

Lattices from Number Fields

For this exercise, while it is possible to do these computations by hand, using SAGE helps greatly (and does not prevent to check the computations by hand later on).

Construction of E_8 from number fields. A Gram matrix for E_8 is

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}.$$

- (a) Suppose we want to try to build E_8 from a cyclotomic field $\mathbb{Q}(\zeta_n)$. Provide some suitable values of n .
- (b) Compute the discriminant of $\mathbb{Q}(\zeta_n)$ for the found values of n .
- (c) The determinant of the Gram matrix of E_8 is 1. We will thus need a scaling factor. To have a scalar factor such that the resulting determinant is 1, we are looking for a lattice whose volume is a integral power of 8. For the case where we already have a lattice satisfying this constraint, which lattice is obtained?
- (d) For the other cases, we need to introduce a twisting element α . Look for such a suitable element for $n = 20$ and $n = 24$ and try to construct E_8 accordingly.

Finding constructions of lattices from number fields is still fairly open. For example, <http://www.math.rwth-aachen.de/~Gabriele.Nebe/LATTICES/> lists interesting lattices, and how they can be constructed can be reported.