] & H^*(G,A) \end{tikzcd}$$

Cor $\circ$ Res

is a multiplication by $[G:H]$.

Proof cor $\circ$ Res comes from:

$A \longrightarrow M_G^H(A) \longrightarrow A$.

$a \longmapsto \left(\varphi_a : G \to A,\ x \mapsto x.a\right) \longmapsto \sum_{\bar{x} \in G/H} a.\varphi_a(x^{-1})$.

and $\displaystyle\sum_{\bar{x} \in G/H} x.\varphi_a(x^{-1}) = \sum_{\bar{x} \in G/H} x.x^{-1}.a$

$\displaystyle = \sum_{\bar{x} \in G/H} a = |G/H| \cdot a = [G:H] \cdot a$.

Chap Spectral Sequences

Def First-quadrant cohomological spec. seq.

① collection of $E_r^{p,q}$ groups, $p,q \in \mathbb{N}$.
$r \geq r_0$ fixed.

② collection of differentials

$$\cdots \longrightarrow E_r^{p-r,q+r-1} \xrightarrow{d_r^{p-r,q+r-1}} E_r^{p,q} \xrightarrow{d^{p,q}} E^{p+r,q-r+1} \longrightarrow \cdots$$

such that $d_r^{\circ} \circ d_r^{\circ} = 0$.

③ $E_{r+1}^{p,q}$ = cohomology at $E_r^{p,q}$ with respect to the above diff.

⊛ $E_r = \left(E_r^{p,q}, d_r^{p,q}\right)_{p,q}$: r^{th}-page.

⊛ the spectral sequence E converges if (stationary).
$\exists s \mid E_s = E_{s+1} = \cdots$

such page E_s is denoted E_∞.

Def A cochain $(C^i, d^i)_i$ is filtered if $\forall i$, C^i has a diff-preserving filtration.

i.e. $F_0 C^i \subset F_1 C^i \subset \cdots \subset F_p C^i \subset F_{p+1} C^i \subset \cdots$
&
$\bigcup_p F_p C^i = C^i$; $\bigcap_p F_p C^i = 0$; $d^i(F_p C^i) \subset F_p C^{i+1}$.

Rem
① A filtration on (C^i, d^i) induces a filtration $F_p H^*(C)$ on cohomology grps.

② Associated graded pieces
$$Gr_p H^n(C) = F_p H^n(C) \big/ F_{p+1} H^n(C)$$

Def Spect. seq E converges to $H^*(C)$ ($E_\infty^{p,q} \Longrightarrow H^{p+q}(C)$) if.

E converge. (stationary at E_s)
$E_s^{p,q} = E_\infty^{p,q} = Gr_p H^{p+q}(C)$.

Rem Convergent spect. seq does not compute $H^*(C)$, rather a filtration of it.

Def <u>Double-complex</u>

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & C^{p-1,q+1} \arrow[r] \arrow[u] \arrow[dr, phantom, "\circlearrowleft"] & C^{p,q+1} \arrow[r] \arrow[u] \arrow[dr, phantom, "\circlearrowleft"] & C^{p+1,q+1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & C^{p-1,q} \arrow[r] \arrow[u] \arrow[dr, phantom, "\circlearrowleft"] & C^{p,q} \arrow[r] \arrow[u] \arrow[dr, phantom, "\circlearrowleft"] & C^{p+1,q} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & C^{p-1,q-1} \arrow[r] \arrow[u] & C^{p,q-1} \arrow[r] \arrow[u] & C^{p+1,q-1} \arrow[r] \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

<u>THM</u> Double-<u>complex</u> spect. seq

$C = (C^{p,q}, d^{p,q})_{p,q}$ double-complex.

$\Rightarrow \exists\, \underline{2}$ spectral sequences

$E_0^{p,q} = C^{p,q}$; $E_0^{p,q} = C^{q,p}$.

both converging to $H^{p+q}(\mathrm{Tot}\, C^{\bullet\bullet})$

(Diff in the resp. $E_0^{p,q}$ pages are induced from the maps in $(C^{p,q}, d^{p,q})$).

Lem (snake)

$$\begin{tikzcd}
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h & \\
0 \arrow[r] & A' \arrow[r, "\alpha'"] \arrow[u, two heads] & B' \arrow[r, "\beta'"] \arrow[u, two heads] & C' \arrow[r] \arrow[u, two heads] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[u, "f"] & B \arrow[r, "\beta"] \arrow[u, "g"] & C \arrow[r] \arrow[u, "h"] & 0 \\
 & \ker f \arrow[r] \arrow[u, tail] & \ker g \arrow[r] \arrow[u, tail] & \ker h \arrow[u, tail] \arrow[uuull, bend right, "\delta"'] &
\end{tikzcd}$$

with exact rows.

$\Rightarrow \exists$ connecting δ. such that:

$$0 \to \ker f \to \ker g \to \ker h \overset{\delta}{\dashrightarrow} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0$$

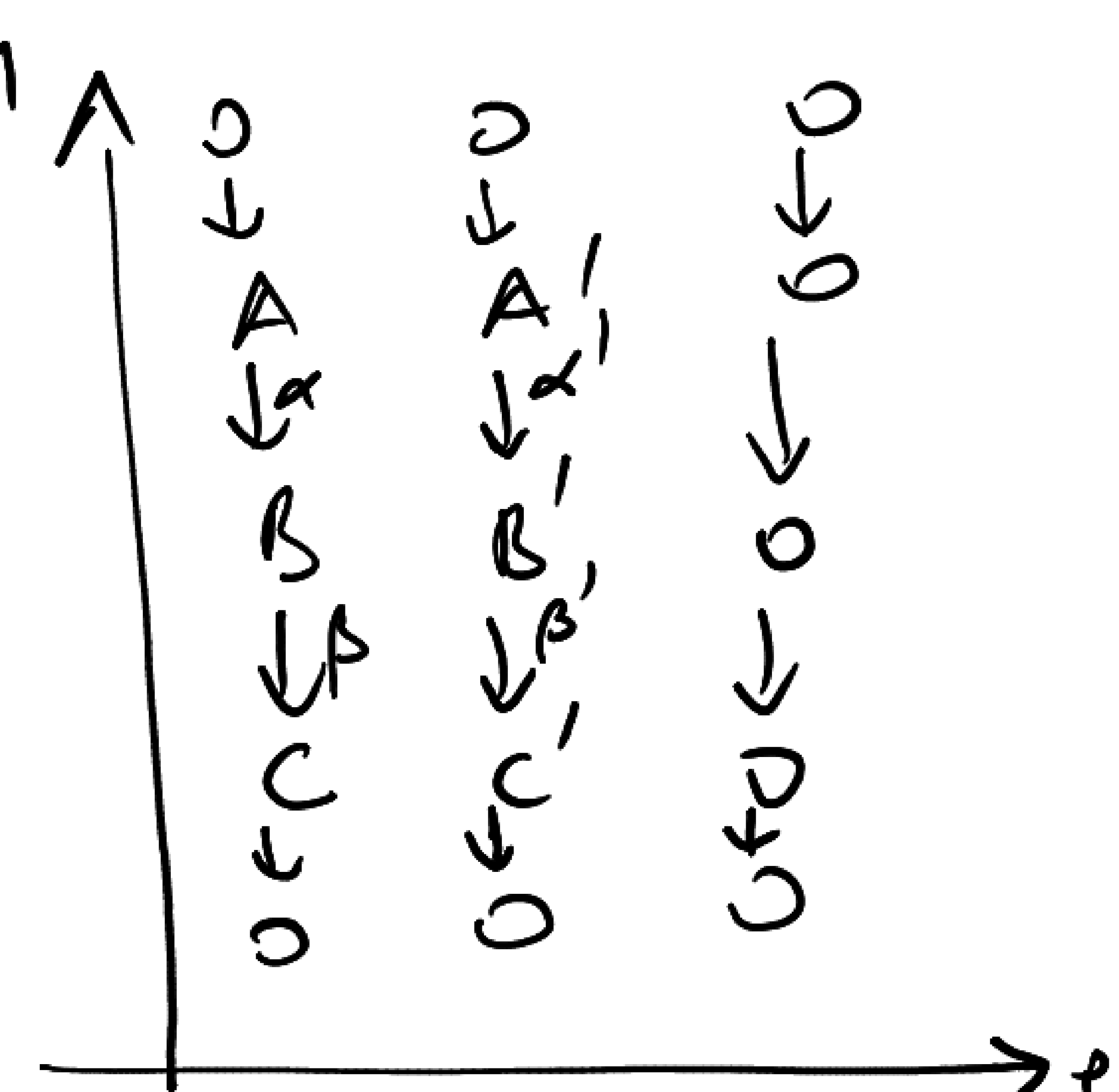
exact.

Proof ① By hand (very long: movie "it's my turn")

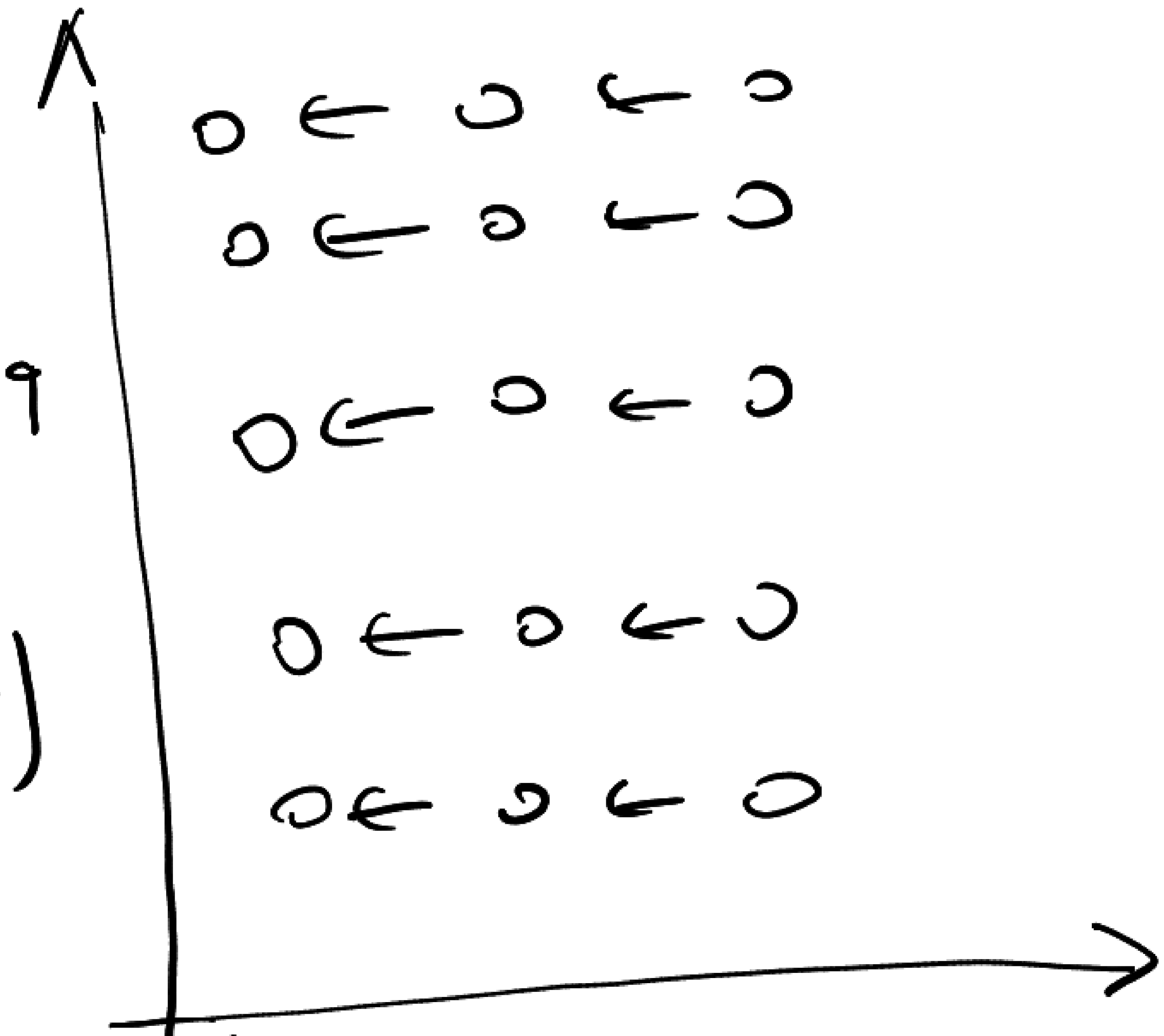
② alternative proof

(*) $E_0^{p,q} = C^{q,p}$

$E_0^{p,q}$



$\Rightarrow$ $E_1^{p,q} = H^*(E_0^{p,q})$



By exactness this converges to 0.