Appendix: on Shanin's topological theorems

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The time of ideological battles in mathematics has long passed. We know that constructivism has achieved much more than Alexandrov could have expected. But much less than the constructivists expected themselves.

In this article, however, the discussion was more about the birth of constructivism as a trend, and not about the results of several decades of its development. Instead of a conclusion, it is advisable to touch upon some of the main results of N.A. Shanin's dissertation, which were discussed in the documents cited above (adding the necessary explanations regarding terminology). We will focus on two theorems - Theorem 16 and Theorem 51, the importance of which is emphasized in the reports written by Alexandrov and Markov.

Recall that according to the original definition by P.S. Aleksandrov and P.S. Urysohn (1923), a topological space is called bicompact if each of its open covers contains a finite subcover. In this case, a bicompact Hausdorff space was called a bicompact. The terms "compact space" and "compact" were originally used in the case of "countable compactness".

Because of the coincidence of the properties of compactness, countable compactness and sequential compactness for subsets of Euclidean spaces (as well as for other classes of "good" spaces: metric spaces, manifolds, spaces studied in algebraic topology), it did not immediately become obvious that compact spaces are the correct extension of the class of metric compacta. However, further developments in mathematics and its applications have confirmed the fundamental importance of the concept of compactness[1].

Below, the terminology and notations from [5].

Set-theoretic concepts play a major role in N.A. Shanin's dissertation. Let us present some agreements, definitions and notations used there.

[1] See, e.g., http://www.lomonosov-fund.ru/enc/ru/encyclopedia:0135684 (acc. 11/12/2023). Cf. also [1].

"To simplify the formulations of a number of definitions and theorems, we introduce the statement: "the set A is a cardinal number" and introduce the following axiom: for any set there exists one and only one cardinal number equivalent to A[2].

A cardinal number equivalent to a set M is called the cardinality of the set M and will be denoted by $|M|$.

If A and B are two sets, then the expressions $A \sim B$, $A \lesssim B$ and $A < B$ will denote respectively: "the set A has the same cardianlity as B", "the set A has the same cardinality as some subset of B" и "the set has A the same cardinality as some subset of the set B, but not as B itself".

If $\mathfrak{N}$ is some family of sets, by $\mathrm{Sup}(\mathfrak{N})$ we shall denote the cardinal number that has the following properties: $N \lesssim Sup(\mathfrak{N})$ for all $N \in \mathfrak{N}$ and, for any cardinal number $\mathfrak{r}$ such that $\mathfrak{r} < Sup(\mathfrak{N})$, there exists a set $N \in \mathfrak{N}$ such that $\mathfrak{r} < N$.»([5], p.10-11).

"The cardinal number $\mathfrak{m}$ is called caliber of the topological space R if $\mathfrak{m} > 1$ and every family of nonempty open subsets R that has the cadinality $\mathfrak{m}$, has a subfamily of the same cardinality such that the intersection of all the elements of this subfamily is nonempty set." ([5], p.6).

"The family $\mathfrak{N}$ is called a skeleton of the set A, if the following conditions hold:

1. $\cup\mathfrak{N} = A$,

2. $N < A$ for all $N \in \mathfrak{N}$.

We shall call character of the set A the minimum of cardinalities of all possible skeletons of the set A. The symbol χ will denote the function that associates with each set A the character of this set $\chi(A)$.

Obviously the sets that have the same cardinality also have the same characters. We shall say that the set A is regular, if $\chi(A) \sim A$. In particuler, the cardinal number $\mathfrak{m}$ will be called regular if $\chi(\mathfrak{m}) = \mathfrak{m}$.

Let us notice the following obvious fact: if there is an infinite cardinal number that is an immediate predecessor of the cardinal number $\mathfrak{m}$, then $\mathfrak{m}$ is a regular cardinal number."([5], p.14)

Below Ξ is some set of indexes and $\{X(\xi)\}_{\xi\in\Xi}$ a family of topological spaces indexed by Ξ.

"**Theorem 16**. Let an infinite cardinal number $\mathfrak{m}$ satisfy the following three conditions:

[2] As usual, equivalence is understood as existence of a bijection.

1. For every $\xi \in \Xi$ the cardinal number $\mathfrak{m}$ is a universal caliber of the T-space $X(\xi)$ [in other words: for every $\xi \in \Xi$ the cardinal numbers $\mathfrak{m}$ and $\chi(\mathfrak{m})$ are calibers of the T-space $X(\xi)$].

2. If the set Ξ is infinite, then

$$\chi(\mathfrak{m}) > \aleph_0$$

3. At least one of the following two conditions is satisfied:

   (a) $\mathfrak{m}$ is a regular cardinal number;

   (b) for any cardinal number $\mathfrak{a} < \mathfrak{m}$, there exists a cardinal number $\mathfrak{c}$ such that

$$\mathfrak{a} \lesssim \mathfrak{c} < \mathfrak{m}$$

   and

$$J(\mathfrak{c}) < \chi(\mathfrak{m})$$

   where J($\mathfrak{c}$) denotes the set of all elements $\xi \in \Xi$, to which $\mathfrak{c}$ is not a universal caliber of the T-space $X(\xi)$.

In this case $\mathfrak{m}$ is a caliber (and even universal caliber) of the product X^{Ξ}.»([5], p.60)

"**Definition 13.** Let us say that the topological space R is a diadic space if R is a T_2-space representable as a continious image of a product of compacts. Obviously, any diadic space is bicompact; therefore we shall also call diadic spaces d i a d i c b i c o m p a c t s".([5], p. 79.)

"**Theorem 51**. Every ordered dyadic bicompact is similar[3] to some closed bounded subset of the space of all real numbers".([5], p.92.)

What exactly seems unacceptable in these definitions and theorems from the point of view of the (not yet created at that time) "constructive direction in mathematics"? In a sense, "everything", i.e. the conceptual apparatus used in them, and above all the "classical" concepts of sets and functions. But this is a view from the future, when the "constructive direction" had already been formed.

During the period of work on his dissertation, N.A. Shanin could have been dissatisfied with much more specific aspects of these definitions and theorems, for example, the non-verifiability of the basic conditions, say, of the regularity of uncountable cardinals.

[3] In the sense of the theory of partially ordered sets.

This leads to questions about the acceptability of the "classical" definition of a function in such cases as the definition of $\chi(A)$ as a minimum over all "skeletons".

Consideration of the definition of a function emphasizes the difference between the classical and constructive understanding of existence - in what sense does this minimum exist? How is the set of "all" skeletons of a given set defined? And this "given set" itself.

The reaction to such difficulties was the priority development of the theory of algorithms, in our case - the theory of "normal algorithms" by A.A. Markov, the development of the theory of constructive real numbers based on the algorithmic approach, works on constructive metric and topological spaces (within the Markov-Shanin school)[4].

Sometimes it seems (recalling the famous words of V.I. Arnold) that the main contribution of constructivism is a number of key examples that emphasize the difference between the constructive and classical approaches, and which, probably, every educated mathematician should know.

A striking example of this type is the example of a non-compact closed ball in a compact constructive metric space, obtained by V.A. Lifshitz and V.P. Chernov [2], although in classical set-theoretic topology the statement that "a closed subspace of a bicompact space is bicompact"[1] is considered a well-known fact.

References

[1] A.V. Arkhangelsky. Bicompact Space. Mathematical Encyclopedy. V. 1 (A - Γ). (In Russian.) Editorial Board: I. M. Vinogradov (chef editor) [et al.] - Moscow, "Soviet Encyclopedy", 1977.

[2] V.A. Lifshits, V.P. Chernov, "A non-compact closed sphere in a constructive compact metric space". Journal of Soviet Mathematics, 1976, 6(4), p.390 - 394.

[3] V. I. Malykhin, V.I. Ponomarev. General Topology (set-theoretical direction). (Russian.), "Algebra, Topology, Geometry". Itogi Nauki i Tekniki, 1975, ,Vol. 13, 149-229.

[4] Note that the set of constructive real numbers is not uncountable. Within the framework of this theory, such statements as Theorem 51, cited above, turn out to be unfounded.

[4] A.A. Markov (Jr.). Selected works. Vol.II. Theory of Algorithms and Constructive Mathematics, Mathematical Logics, Informatics and Related Questions. - Moscow: MCCME Publishing, 2003.

[5] N. A. Shanin, On the product of topological spaces (in Russian). Trudy Mat. Inst. Steklov, Academy of Sciencies of the USSR, 24 (1948), 1-112.