BÁLINT TÓTH
(Rényi Institute Budapest and University of Bristol)

LARGE-SCALE BEHAVIOUR OF RANDOM MOTIONS WITH LONG MEMORY

-4-
LORENTZ GAS - BEYOND KINETIC TIME-SCALES
(A) LOW DENSITY / BOLTZMANN-GRAD LIMIT
(B) WEAK COUPLING LIMIT

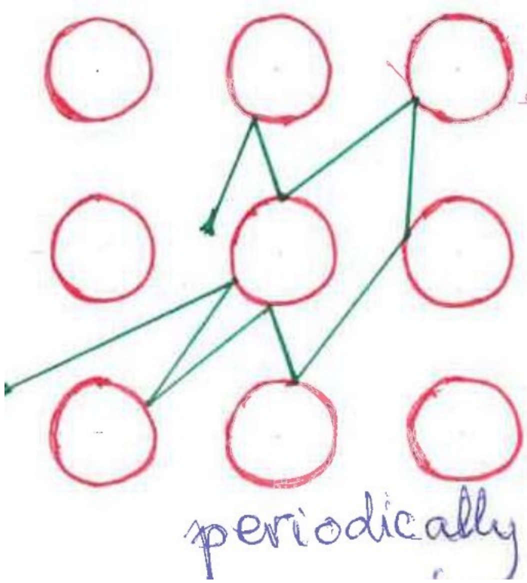
PDE AND PROBABILITY - SUMMER SCHOOL
Sorbonne Université, Paris, June 16-25, 2025

The Lorentz / Ehrenfest Gas - genesis

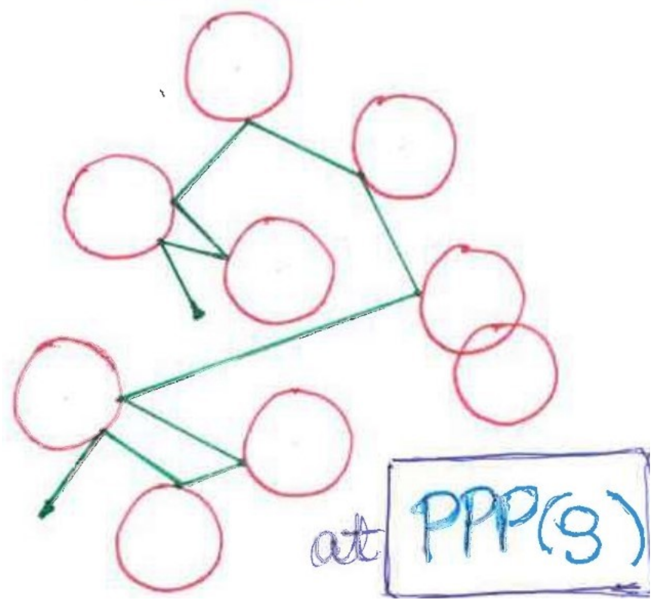
Physics. — "*The motion of electrons in metallic bodies.*" II. By Prof. H. A. LORENTZ.

(Communicated in the meeting of January 28, 1905).

Periodic



Random



Detour:

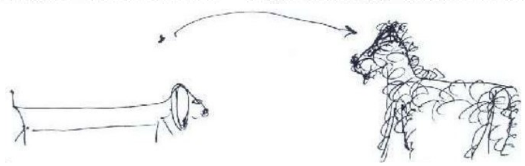
Tatyana Afanasieva (1876-1964)

Paul Ehrenfest (1880-1933)

1907:

Über zwei bekannte Einwände gegen das Boltzmannsche *H*-Theorem.

Von Paul u. Tatiana Ehrenfest.



Genesis of Markov Chains: . . . ,

AA Markov (1906), EH Bruns (1906),

P&T Ehrenfest (1907), O Perron (1907),

G Frobenius (1908), . . .

1911:

IV 32. BEGRIFFLICHE GRUNDLAGEN DER STATISTISCHEN AUFFASSUNG IN DER MECHANIK.

Von

P. u. T. EHRENFEST*)

IN ST. PETERSBURG.

In: F Klein (ed): *Encyklopädie der math. Wissenschaften* vol. 4-4

extended book in 1912

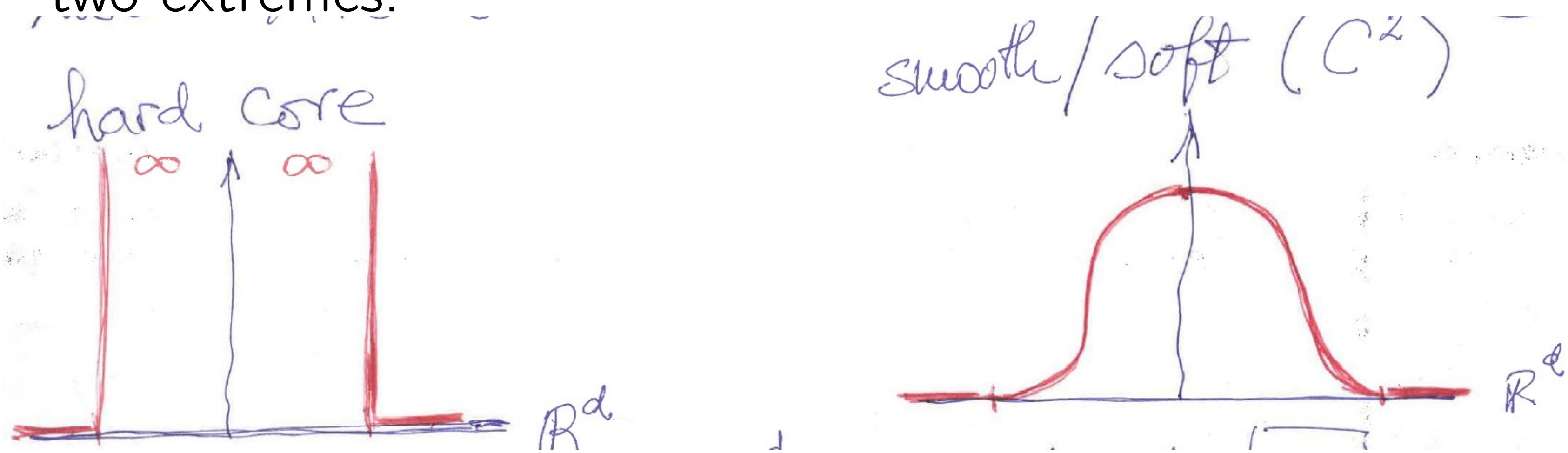
The random Lorentz gas:

Ingredients:

- A spherically symmetric finite range potential:

$$\varphi : \mathbb{R}^d \mapsto \mathbb{R} \cup \{+\infty\}, \qquad \varphi(x) = \varphi(|x|e) = \varphi(x)\mathbb{1}_{\{|x| \leq r\}}$$

two extremes:



- A PPP ω in $\mathbb{R}^d \setminus \{x : |x| \leq r\}$, of density ϱ.
  Points $q \in \omega$ will be the centres of fixed (∞-mass) scatterers.

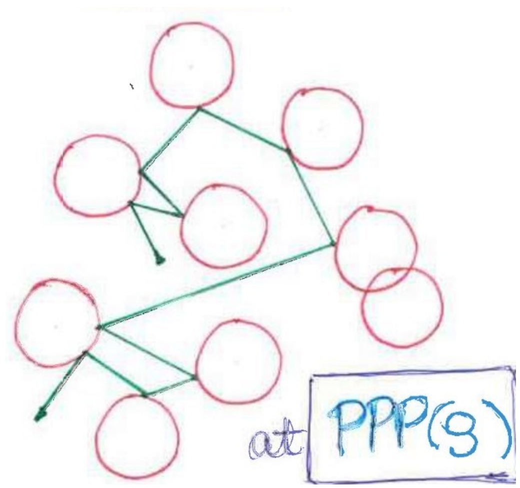
The Lorentz/Ehrenfest trajectory: Particle of mass 1 moves among the fixed scatterers, according to Newtonian dynamics $t \mapsto (V(t)), X(t))$, with i.c. $X(0) = 0 \in \mathbb{R}^d$, $V(0) \in \S^{d-1}$.

Soft case:

$$\Phi(x) := \sum_{q\in\omega} \varphi(x-q), \qquad F(x) = -\nabla\Phi(x) = -\sum_{q\in\omega} \nabla\varphi(x-q)$$

$$\dot V(t) = F(X(t)), \qquad \dot X(t) = V(t), \qquad + \text{i.c.}$$

Hard core case: the ODE is formal, nevertheless the dynamics is still (a.s.) well defined



No trapping:

hard core: $r^d\varrho < \theta_c$,

soft: $\max|\varphi| < m_c(r^d\varrho)$.

Sources of randomness:

- environment: random placement of scatterers, $\omega \sim \mathsf{PPP}(\varrho)$.
- random direction of initial velocity, e.g., $V(0) \sim \mathtt{UNI}(\mathbb{S}^{d-1})$.

and **nothing more**. Dynamics: fully deterministic, Newtonian.

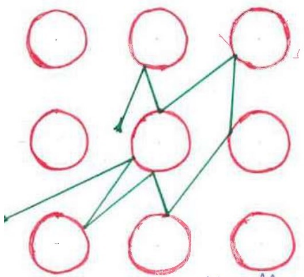
Wanted: $t \gg 1$ scaling behaviour of the trajectory $t \mapsto (V(t), X(t))$

Holy Grail: **?** $T^{-1/2} X(Tt) \Rightarrow W(t)$ **?**
(conditioned on no trapping)

annealed CLT/IP: averaged over $V(0)$ *and* scatterer config
semi-quenched: ave over $V(0)$, in prob. w.r.t. scatt. config
(fully) quenched: ave over $V(0)$, a.s. w.r.t. scatterer config

(A) Hard Core scatterers

Periodic (detour): Factorize on cell:
Sinai billiard, hyperbolic dynamics.
Big theory, since the 1970s



Source of randomness: $V(0) \sim \mathtt{UNI}(\mathbb{S}^{d-1})$

Finite horizon: [L Bunimovich, Ya Sinai (1980)]: $d = 2$
(conditional) [N Chernov, D Dolgopyat (2009)]: $d \geq 3$

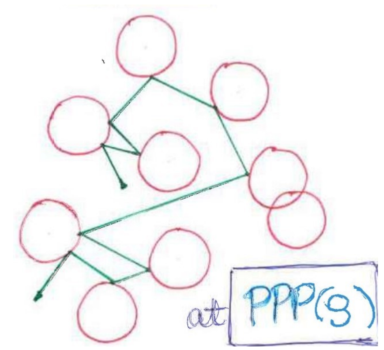
$$\frac{X(T\cdot)}{\sqrt{T}} \Rightarrow W(\cdot)$$

Infinite horizon: [P Bleher (1992)]: conjecture
[D Szász, T Varjú (2007)], [N Chernov, D Dolgopyat (2008)]: $d = 2$

$$\frac{X(T\cdot)}{\sqrt{T \log T}} \Rightarrow W(\cdot)$$

$d \geq 3$: wide open

Random: No dynamics tools – less understood



Source of randomness:

$\omega \sim \texttt{PPP}(\varrho)$ & $V(0) \sim \texttt{UNI}(\mathbb{S}^{d-1})$

Wanted: $t \gg 1$ scaling behaviour of the trajectory $t \mapsto (V(t), X(t))$

Holy Grail: ? $T^{-1/2} X(Tt) \Rightarrow W(t)$?

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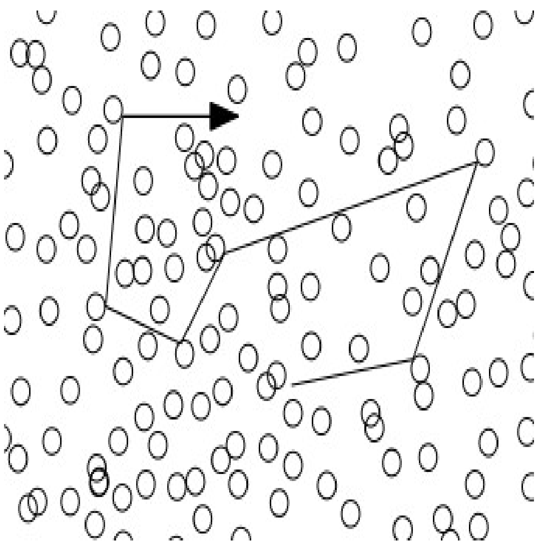
Kinetic limits I: Boltzmann-Grad / Low Density:

$$\varrho = \varepsilon^{-d}, \qquad r = \varepsilon^{d/(d-1)}, \qquad \underbrace{\varrho r^d = \varepsilon^{d/(d-1)}}_{\text{low density}}, \qquad \varepsilon \to 0 \qquad \text{(BGLIM)}$$

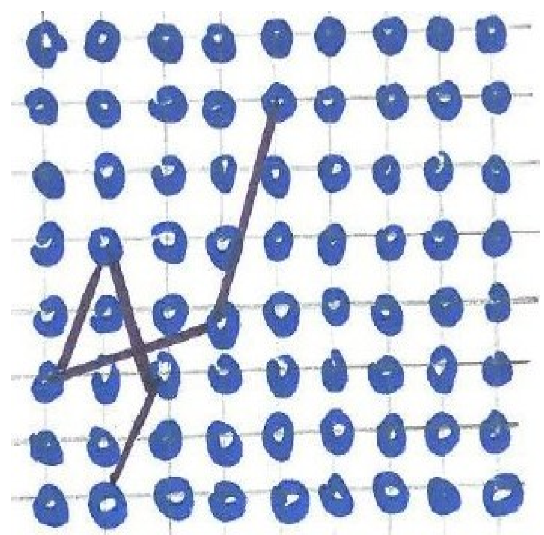
ε: microscopic (linear) length scale

In this limit the free flight between successive collisions is $\asymp 1$

Random:



Periodic:

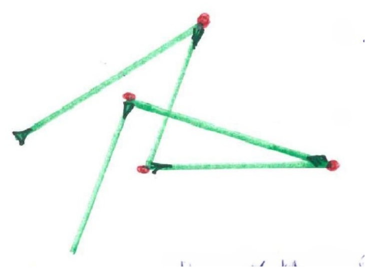


Random: Easy to guess . . .

$$\Big\{t \mapsto X(t) : t \in [0,T]\Big\} \overset{(\texttt{BGLIM})}{\Rightarrow} \Big\{t \mapsto Y(t) : t \in [0,T]\Big\}$$

$t \mapsto Y(t)$ = **Markovian random flight process:**

- i.i.d $\texttt{EXP}(1)$ flights, with $|v| = 1$
- Markovian scatterings with differential cross section $\sigma(v, v') \sim |v - v'|^{3-d}$
  Note: $d = 3$ is very special! (Archimedes ☺)

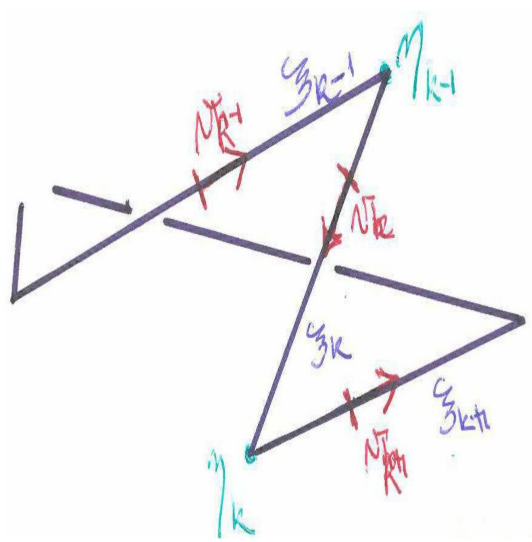


Hard to prove.
annealed, $d \geq 2$: [G Gallavotti (1970)], [H Spohn (1978)]
quenched, $d = 2$: [C Boldrighini, L Bunimovich, Ya Sinai (1982)]

Periodic: **Not so easy to guess . . .**

$$\Big\{t \mapsto X(t) : t \in [0,T]\Big\} \stackrel{\text{(BGLIM)}}{\Rightarrow} \Big\{t \mapsto Y(t) : t \in [0,T]\Big\}$$

$t \mapsto Y(t)$ = **"Hidden Markovian" random flight process:**



- $\eta_k \in \mathbb{B}^{d-1}$, $\quad v_k \in \mathbb{S}^{d-1}$, $\quad \xi_k \in \mathbb{R}_+$
- $(\eta_k)_{k\geq 0}$: Markov chain
- $v_k = R(v_0)\, S(\eta_0)\, \ldots\, S(\eta_{k-1})\, e_1$
- $\mathbf{E}\left(\prod_{k=1}^n f_k(\xi_k) | \underline{\eta}\right) = \prod_{k=1}^n \mathbf{E}\left(f_k(\xi_k) | \eta_{k-1}, \eta_k\right)$
- $\mathbf{P}\Big(\xi_k > x\Big) \sim x^{-2}$

[E Caglioti, F Golse (2008)] $d = 2$ [explicit formulas] . . . . . .

[J Marklof, A Strömbergsson (2011)] $d \geq 2$ [qualitative formulas]

Two-steps limit I.: first (BGLIM) then diffusive

Random: Since $t \mapsto Y(t)$ is essentially a rw, by Donsker's Thm

$$T^{-1/2} Y(T\cdot) \Rightarrow W(\cdot)$$

Periodic: more subtle. [Marklof-T (2016)]: For $d \geq 2$

$$(T \log T)^{-1/2} Y(T\cdot) \Rightarrow W(\cdot)$$

Log-correction due to the heavy tails.

Note: Two paths to superdiffusivity ...

Can one do better?

Random: Interpolate between the the (fully open) Holy Grail and the two-steps limit.

Periodic: For $d \geq 3$, infinite horizon: Interpolate between the conjectured (fully open) super-diffusive limit (with fixed scatterer-size) and the two-steps limit.

Interpolating IP for the rnd Lorentz gas (8 slides)

Theorem 1. `[Annealed IP]` [C Lutsko, BT (2020)]
Let $d = 3$, (`BGLIM`) *hold and* $T = T_\varepsilon$ *be such that* $\lim_{\varepsilon\to 0} T = \infty$ *and* $\lim_{\varepsilon\to 0} r^2|\log r|^2 T = 0$. *Then*

$$T^{-1/2}\, X(T\cdot) \overset{(\texttt{BGLIM})}{\Rightarrow} W(\cdot)$$

in the *<u>annealed</u>* *sense.*

Remarks:

◦ Up to $T = o(r^{-1})$ purely probabilistic: Green's fnc arguments. Still goes beyond [Gallavotti (1969)], [Spohn (1978)].

◦ For $r^{-1} \ll T \ll (r|\log r|)^{-2}$ geometry & dynamics matter.

◦ Can be extended to $d \geq 2$, up to $T \ll r^{1-d}|\log r|^{-\alpha}$.

◦ Can be extended to other short-range interactions.

Idea: Coupling

$t \mapsto Y(t)$ the Markovian flight process. $U(t) := \dot{Y}(t)$

$t \mapsto X(t)$ the Lorentz *exploration* process, constructed from $Y(\cdot)$, adapted to the filtration of $Y(\cdot)$. $V(t) := \dot{X}(t)$.

The construction is such that w.h.p.

- ∘ mismatches between $U(t)$ & $V(t)$ occur w' frequency $\sim r$
- ∘ after mismatches $U(t)$ & $V(t)$ are recoupled (to $U = V$) within an $\mathtt{EXP}(1)$ time

Up to $t < T(r) = o(r^{-1})$**:** no mismatch of $U(t)$ & $V(t)$ w.h.p.

$$\lim \mathbf{P}\Big(\inf\{t : X(t) \neq Y(t)\} < T \Big) = 0$$

Up to $t < T(r) = o((r|\log r|)^{-2})$**:** (hand waving argument)

$$\max_{0 \le t \le 1} \left| \frac{X(Tt)}{\sqrt{t}} - \frac{Y(Tt)}{\sqrt{t}} \right| \le \frac{1}{\sqrt{T}} \int_0^T |V(s) - U(s)| ds \approx \frac{1}{\sqrt{T}} Tr \to 0$$

The coupling - in plain words:

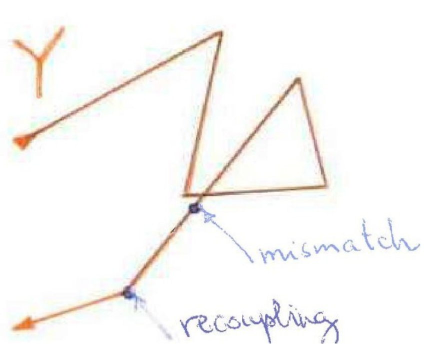
- $X(t)$ explores the environment on its way, trying to fly parallel with $Y(t)$ [trying to keep $V(t) = U(t)$] whenever possible.
- Explored areas are recorded and kept unchanged for ever.
- When in not-yet-explored "virgin" area, $X(t)$ behaves like $Y(t)$.
- When in already-explored-in-the-past area, $X(t)$ observes Newton's Laws.

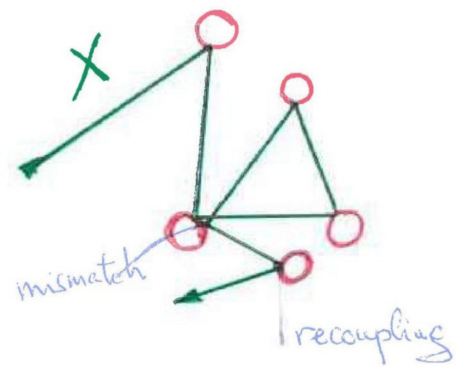
What can go wrong? . . . and the remedy . . .

See next slide.

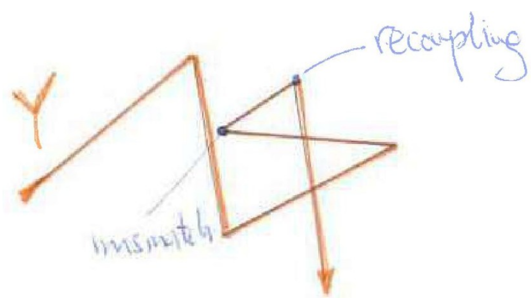
Mismatches and recouplings

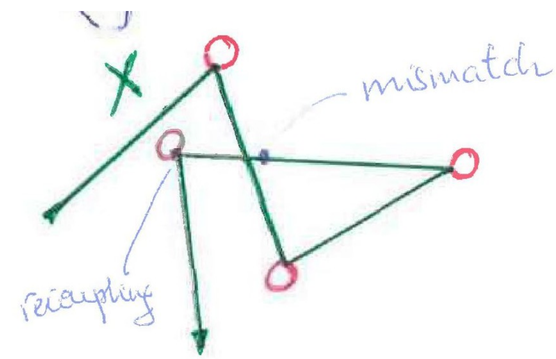
◦ Recollisions with past scatterers





◦ Shadowed scatterings





◦ Note: {recollision} $\leftrightarrow$ {shadowed scattering}, by time-reversal.

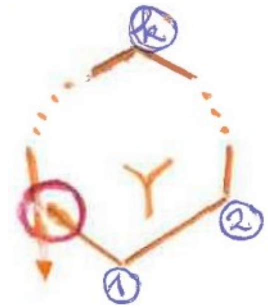
Theorem 2. [C Lutsko - BT (2020)], main thm
Setting: $d = 3$*, this coupling,* (BGLIM).
(i) $T(r) = o(r^{-1})$*:*

$$\lim \mathbf{P}\Big(\inf\{t : X(t) \neq Y(t)\} < T \Big) = 0.$$

(ii) $T(r) = o((r|\log r|)^{-2})$*:* $\forall \delta > 0$

$$\lim \mathbf{P}\Big(\max_{0 \le t \le T} |X(t) - Y(t)| > \delta\sqrt{T} \Big) = 0.$$

(i) Up to $T = o(r^{-1})$: purely probablilistic, no dynamical or geometric argument



$$\mathbf{P}\begin{pmatrix} Y \text{ returns to } r\text{-nb'hood} \\ \text{of starting point} \\ \text{after } \geq k \text{ scatterings} \end{pmatrix} \leq \begin{cases} C_k r^k & \text{if } k \leq d-2 \\ C_k r^k |\log r| & \text{if } k = d-1 \\ C_k r^{d-1} & \text{if } k \geq d \end{cases}$$

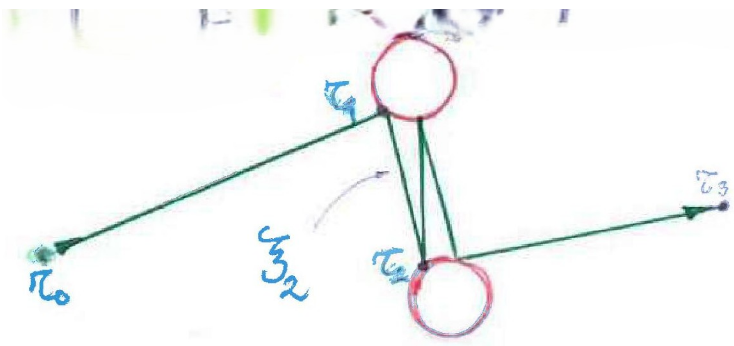
Plus: Green function estimates for the random walk Y

Plus: Union bounds.

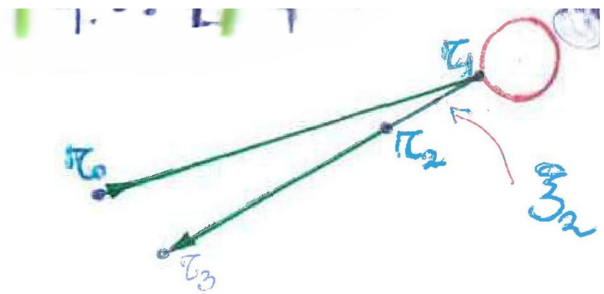
(Will see similar arguments in the wak coupling limit . . .)

(ii) Up to $T = o((r|log r|)^{-2})$ expect only

direct recollisions



direct shadowings



Moreover: $\xi_2 = o(1)$ (actually: $\xi_2 = \mathcal{O}(r)$)

We construct a **triple coupling** $t \mapsto (Y(t), Z(t), X(t))$ s.t.

- ⋆ Y is the Markovian flight process
- ⋆ X is the Lorentz exploration process
- ⋆ Z is a *myopic* version of X, which considers only direct recollisions & direct shadowings with $\xi_2 \leq 1$

- ∘ Data: $(\xi_j, v_j)_{0\le j<\infty}$ i.i.d, $(\xi_j, v_j) \sim \mathtt{EXP(1)} \times \mathtt{UNI}(\mathbb{S}^2)$.
- ∘ Break up the sequence into *independent legs*:

$$\dots\Big],\ \Big[\xi > 1, \xi > 1, \xi, \dots, \xi, \xi > 1, \xi > 1\Big],\ \Big[\dots$$

- ∘ Construct $t \mapsto (Y(t), Z(t), \mathcal{X}(t))$ *within each leg* and *concatenate*. Note: concat. $\mathcal{X} \neq X$!
- • [P&G&D] Within one leg: $\mathbf{P}\Big(\mathcal{X} \not\equiv Z\Big) < C(r|\log r|)^2$
- • [P] Interference between legs:

$$\mathbf{P}\Big(\text{a leg of } Z \text{ interferes with a past leg}\Big) < Cr^2$$

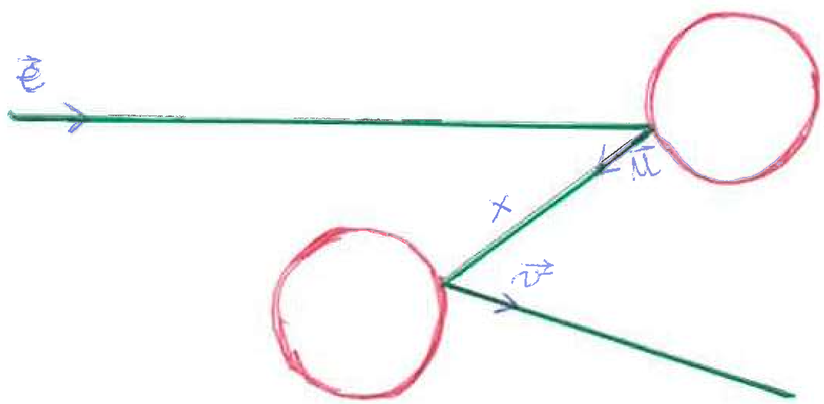
- • [P] $1 \ll T \ll r^{-2}$: $\lim \mathbf{P}\Big(\max_{0\le t\le T} |Y(t) - Z(t)| > \delta T\Big) = 0$.
- ∘ Putting together •&•&•, the result follows.

(17)

A geometric element:

Conditioning on recoll./shadowing

Space of double collisions



$\mathcal{D} = \{(\vec{u}, x, \vec{v}) \in S^{d-1} \times \mathbb{R}_+ \times S^{d-1}\}$

$d\nu = du \cdot dx \cdot dv$

$d\mu \sim e^{-x}\, d\nu$

$D_r : \mathcal{D} \to \mathcal{D},$

$D_r(\vec{u}, x, \vec{v}) = (\vec{u}, rx, \vec{v})$

(18)

$$\mathcal{A}_r := \{(\vec{u}, x, \vec{v}) : \text{second scattering shadowed}\}$$

$$= \left\{(\vec{u}, x, \vec{v}) : \operatorname{dist}\left(x \cdot \vec{u} + r\frac{\vec{u}-\vec{v}}{|\vec{u}-\vec{v}|}, \{-\vec{e}y : y > 0\}\right) < r\right\}$$

$$\mathcal{B}_r := \{(\vec{u}, x, \vec{v}) : \text{recollision would occur}\}$$

$$= \left\{(\vec{u}, x, \vec{v}) : \operatorname{dist}\left(r\frac{\vec{e}-\vec{u}}{|\vec{e}-\vec{u}|}, \{x\vec{u} + y\vec{v} : y \geq 0\}\right) < r\right\}$$

$$\mathcal{C}_r := \mathcal{B}_r \setminus \mathcal{A}_r$$

Note: $\mathcal{A}_r = D_r \mathcal{A}_1$, $\mathcal{B}_r = D_r \mathcal{B}_1$, $\mathcal{C}_r = D_r \mathcal{C}_1$

Relevant random variables

(19)

on $(\mathcal{D}, \mu)$:

$\vec{w}_r :=$ escape velocity | conditioned on

$\beta_r :=$ trapping time | $\mathcal{A}_r$ resp. $\mathcal{C}_r$

<u>Fact</u>: $\boxed{d=2}$ $\nu(\mathcal{A}_r) = \nu(\mathcal{B}_r) = \nu(\mathcal{C}_r) = \infty$

$\boxed{d=3}$ $\nu(\mathcal{A}_r), \nu(\mathcal{B}_r), \nu(\mathcal{C}_r) < \infty$

$$\nu_{\mathcal{A}} := \frac{\nu|_{\mathcal{A}_1}}{\nu(\mathcal{A}_1)} \qquad \nu_{\mathcal{C}} := \frac{\nu|_{\mathcal{C}_1}}{\nu(\mathcal{C}_1)}$$

are bona fide probability measures

(20)

... and

$$D_r^{-1}\mu(\ |\mathcal{A}_r) \Rightarrow \nu_{\mathcal{A}}\ ,\quad D_r^{-1}\mu(\ |\mathcal{C}_r) \Rightarrow \nu_{\mathcal{C}}$$

Follows:

$$(r^{-1}\beta_r\,|\,\mathcal{A}_r)_\mu \Rightarrow (\beta\,|\,\mathcal{A})_\nu\ ,\quad (r^{-1}\beta_r\,|\,\mathcal{C}_r)_\mu \Rightarrow (\beta\,|\,\mathcal{C})_\nu$$

$$(\vec{w}_r\,|\,\mathcal{A}_r)_\mu \Rightarrow (\vec{w}\,|\,\mathcal{A})_\nu\ ,\quad (\vec{w}_r\,|\,\mathcal{C}_r)_\mu \Rightarrow (\vec{w}\,|\,\mathcal{C})_\nu$$

Wanted:

$$\mathbb{E}_\nu(\beta\,|\,\mathcal{A}) < \infty\ ,\quad \mathbb{E}_\nu(\beta\,|\,\mathcal{C}) < \infty$$

$\text{distrib}_\nu(\vec{w}\,|\,\mathcal{A})$, $\text{distrib}_\nu(\vec{w}\,|\,\mathcal{C})$ not too strongly concentrated around $w = -e$

genuine geometric arguments

Other interactions, and/or $d \neq 3$: (2 slides)

Spherical scatterers in $d = 3$ are special (Archimedes ☺) since

$$\sigma(v, v')dv' = |v - v'|^{3-d} dv'$$

If Döblin's condition $\sigma(v, v')dv' > c dv'$ holds, apply Döblin's trick: *Break up Y into independent legs*. Essentially the same probabilistic estimates work.

Applications:

(1) Ehrenfest's Wind-Tree model:

$d = 2$ $\Diamond$-scatterers $v \in \{\rightarrow, \uparrow, \leftarrow, \downarrow\}$

[Lutsko - T (2021)]: IP up to $T = o(r^{-1})$.
Compare with the "mirror model" on $\mathbb{Z}^d$.
[D Elboim, A Gloria, P Hernandez (2025)]: IP for the mirror model under (BGLIM): $d \geq 5$, up to $T = \mathcal{O}(r^{-N})$, $N < \infty$!

(2) Spherical scatterers, $d \geq 4$. [Not written up]
Note however, that $T = o(r^{1-d}|\log r|^{-\alpha})$ is a strict borderline for this method.

(3) Lorentz gas in $d = 2$, *in transversal magnetic field*.
Kinetic time scale, $T = \mathcal{O}(1)$: [Bobylev et al. (1995)] . . .
[A Nota, C Saffirio, S Simonella (2021)]
alt. proof & IP up to $T(r) = o((r|\log r|^2)^{-1})$ [L-T (2024)]

If Döblin's condition does not hold for σ but holds for $\sigma * \sigma$:
Break up Y into one-dependent legs. More tricky:
Green's fnc estimates for RWs with one-dependent steps needed.

Application:

(4) $d = 2$, spherical scatterers, up to $T(r) = o((r|\log r|^2)^{-1})$
[Not written up]

Semi-quenched: (4 slides)

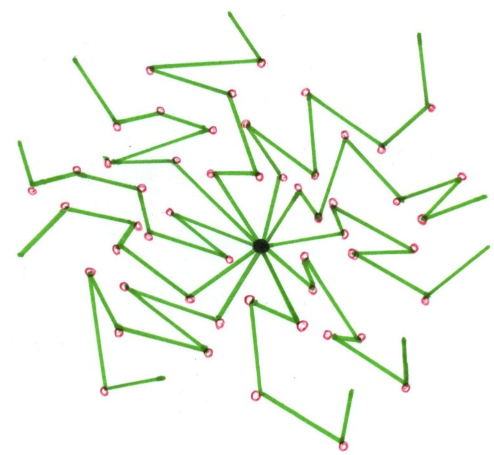
- $d = 3$; scatterers: $r = \varepsilon^{3/2}$ centred at $\{\varepsilon q : q \in \varpi \sim \texttt{PPP}(1)\}$,
- $t \mapsto X(t)$: the Lorentz traj. with $\dot{X}(0) = \texttt{UNI}(\{v : \angle(v, e) \le \beta\})$.

Theorem 3. `[Semi-quenched IP]` [T (2025)]
If $\varepsilon_n \to 0$, $T_n \to \infty$, $\beta_n \in (0, \pi]$ *are such that*

$$\sum_{n=1}^{\infty} \left(r_n T_n \log n + (r_n \beta_n^{-1})^{2/3} (\log n)^2 \right) < \infty$$

then for almost all realizations of the PPP ϖ,

$$T^{-1/2} X(T\cdot) \overset{(\texttt{BGLIM})}{\Rightarrow} W(\cdot)$$



Convert annealed to (semi)quenched IP by *joint exploration*.

Quenched coupling: On an enlarged $(\Omega, \mathcal{F}, \mathbf{P})$ realize jointly

$$\Big((\varpi, (X_j(t) : 1 \le j \le N, 0 \le t \le T)), \Big((Y_j(t) : 1 \le j \le N, 0 \le t \le T)\Big)$$

- ϖ: a $\mathtt{PPP}(1)$ in $\mathbb{R}^3$
- X_j: Lorentz trajectories among scatterers of rad. $r = \varepsilon^{3/2}$ centred at $\varepsilon\varpi$, with i.c. $X_j(0) = 0$, $\dot{X}_j(0) = v_j \in \mathbb{S}^2$ (possibly also random). $w := \min\{\angle(v_i, v_j) : 1 \le i < j \le N\}$
- Y_j: i.i.d flight processes, with the same i.c.
- the coupling: at the blackboard . . .
- time of first mismatch: $\tau := \min\{t \in [0, T] : X(t) \ne Y(t)\}$. Note: τ is (actually) $\mathcal{F}(Y)$-measurable!
- Key bound: $\mathbf{P}\Big(\tau < T\Big) \le C(NrT + N^2 r w^{-1})$

Putting the bits together:

◦ **Choose** N_n such that

$$\lim_{n\to\infty} N_n(\log n)^{-1} = \infty \qquad (*)$$

$$\sum_n \Big(N_n r_n T_n + N_n^2 \left(r_n \beta_n^{-1}\right)^{(d-1)/d} \Big) < \infty$$

• **Borel-Cantelli:** With $\alpha_n := r_n^{1/d} \beta_n^{(d-1)/d}$ we get

$$\mathbf{P}\Big(\tau_n < T_n\Big) \le \mathbf{P}\Big(w_n < \alpha_n\Big) + \mathbf{P}\Big(\{\tau_n < T_n\} \cap \{w_n \ge \alpha_n\}\Big)$$

$$\le C N_n^2 (\alpha_n \beta_n^{-1})^{d-1} + C(N_n r_n T_n + N_n^2 r_n \alpha_n^{-1})$$

$$\le C(N_n r_n T_n + N_n^2 (r_n \beta_n^{-1})^{(d-1)/d})$$

and $$\mathbf{P}\Big(\max\{n : \tau_n < T_n\} < \infty\Big) = 1$$

• **SLLN for △-ar arrays:** Under (*), a.s., for any $F \in \mathcal{C}_0(\mathcal{C})$,

$$\lim_{n\to\infty} \left(N_n^{-1} \sum_{j=1}^{N_n} F(T_n^{-1/2} Y_{n,j}(T_n \cdot)) - \mathbf{E}\left(F(T_n^{-1/2} Y_{n,1}(T_n \cdot)) \right) \right) = 0$$

$$\lim_{n\to\infty} \left(N_n^{-1} \sum_{j=1}^{N_n} F(T_n^{-1/2} X_{n,j}(T_n \cdot)) - \mathbf{E}_\omega \left(F(T_n^{-1/2} X_{n,1}(T_n \cdot)) \right) \right) = 0$$

• **Donsker:** $\lim_{n\to\infty} \mathbf{E} \left(F(T_n^{-1/2} Y_{n,1}(T_n \cdot)) \right) = \mathbf{E}\left(F(W(\cdot)) \right)$

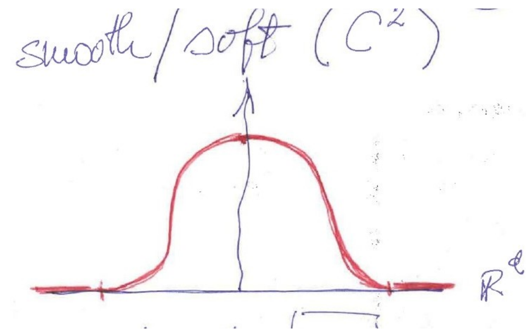
∘ Putting together •&•&•, the result follows. □Thm 3

Remarks:

(1) Theorem 3 [$d = 3$, $T = o(r^{-1})$, semi-quenched]
to be compared with [Boldrighini-Bunimovich-Sinai (1982)]
[$d = 2$, $T = \mathcal{O}(1)$, fully quenched]

(2) With harder work (Döblin trick) Thm 3 extendsed to $d \geq 2$.

(B) Soft scatterers and weak coupling



- A spherically symmetric, smooth finite range potential: $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$,

$$\varphi(x) = \varphi(|x|e) = \varphi(x)\mathbb{1}_{\{|x|\le r\}}$$

- A $\mathtt{PPP}(\varrho)$, $\omega \subset \mathbb{R}^d \setminus \{x : |x| \le r\}$. Points $q \in \omega$ will be the centres of fixed (∞-mass) scatterers.
- The (overall) potential and force field

$$\Phi(x) := \sum_{q\in\omega} \varphi(x-q), \qquad F(x) = -\nabla\Phi(x) = -\sum_{q\in\omega} \nabla\varphi(x-q)$$

- The Lorentz/Ehrenfest trajectory: $t \mapsto (V(t)), X(t))$

$$\dot{V}(t) = F(X(t)), \qquad \dot{X}(t) = V(t), \qquad + \text{i.c.}$$

- Condition on no trapping: $\max|\varphi| < m_c(r^d\varrho)$.

Kinetic limits II. Weak Coupling:

$$\varrho = \varepsilon^{-d}, \qquad r = \varepsilon, \qquad \underbrace{\text{intensity of potential} \sim \varepsilon^{1/2}}_{\text{weak coupling}} \qquad (\texttt{WCLIM})$$

$$\Phi_\varepsilon(x) := \varepsilon^{1/2} \sum_{q \in \varepsilon \cdot \omega} \varphi(\frac{x-q}{\varepsilon}) \qquad \sim \varepsilon^{1/2},$$

$$F_\varepsilon(x) = -\varepsilon^{-1/2} \sum_{q \in \varepsilon \cdot \omega} \nabla\varphi(\frac{x-q}{\varepsilon}) \quad \sim \varepsilon^{-1/2},$$

The trajectory under (WCLIM):

$$\dot V_\varepsilon(t) = F_\varepsilon(X_\varepsilon(t)), \qquad \dot X_\varepsilon(t) = V_\varepsilon(t), \qquad + \text{i.c.}$$

Let's guess the limit together.

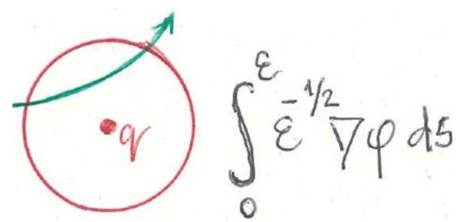
∘ Conservation of energy:

$$\underbrace{|V_\varepsilon(t)|^2}_{E_{\text{kin}}} + \underbrace{\Phi_\varepsilon(X_\varepsilon(t))}_{E_{\text{pot}} \sim \varepsilon^{1/2}} = 1$$

The particle travels with speed $|V_\varepsilon(t)| = 1 - \mathcal{O}(\varepsilon^{1/2})$.

∘ In (infinitesimal) time dt it encounters $\sim \varepsilon^{-1} dt$ scatterers.

∘ Each scatterer has impact $\sim \varepsilon^{1/2}$ on $V_\varepsilon(t)$:



The expected limit: *Spherical Langevin Process*:

$t \mapsto U(t)$: Wiener ("BM") on $\mathbb{S}^{d-1}$, $\qquad Y(t) = \int_0^t U(s)ds$.

Not so easy to guess. Even harder to prove.

[H Kesten, G Papanicolaou (1980)]

Two-steps limit II.: first (WCLIM) then diffusive

[KP80] : $(V_\varepsilon(t), X_\varepsilon(t)) \Rightarrow \underbrace{(U(t), Y(t))}_{\text{spherical Langevin proc.}}$

Doeblin : $T^{-1/2} Y(Tt) \Rightarrow W(t).$

Can one do better? in the (WCLIM) setting

$$? \quad T(\varepsilon)^{-1/2} X_\varepsilon(T(\varepsilon)t) \Rightarrow W(t) \quad ? \qquad \text{(INTERPOL)}$$

with $T(\varepsilon) \to \infty$ – the faster the better.

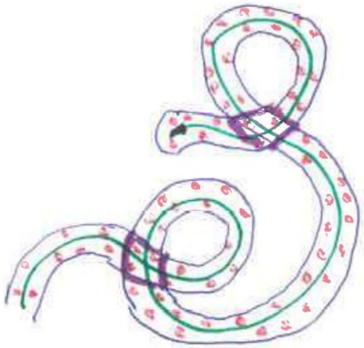
- [T Komorowski, L Ryzhyk (2006)]: $d \geq 3$, $T(\varepsilon) = \varepsilon^{-\kappa}$, $\kappa > 0$
- [L Erdős, M Salmhofer, H-T Yau (2007)]: q-setting, $\kappa \approx 1/370$.

Theorem 4. [annealed IP in WC setting] [BT (2025+)]
Let $d \geq 3$. *Under* (WCLIM), (INTERPOL) *holds with* $T(\varepsilon) = \varepsilon^{-(d-2)}$

Explore!

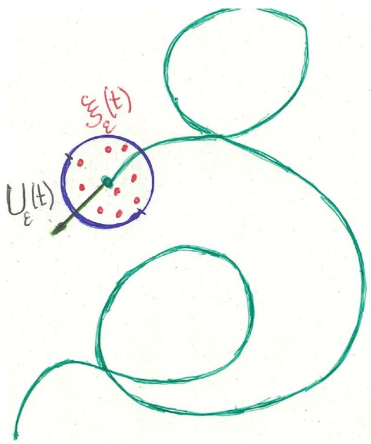
Rather than sample . . .

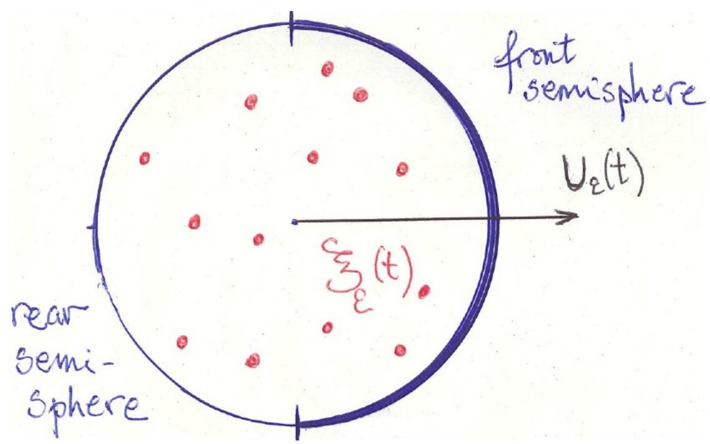
better explore the environment!



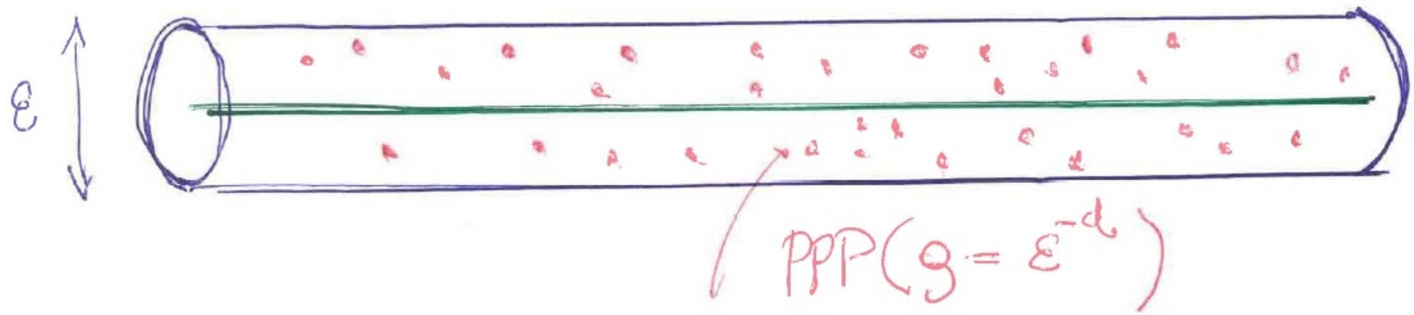
Markovize!

$t \mapsto (U_\varepsilon(t), \xi_\varepsilon(t))$





Probabilistic ingredient for the construction of the Markovized process:



- explicit construction
- the MP $t \mapsto (U_\varepsilon(t), \xi_\varepsilon(t))$ is well-behaved due to
- $\theta_{\varepsilon,n} =$ successive times when $\xi_\varepsilon(t) = \emptyset$.

  $|\theta_{\varepsilon,n+1} - \theta_{\varepsilon,n}| \sim \varepsilon, \quad n \mapsto U_\varepsilon(\theta_{\varepsilon,n})$ is a $O(d)$-invar. RW on $\S^{d-1}$

Limit theorems for the Markovized process.

(i) *Fix* $0 < T < \infty$. *Then, as* $\varepsilon \to 0$,

$$(U_\varepsilon(t), Y_\varepsilon(t)) \Rightarrow \underbrace{(U(t), Y(t))}_{\text{spherical Langevin proc.}}$$

[Key: CLT for RW on $O(d)$.]

(ii) *Let* $T(\varepsilon) \to \infty$ (no matter how fast or slow). *Then, as* $\varepsilon \to 0$,

$$T(\varepsilon)^{-1/2} Y_\varepsilon(T(\varepsilon)t) \Rightarrow W(t)$$

[Key: Martingale approximation + martingale CLT.]

Nothing new or surprising here.

Couple! (the physical and the Markovized processes)

To be proven: Up to $t < T(\varepsilon) = o(\varepsilon^{-d+2})$, with high probability, no ε-neighbourhood of a point left behind is revisited by the Markovized process $t \mapsto Y_\varepsilon(t)$:

$$\Sigma_\varepsilon := \inf\{t : 0 < \exists\, r < \exists\, s < t, \text{such that}$$

$$B_\varepsilon(Y_\varepsilon(r)) \cap B_\varepsilon(Y_\varepsilon(s))^{\mathsf{c}} \cap B_\varepsilon(Y_\varepsilon(t)) \neq \emptyset\}$$

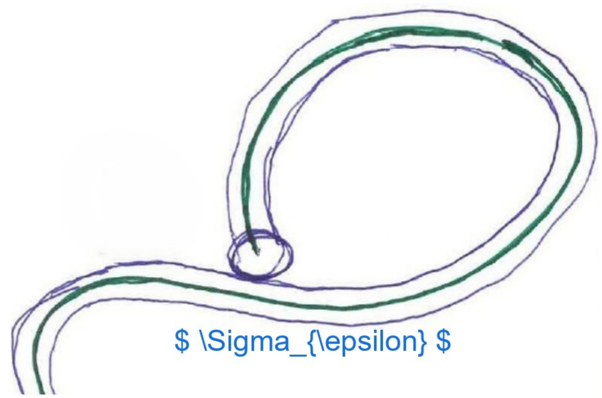
$=$ the first time (t) when a point which was within range ε some time (r) in the past, and left behind (at time $s > r$), is revisited within range ε (at time $t > s$).

By construction (coupling): $\inf\{t : V_\varepsilon(t) \neq U_\varepsilon(t)\} \geq \Sigma_\varepsilon$

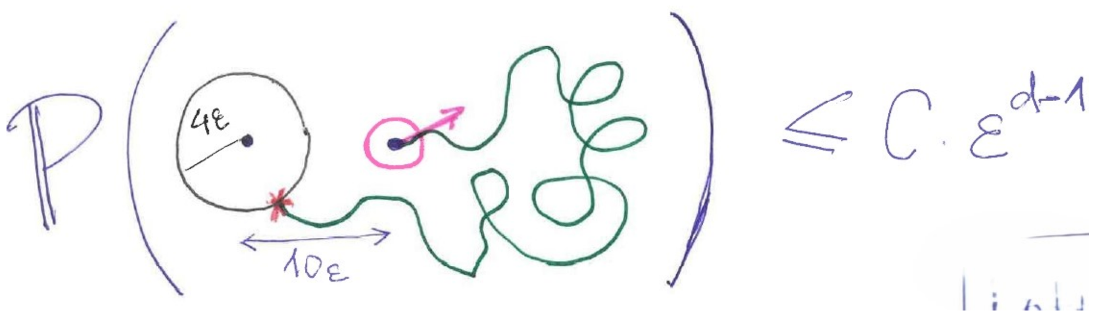
Lower bound on Σ_ε is needed. However, Σ_ε can in principle be very small, if the trajectory $t \mapsto Y_\varepsilon(t)$ is too rough.

Geometry helps:

$$|\ddot{Y}_\varepsilon| = |\dot{U}_\varepsilon| \sim \varepsilon^{-1/2} \ll \varepsilon^{-1}$$



The main probabilistic input



(note the difference from BM)

relies on Green-function (for $t \mapsto Y_{\varepsilon(t)}$) and geometric estimates

Hence (by union bounds and some massaging) the key estimate

$$\mathbf{P}\Big(\Sigma_\varepsilon < T\Big) < CT\varepsilon^{d-2}. \qquad ☺$$