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LARGE-SCALE BEHAVIOUR OF RANDOM MOTIONS WITH LONG MEMORY

-1-
INTRODUCTION, MOTIVATION, EXAMPLES

PDE AND PROBABILITY - SUMMER SCHOOL
Sorbonne Université, Paris, June 16-25, 2025

Goal: Mathematical understanding of physical diffusion.

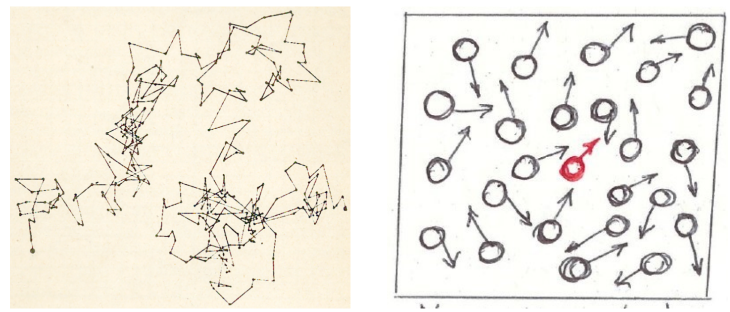
Three very different examples, at very different scales.

(A) Molecular diffusion, Brownian motion:

Macro

[Wikipedia/public]

Micro [$\sim 10^{-9}$-10^{-8} m]



[J Perrin, Les atomes, 1910]

Empirical:

. . . [J Ingenhousz (1785)] . . . [R Brown (1827)] . . .

Theoretical:

. . . [A Fick (1855)] . . . [A Einstein (1905)] . . .

. . . [M Smoluchowski (1906)] . . . [P Langevin (1908)] . . .

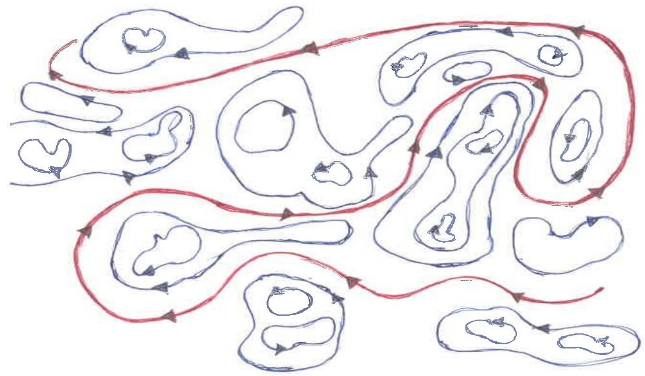
. . . [J Perrin (1910)] . . .

(B) Convective (or, "turbulent") diffusion:

Macro

Hommage à Magritte

Micro [$\sim 10^{-5}$-10^{-4}m]



Empirical:

[Lucretius Carus ($\sim$ 60 BC)] . . .

. . . [L da Vinci (1510-12)] . . .

Theoretical:

. . . [JV Boussinesq (1870s)] . . . [H Bénard (1900)] . . .

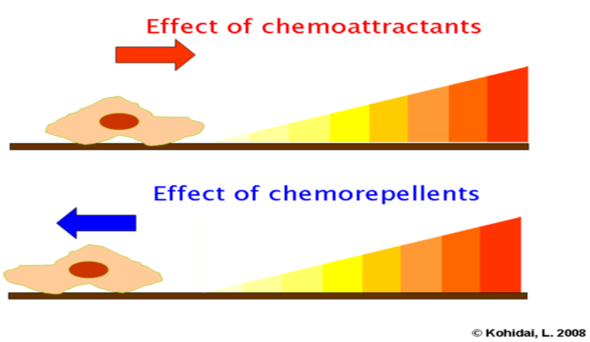
. . . [M Smoluchowski (1906)] . . . [T v. Kármán (1930s)] . . .

(C) (Negative) Chemotaxis (or, chemorepulsion - in bio):

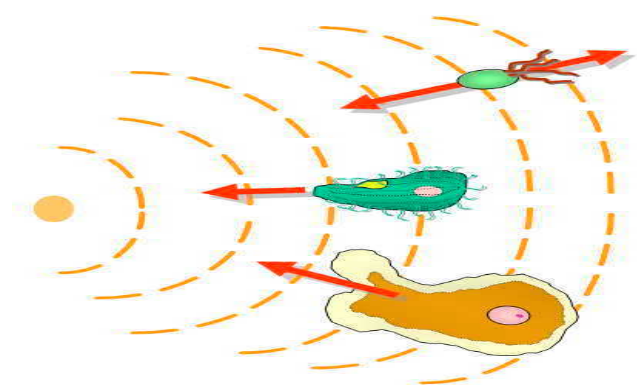
"Negative chemotaxis (a.k.a. chemorepulsion) occurs when cells are exposed to a spatial gradient of a signaling molecule and move down-gradient, toward regions with lower concentration."

The medium may be influenced by the active population.

Schematic



Micro [$\sim 10^{-4}$m]



[from Wikipedia, by L Kohidai]

Denote $t \mapsto X(t) \in \mathbb{R}^d$ the trajectory of the *tracer*. This is a *stochastic process*, with randomness coming from

- ∘ initial conditions, and
- ∘ [possibly] randomized dynamics
- ∘ thermal fluctuations

Goal: Understand the *scaling limit*, as $\varepsilon \to 0$, of

$$X_\varepsilon(t) := \varepsilon^\nu X(\varepsilon^{-1} t).$$

- ∘ **diffusive**: $\nu = 1/2, \quad X_\varepsilon(\cdot) \Rightarrow W(\cdot) \quad (t \mapsto W(t)$ "BM")
- ∘ **super-diffusive**: $\nu > 1/2$ (robust), $\varepsilon^{1/2} |\log \varepsilon|^\gamma$ (borderline)
- ∘ **sub-diffusive**: $\nu < 1/2$ (robust), $\varepsilon^{1/2} |\log \varepsilon|^{-\gamma}$ (borderline)

Why $\nu = 1/2$? Why "BM"?:

The process $t \mapsto W(t) \in \mathbb{R}^d$ is characterized by

∘ *independent and stationary increments*:
Let $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, then the increments $\Big(W(t_j) - W(t_{j-1})\Big)$, $1 \leq j \leq n$, are independent and their distribution doesn't depend on t_0.

∘ *finite second moment*

$$\begin{aligned} \mathbf{E}\Big(W(t_j) - W(t_{j-1})\Big) &= 0 \\ \mathbf{Var}\Big(W(t_j) - W(t_{j-1})\Big) &= \sigma^2(t_j - t_{j-1}) \end{aligned} \tag{1}$$

∘ the path $t \mapsto W(t)$ is almost surely *continuous*.

Bonus: *A fortiori*: $t \mapsto W(t) \in \mathbb{R}^d$ is a *Gaussian process*, specified by (1).

. . . the bottomless well of the past . . .

. . . [J Bernoulli (1713)] . . . [A de Moivre (1711-1738)] . . .

. . . [PS Laplace (....-1812)] . . . [A Lyapunov (1901)] . . .

. . . [C Pearson & Lord Rayleigh (1905)] . . . [G Pólya (1922)] . . .

. . . [N Wiener (1920-1933)] . . . [P Lévy (1920-1940)] . . .

. . . [M Kac (1946-....)] . . . [M Donsker (1952)] . . .

However:
The increments of the physical processes are by no means independent. The (simple) random walk picture is too naive.

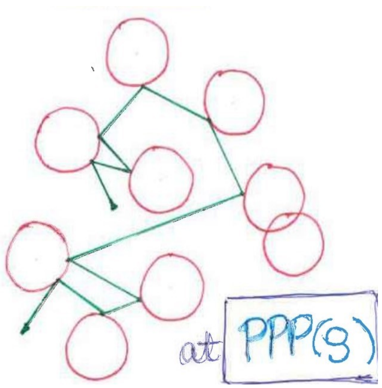
Examples of models which are mathematically tractable at various levels

(A) Molecular diffusion, Brownian motion.

(A1) Hard ball gas. Cf. Thierry's lectures.

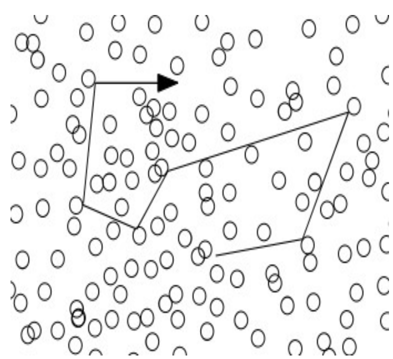
(A2) The random Lorentz gas. Hamiltonian dyn., random i.c.

(A2-BG) low density (Boltzmann-Grad) + diffusive limit.



$$\varrho r^d < \theta_{crit}$$

$$t \mapsto X(t)$$



The BG limit:

$$\varrho = \varepsilon^{-d}$$

$$r = \varepsilon^{d/(d-1)}$$

$$t \mapsto X_\varepsilon(t)$$

[G Gallavotti (1970)], [H Spohn (1979)]: $d \geq 2$.

$$\Big((V_\varepsilon(t), X_\varepsilon(t)) : t \in [0,T]\Big) \overset{\text{a}}{\Rightarrow} \underbrace{\Big((U(t), Y(t) : t \in [0,T])\Big)}_{\text{random flight proc.}} =$$

[C Boldrighini, L Bunimovich, Y Sinai (1982)]: $d = 2$.

$$\Big((V_\varepsilon(t), X_\varepsilon(t)) : t \in [0,T]\Big) \overset{\text{q}}{\Rightarrow} \underbrace{\Big((U(t), Y(t) : t \in [0,T])\Big)}_{\text{random flight proc.}} =$$

[C Lutsko, BT (2020)]: $d = 3$, $1 \ll T_\varepsilon \ll \varepsilon^{-3} |\log \varepsilon|^{-2}$.

$$\Big(T_\varepsilon^{-1/2} X_\varepsilon(T_\varepsilon t) : t \in [0,\infty)\Big) \overset{\text{a}}{\Rightarrow} \Big(W_\sigma(t) : t \in [0,\infty)\Big)$$

[BT (2025)]: $d = 3$, $1 \ll T_\varepsilon \ll \varepsilon^{-2}$.

$$\Big(T_\varepsilon^{-1/2} X_\varepsilon(T_\varepsilon t) : t \in [0,\infty)\Big) \overset{\text{sq}}{\Rightarrow} \Big(W_\sigma(t) : t \in [0,\infty)\Big)$$

(A2-WC) Weak coupling + diffusive limit.
Same, with soft/smooth finite range potential, φ.

$$\varrho = \varepsilon^{-d}, \qquad r = \varepsilon, \qquad \underbrace{\text{intensity of potential} \sim \varepsilon^{1/2}}_{\text{weak coupling}}$$

$$\Phi_\varepsilon(x) := \varepsilon^{1/2} \sum_{q \in \varepsilon \cdot \omega} \varphi(\frac{x-q}{\varepsilon}) \qquad \sim \varepsilon^{1/2},$$

$$F_\varepsilon(x) = -\varepsilon^{-1/2} \sum_{q \in \varepsilon \cdot \omega} \nabla \varphi(\frac{x-q}{\varepsilon}) \sim \varepsilon^{-1/2},$$

$$\dot{V}_\varepsilon(t) = F_\varepsilon(X_\varepsilon(t)), \qquad \dot{X}_\varepsilon(t) = V_\varepsilon(t), \qquad + \ \text{i.c.}$$

Note: (1) $|V_\varepsilon(t)|^2 = |V_\varepsilon(0)|^2 + \mathcal{O}(\varepsilon)$. (2) In time dt, $\sim \varepsilon^{-1} dt$ scatterings of impact $\sim \varepsilon^{1/2}$ on the travelling velocity.

[H Kesten, G Papanicolaou (1980)]: $d \geq 3$.

$$\Big((V_\varepsilon(t), X_\varepsilon(t)) : t \in [0,T] \Big) \overset{\text{a}}{\Rightarrow} \underbrace{\Big((U(t), Y(t) : t \in [0,T]) \Big)}_{\text{spherical Langevin proc.}}$$

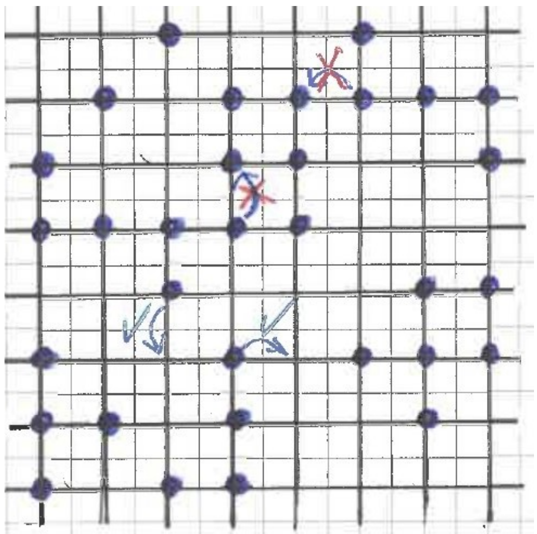
SLP: $\quad dU(t) = U(t) \times dB(t), \qquad dY(t) = U(t) dt.$

[T Komorowski, L Ryzhik (2007)], [BT (2025+)]:
$d \geq 3$, $1 \ll T_\varepsilon \ll \varepsilon^{-(d-2)}$. ([KR]: $1 \ll T_\varepsilon \ll \varepsilon^{-\kappa}$, $\kappa > 0$.)

$$\Big(T_\varepsilon^{-1/2} X_\varepsilon(T_\varepsilon t) : t \in [0,\infty) \Big) \overset{\text{a}}{\Rightarrow} \Big(W_\sigma(t) : t \in [0,\infty) \Big)$$

(A3) Tagged particle diffusion in stochastic interacting particle systems. The *exclusion process* (random dynamics!)

$$t \mapsto \eta_t \in \Omega := \{0,1\}^{\mathbb{Z}^d} \qquad \text{MP}$$



- particle configurations: $\omega \in \Omega$
- $(p_z)_{z\in\mathbb{Z}^d}$: a finite range rw kernel
- particles attempt jumps $x \rightsquigarrow y$

  independently, with rate p_{y-x}. If the target site is free then the jump is performed, otherwise it is suppressed.

Fact: $\mathtt{BER}(\varrho)$, $\varrho \in (0,1)$, are time-wise ergodic measures for the MP $t \mapsto \eta_t$. These are the only ones which are also space-wise ergodic. Also, seen from the position of a tagged particle.

Goal: Scaling limit for the trajectory of $t \mapsto X(t)$ of a tagged particle.

SEP: $p_z \equiv p_{-z}$ (**symmetric**/reversible/self-adjoint)

[R Arratia (1983)]: $d = 1$, only nearest neighbour jumps.

$$T^{-1/4} X(T\cdot) \Rightarrow \text{frBM} \qquad \text{subdiffusive!}$$

Method: "Combinatorial-probabilistic" reminiscent of [T Harris (1965)].

[C Kipnis, SRS Varadhan (1986-)]: $d = 1$ and not n.n., or $d \geq 2$.

$$T^{-1/2} X(T\cdot) \Rightarrow W_\sigma(\cdot).$$

Method: KV martingale approximation - reversible setting.

ASEP: $p_z \neq p_{-z}$ (**asymmetric**/not reversible/not self-adjoint)

[SRS Varadhan (1996)] $d \geq 1$, zero mean $\sum_z z p_z = 0$.

$$T^{-1/2} X(T\cdot) \Rightarrow W_\sigma(\cdot).$$

Method: KV martingale approximation - nonreversible setting,
strong sector condition.

[S Sethuraman, SRS Varadhan, H-T Yau (2000)] $d \geq 3$.
non-zero mean $\sum_z z p_z \neq 0$.

$$T^{-1/2}(X(T\cdot) - \mathbf{E}(X(T\cdot)) \Rightarrow W_\sigma(\cdot).$$

Method: KV martingale approximation - nonreversible setting,
graded sector condition.

Open: $d = 1, 2$, $\sum_z z p_z \neq 0$.

(B) Convective diffusion.

AS Monin, AM Yaglom (1965): `Statistical Fluid Mechanics - Ch. 10: Turbulent Diffusion.`

$(t,x) \mapsto v(t,x) \in \mathbb{R}^d$ velocity field: e.g., (turbulent) sln of *incompressible* E/NS eq. In particular, $\nabla \cdot v \equiv 0$.

$(t,x) \mapsto \vartheta(t,x) \in \mathbb{R}_+$ concentration of "*passive tracers*", carried by the flow:

$$\partial_t \vartheta + \nabla \cdot (\vartheta v) = \frac{a}{2} \Delta \vartheta$$

This is exactly the bwK equation of the diffusion process

$$dX(t) = v(t, X(t))\, dt + \sqrt{a}\, dB(t).$$

Replace the "incompressible turbulent flow" with steady state "incompressible random flow" $v(x) = v(x,\omega)$ - stationary and ergodic w.r.t. spatial shifts.

(B1) Diffusion in divergence-free random drift field

$$x \mapsto a(x,\omega) \in \mathbb{R}^{d\times d}_{\text{sym},+} \qquad\qquad x \mapsto v(x,\omega) \in \mathbb{R}^d$$

stationary & ergodic w.r.t. spatial shifts, both $L^\infty \cap C^1$ a.s.
Assume (for now . . .)

DIVFREE : $\nabla \cdot v \equiv 0$ a.s.

NO DRIFT : $\mathbf{E}(v) = 0$

The diffusion:

$$dX(t) = \Big(\frac{1}{2}\nabla \cdot a(X(t)) + v(X(t))\Big) + a^{1/2}(X(t))\, dB(t)$$

Q: CLT $T^{-1/2}X(T) \Rightarrow \mathcal{N}(0,\sigma^2)$, or **IP** $T^{-1/2}X(T\cdot) \Rightarrow W_\sigma(\cdot)$ **?**

A: Yes! Under suitable (not merely technical!) conditions.
No! In relevant cases. (Not merely strange counterexamples!)

(B2) RW in $\{$Divfree/Doubly Stochastic$\}$ rnd environment

Let $\Big((p_k(x,\omega)_{k\in\mathcal{U}}) \Big)_{x\in\mathbb{Z}^d}$ stationary+ergodic rnd jump rates.
Assume (for now . . .)

$$\texttt{DIVFREE}: \qquad \sum_{k\in\mathcal{U}} p_k(x) = \sum_{k\in\mathcal{U}} p_{-k}(x+k)$$

$$\texttt{NODRIFT}: \qquad \sum_{k\in\mathcal{U}} k\mathbf{E}\,(p_k) = 0$$

The (continuous time) random walk:

$$\mathbf{P}_\omega\left(X(t+dt) = x+k | X(t) = x\right) = p_k(x,\omega)\,dt + o(dt)$$

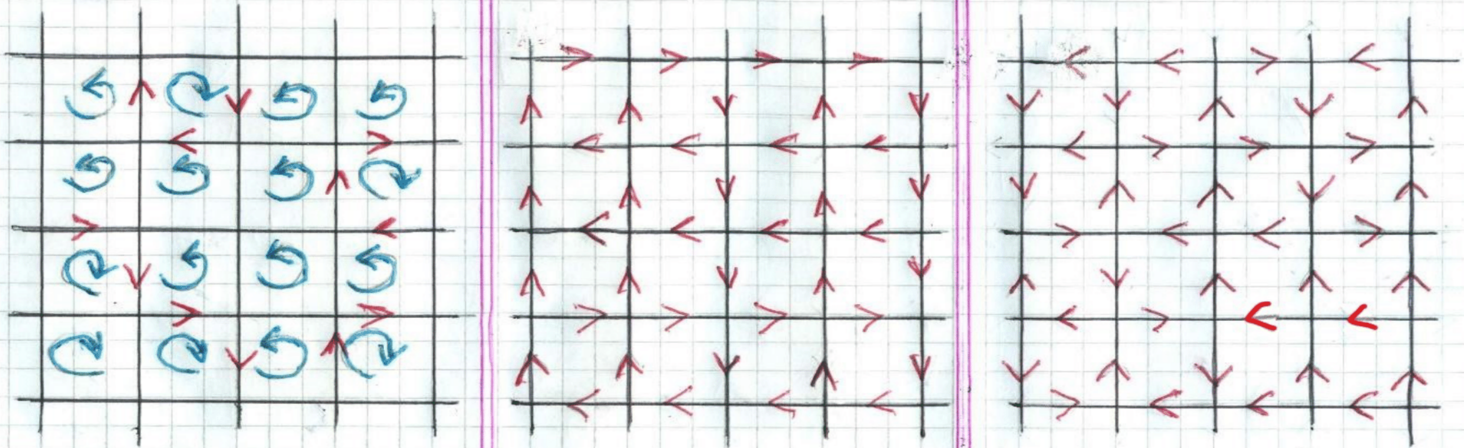
Q: CLT $T^{-1/2}X(T) \Rightarrow \mathcal{N}(0,\sigma^2)$, or **IP** $T^{-1/2}X(T\cdot) \Rightarrow W_\sigma(\cdot)$ **?**

A: Yes! Under suitable (not merely technical!) conditions.
No! In relevant cases. (Not merely strange counterexamples!)

Small cycles with short range dependence

"Manhattan"

Six-vertex / "Square Ice"



$X(t)$: RW, at each step uniform choice between the allowed directions

and also in higher dimension ...

Q: large scale behaviour? diffusive? super diff.?

list of results to be written here

(C) Self-repelling walk/diffusion. Pushed by the negative gradient of its own occupation time measure.

(C1) Self-repelling diffusion (SRBP): $t \mapsto X(t) \in \mathbb{R}^d$

$$\ell(t, A) := \ell(0, A) + |\{0 < s \leq t : X(s) \in A\}|$$

$$dX(t) = dB(t) - \text{grad}\Big(V * \ell(t, \cdot)\Big)(X(t))dt$$

$$V : \mathbb{R}^d \to [0, \infty) \quad \text{approx.-}\delta, \qquad \widehat{V}(p) := \int_{\mathbb{R}^d} e^{ip \cdot x} V(x) dx \geq 0$$

(C2) Self-repelling random walk (TSAW): $t \mapsto X(t) \in \mathbb{Z}^d$,

$$\ell(t, x) := \ell(0, x) + |\{0 < s \leq t : X(s) = x\}|$$

$$\mathbf{P}\left(X(t + dt) = x + k | X(t) = x\right) = w(\ell(t, x) - \ell(t, x + k))dt$$

$$w : \mathbb{R} \to (0, \infty) \quad \text{increasing}$$

Roots:

TSAW, physics:

[D Amit, G Parisi, L Peliti (1983)]
[S Obukhov, L Peliti (1983)]
[L Peliti, L Pietronero (1987)]
. . .

SRBP, probability:

[J Norris, C Rogers, D Williams (1987)]
[R Durrett, C Rogers (1992)]
[M Cranston, Y Le Jan (1995)]
[M Cranston, T Mountford (1996)]
. . .

Conjectures, based on RG and scaling arguments ("physics"):

- $\mathbf{d=1:}$ $X(t) \sim t^{2/3}$, intricate, non-Gausssian scaling limit. (Limit distributions not identified.)

- $\mathbf{d=2:}$ $X(t) \sim t^{1/2}(\log t)^{\zeta}$, Gaussian scaling limit. (Controversy about the value of ζ.)

- $\mathbf{d\geq 3:}$ $X(t) \sim t^{1/2}$, Gaussian scaling limit.

Some results: . . .

- $\mathbf{d = 1}$:
  - **Limit theorem** in some particular (lattice) cases, [T (1995), T-Vető (2011)]:

$$t^{-2/3} X(t) \Rightarrow \mathcal{X}.$$

  - Construction of the **scaling limit process** (TSRM, the Brownian Web, . . .), [T-Werner (1998)]

$$t \mapsto \mathcal{X}(t)$$

  - **"Robust" bounds**, [Tarrés-T-Valkó (2012)]:

$$C_1 t^{5/4} \leq \mathbf{E}\left(|X(t)|^2\right) \leq C_2 t^{3/2}.$$

    Method: resolvent calculus, analysis
  - **Missing:** *Universality* = Fully robust proof of the limit theorem, not relying on "combinatorial details".

- $\mathbf{d = 2}$:
  - **Super diffusive bounds**, [T-Valkó (2012)]:

$$C_1 t \log\log t \le \mathbf{E}\left(|X(t)|^2\right) \le C_2 t \log t.$$

    Method: resolvent calculus, analysis

  - Expected: $$\mathbf{E}\left(X(t)^2\right) \sim t\sqrt{\log t}$$

$$t^{-1/2}(\log t)^{-1/4} X(t) \Rightarrow \mathcal{N}(0, \sigma)$$

- $\mathbf{d = 3}$:
  - **CLT**, [Horváth-T-Vető (2012)]:

$$t^{-1/2} X(t) \Rightarrow \mathcal{N}(0, \sigma).$$

    Method: KV martingale approximation, graded sector condition

Precise conditions and statements: later