BÁLINT TÓTH
(Rényi Institute Budapest and University of Bristol)

LARGE-SCALE BEHAVIOUR OF RANDOM MOTIONS WITH LONG MEMORY

-2-
RANDOM WALK IN DIVERGENCE-FREE RANDOM ENVIRONMENT

PDE AND PROBABILITY - SUMMER SCHOOL
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Random Walk in Random Environment (RWRE)

$\big(\Omega, \pi, (\tau_z : \Omega \to \Omega, z \in \mathbb{Z}^d)\big)$ probab. sp., ergodic $\mathbb{Z}^d$-action

$\mathcal{U} = \{k \in \mathbb{Z}^d : |k| = 1\}$ n.n. steps of the r.w.

$p : \Omega \to [0, \infty)^{\mathcal{U}}$, jump rates of the r.w.

RWRE: Given $\omega \in \Omega$, $t \mapsto X(t) \in \mathbb{Z}^d$ cont. time Markov chain:

$$\mathbf{P}_\omega \left(X(t + dt) = x + k | X(t) = x\right) = \underbrace{p_k(\tau_x \omega)}_{p_k(x,\omega)} dt + o(dt)$$

Separate symmetric and skew-symmetric part of jump rates:

$\sqrt{\texttt{conductances}}$ $\quad r_k(\omega) := \sqrt{(p_k(\omega) + p_{-k}(\tau_k \omega))/2} = r_{-k}(\tau_k \omega)$

$\texttt{flows}$ $\quad b_k(\omega) := (p_k(\omega) - p_{-k}(\tau_k \omega))/2 = -b_{-k}(\tau_k \omega)$

Major issue in probability theory since the 1970s.

[Overwhelming majority of the RWRE literature is about '*random walk among random conductances*': $b \equiv 0$ a.s. – reversible, self-adjoint . . .]

Assumptions I. (minimal)

$$\sum_{k \in \mathcal{U}} b_k(\omega) \equiv 0 \quad \text{a.s.} \tag{DIV-FREE}$$

$$r_k \in \mathcal{L}^2(\Omega, \pi) \tag{UPPER}$$

$$\int_\Omega b_k(\omega) d\pi(\omega) = 0 \tag{NO-DRIFT}$$

$$(r_k)^{-1} \in \mathcal{L}^2(\Omega, \pi) \tag{LOWER}$$

Rather than strong ellipticity

$$r_k(\omega) \geq r_* > 0. \tag{ELLIP}$$

First consequences:

∘ (DIV-FREE) $\Rightarrow$ The *environment process*:

$$t \mapsto \eta_t := \tau_{X(t)}\omega$$

is *stationary* and *ergodic* Markov process in (Ω, π).

∘ (UPPER) & (NO-DRIFT) $\Rightarrow$ zero (annealed) drift & SLLN:

$$\mathbf{E}\left(X(t)\right) := \int_\Omega \mathbf{E}_\omega\left(X(t)\right) d\pi(\omega) = 0$$

$$t^{-1}X(t) \to 0, \quad \text{a.s.}$$

∘ (LOWER) $\Rightarrow$ diffusive lower bound (not totally straightforward):

$$\varliminf_{t\to\infty} t^{-1}\mathbf{E}\left(|X(t)|^2\right) > 0$$

Questions left open:

Diffusive upper bound? CLT? Superdiffusive lower bound?

Analogous: **diffusion in div-free rnd drift field** $t \mapsto X(t) \in \mathbb{R}^d$

$$dX(t) = \Big(\frac{1}{2}\nabla \cdot r^2(X(t)) + b(X(t))\Big)\, dt + r(X(t))\, dB(t)$$

with infinitesimal generator

$$L := \frac{1}{2}\nabla \cdot r^2\, \nabla + b \cdot \nabla,$$

where

$$r = r(\omega) : \mathbb{R}^d \to \mathbb{R}_+^{d\times d} \qquad\qquad b = b(\omega) : \mathbb{R}^d \to \mathbb{R}^d$$

are space-wise stationary & ergodic,

$$\operatorname{div} b \equiv 0 \qquad \pi\text{-a.s.} \qquad\qquad (\texttt{DIV-FREE})$$

with conditions analogous to (UPPER), (NO-DRIFT), (LOWER).

Phyisical motivation: . . . passsive tracer in turbulent flow . . .

Historic notes: . . . later . . .

Martingale decomposition:

$$X(t) = \underbrace{Z(t)}_{\checkmark} + \underbrace{\int_0^t \Big(\varphi(\eta_s) + \psi(\eta_s)\Big)\, ds}_{?}$$

where $\varphi, \psi : \Omega \to \mathbb{R}^d$,

$$\varphi(\omega) := \sum_{k \in \mathcal{U}} k r_k^2(\omega) \qquad \psi(\omega) := \sum_{k \in \mathcal{U}} k b_k(\omega)$$

$$\varphi(\omega) + \varphi(\omega) = \sum_{k \in \mathcal{U}} k p_k(\omega)$$

Then $t \mapsto Z(t) \in \mathbb{R}^d$ is a quenched martingale whose increments are statioanry, ergodic and $\mathcal{L}^2$ in the annealed setting, and thus obeys the *martingale CLT*.

Blueprint: Martingale approx. of additive functionals (integrals above) of ergodic Markov processes – à la Kipnis-Varadhan.

Helmholtz: "div-free vector field = curl of vector potential" – *cum grano salis*

Setting:

$\big(\Omega, \pi, (\tau_z : \Omega \to \Omega, z \in \mathbb{Z}^d)\big)$ erg. $\mathbb{Z}^d$-action, $b : \Omega \to \mathbb{R}^{\mathcal{U}}$ s.t. π-a.s.

$$b_k(\omega) + b_{-k}(\tau_k \omega) = 0 \qquad (\texttt{FLOW})$$

$$\sum_{k \in \mathcal{U}} b_k(\omega) = \frac{1}{2} \sum_{k \in \mathcal{U}} \big(b_k(\omega) - b_k(\tau_{-k}\omega)\big) = 0 \qquad (\texttt{DIV-FREE})$$

Lift to $\mathbb{Z}^d$: $\quad \mathbb{Z}^d \times \mathcal{U} \ni (x,k) \mapsto b_k(x,\omega) = b_k(\tau_x \omega)$ is a stationary & ergodic **divergence-free flow/vector field** on $\mathbb{Z}^d$:

$$b_k(x,\omega) + b_{-k}(x+k,\omega) = 0 \qquad (\texttt{FLOW})$$

$$\sum_{k \in \mathcal{U}} b_k(x,\omega) = \frac{1}{2} \sum_{k \in \mathcal{U}} \big(b_k(x,\omega) - b_k(x-k,\omega)\big) = 0 \qquad (\texttt{DIV-FREE})$$

Proposition 1. ['Helmholtz's thm'] [K-L-O(2012), K-T(2017), T(2025

Let $b : \Omega \to \mathbb{R}^{\mathcal{U}}$ *be a div-free flow/vf as above,* $\underline{b \in \mathcal{L}^1(\Omega, \pi)}$*.*
There exists a *stream tensor field* $(x, k, l) \mapsto H_{k,l}(x, \omega)$, $\underline{H \in \mathcal{L}^{1\mathrm{w}}(\Omega, \pi)}$,

$$H_{-k,l}(x+k, \omega) = H_{k,-l}(x+l, \omega) = H_{l,k}(x, \omega) = -H_{k,l}(x, \omega) \quad (\texttt{STREAM})$$

with stationary increments, a.k.a. a cocycle

$$H_{k,l}(y, \omega) - H_{k,l}(x, \omega) = H_{k,l}(y-x, \tau_y \omega) - H_{k,l}(0, \tau_x \omega), \quad (\texttt{COCY})$$

such that Helmholtz's relation holds

$$\begin{aligned} b_k(x, \omega) &= \sum_{l \in \mathcal{U}} H_{k,l}(x, \omega) = \frac{1}{2} \sum_{l \in \mathcal{U}} (H_{k,l}(x, \omega) - H_{k,l}(x-l, \omega)) \\ &= \sum_{l \in \mathcal{U}} H_{k,l}(0, \tau_x \omega) \qquad (\texttt{HELMHOLTZ}) \end{aligned}$$

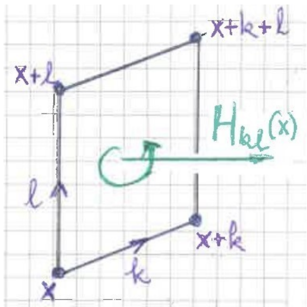
If $\underline{b \in \mathcal{L}^p(\Omega, \pi),\ p \in (1, 2]}$ *than also* $\underline{H \in \mathcal{L}^p(\Omega, \pi)}$*.*

Proof: $b \in \mathcal{L}^2$: soft / $b \in \mathcal{L}^1$: relies on Calderón-Zygmund.

Remarks/comments:

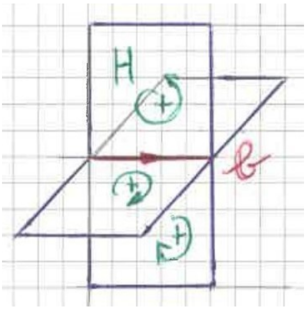
◦ This is *not the 'usual'* Helmholtz Thm.

◦ (STREAM) means that $(x, k, l) \mapsto H_{k,l}(x)$ is a function of the *oriented plaquettes* of $\mathbb{Z}^d$



◦ In 2-dim H is a *height function* on the dual $\mathbb{Z}^2$.
In 3-dim H is a *flow / vector field* on the dual $\mathbb{Z}^3$.

◦ (HELMHOLTZ):



◦ (STREAM), (COCY), (HELMHOLTZ) determine H *uniquely*, up to the additive $H_{k,l}(0)$. The standard choice is: $H_{k,l}(0) = 0$.

◦ **Subtle q.:** Is the (COCY) stream field actually *stationary*?

$$H_{k,l}(x,\omega) = h_{k,l}(\tau_x\omega) - h_{k,l}(\omega) \qquad \text{(STATI-STREAM)}$$

with some $(k,l) \mapsto h_{k,l}(\omega)$, such that

$$h_{-k,l}(\tau_k\omega) = h_{k,-l}(\tau_l\omega) = h_{l,k}(\omega) = -h_{k,l}(\omega) \qquad \text{(STATI-STREAM)}$$

Could be

◦◦ (Y1) Yes, with h in the same $\mathcal{L}$-class as H

◦◦ (Y2) Yes, with h in weaker $\mathcal{L}$-class as H

◦◦ (N) No.

Given $b \in \mathcal{L}^p$, $p \in [1,2]$, div-free, it can be hard to see whether the case (Y1), (Y2) or (N) holds.

Except: $b \in \mathcal{L}^2$: $h \in \mathcal{L}^2$ (i.e. (Y1) holds) iff $b \in \mathcal{H}_{-1}(\Delta)$.

More about this - later.

Back to the RWRE: Recall notation:

$$r_k(\omega) := \sqrt{(p_k(\omega) + p_{-k}(\tau_k\omega))/2} \qquad b_k(\omega) := (p_k(\omega) - p_{-k}(\tau_k\omega))/2$$

Assumptions II. (final form)

- (UPPER)+(LOWER):

$$r_k \in \mathcal{L}^2(\Omega,\pi) \qquad (r_k)^{-1} \in \mathcal{L}^2(\Omega,\pi)$$

- (STATI-STREAM): $\exists\ h : \Omega \to \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ s.t.

$$h_{k,l}(\omega) = -h_{-k,l}(\tau_k\omega) = -h_{k,-l}(\tau_l\omega) = -h_{l,k}(\omega)$$

$$b_k(\omega) = \sum_{l\in\mathcal{N}} h_{k,l}(\omega)$$

- (H-1): (new!)

$$(r_k)^{-1}h_{k,l} \in \mathcal{L}^2(\Omega,\pi) \tag{H-1}$$

Remarks:

◦ Bonus: (LOWER)+(UPPER)+(H-1) imply

$$h_{k,l} \in \mathcal{L}^1(\Omega, \pi) \qquad\qquad (r_k)^{-1}\, h_{k,l}\, (r_l)^{-1} \in \mathcal{L}^1(\Omega, \pi)$$

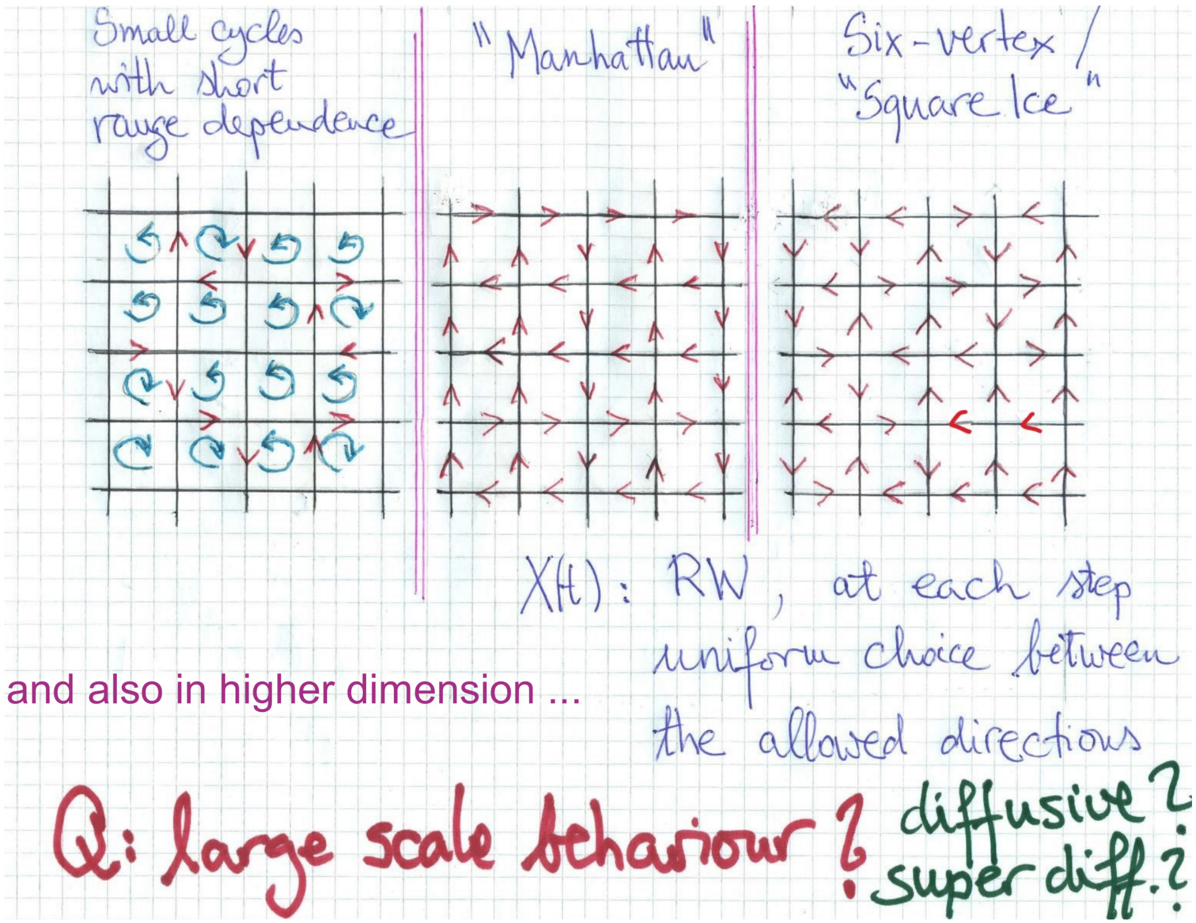
◦ Assuming (ELLIP) [rather than (LOWER)] (H-1) reduces to

$$h_{k,l} \in \mathcal{L}^2(\Omega, \pi) \qquad \Leftrightarrow \qquad b \in \mathcal{H}_{-1}(\Delta) \qquad\qquad \text{(H-1/ELLIP)}$$

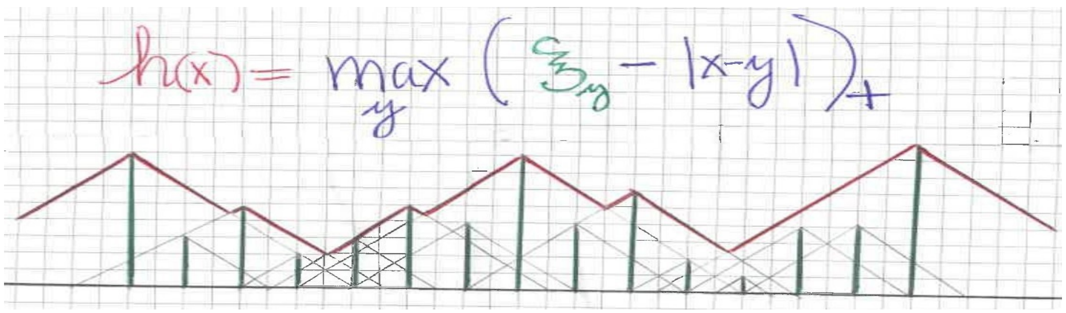
More on this - later.

◦ Examples - on next three slides . . .

Examples $r, b \in \mathcal{L}^{\infty}(\Omega, \pi)$ + (ELLIP)



and another example on the next slide



This example is due to Ron Peled. Do the same on $\mathbb{Z}_*^2$

Let $(\xi_y)_{y\in\mathbb{Z}_*^2}$ i.i.d., $\xi_y \in \mathbb{N}$, and $h : \mathbb{Z}_*^2 \to \mathbb{N}$,

$$h(x) := \sup \left\{ (\xi_y - |x-y|)_+ : y \in \mathbb{Z}_*^2 \right\}.$$

By Borel-Cantelli: $h(x) < \infty$ a.s. iff $\mathbf{E}\left(\xi^2\right) < \infty$. In this case, $x \mapsto h(x)$ is (obviously) stationary and $\mathrm{Lip}(1)$. Furthermore, $\mathbf{E}\left(|h(x)|^p\right) < \infty$ iff $\mathbf{E}\left(|\xi|^{??}\right) < \infty$.

Let $$b_k(x) := h(x + (1+i)k/2) - h(x + (1-i)k/2)$$

Then $(x,k) \mapsto b_k(x)$ is (DIV-FREE) and $|b_k(x)| \leq 1$ a.s. Finally,

$$p_k(x) := 1 + b_k(x).$$

Examples ctd. Why bother with non-ellipticity?

Let $t \mapsto X(t)$ be a RWRE as on page 2. Without (DIV-FREE) the environment process $t \mapsto \eta_t := \tau_{X(t)}\omega$ won't be stationary.

The equation for stationary RN derivative:

$$\sum_{k\in\mathcal{N}} p_{-k}(\tau_k\omega)\varrho(\tau_k\omega) = \sum_{k\in\mathcal{N}} p_k(\omega)\varrho(\omega)$$

This is a hard problem! See, e.g. [Sabot (2013)].

Assume sln $\varrho \in \mathcal{L}^1(\Omega, \pi)$ exists. Then, $t \mapsto \widetilde{X}(t)$ with jump rates

$$\widetilde{p}_k(\omega) := p_k(\omega)\varrho(\omega)$$

will be (obviously) (DIV-FREE).

However, this is a time-changed version of the original RWRE. So, proving CLT/non-CLT for $t \mapsto X(t)$ reduces to checking/refuting the stated conditions for the rates $(\widetilde{p}_k(\omega))_{k\in\mathcal{U}}$. ...

Theorem 1. [CLT in probability w.r.t. the environment]
[G Kozma, BT (2017)], [BT (2025)]
Assume (UPPER) *&* (LOWER) *&* (STATI-STREAM) *&* (H-1). *Then*

$$X(t) = Y(t) + Err(t)$$

so that the following limits hold as $N \to \infty$.
(i) For π*-almost all* $\omega \in \Omega$, $t \mapsto Y(t)$ *is a square integrable martingale whose increments are stationary and ergodic in the annealed setting. Thus (due to the martingale IP) for* π*-almost all* $\omega \in \Omega$,

$$N^{-1/2} Y(N\cdot) \Rightarrow W_\sigma(\cdot)$$

in $D([0,1])$ *under the quenched probab. measure* $\mathbf{P}_\omega(\ldots)$, *where* $W_\sigma(\cdot)$ *is a BM with* $\sigma^2 \in (0,\infty)$.
(ii) For any $t \in [0,1]$ *and* $\delta > 0$,

$$\lim_{N\to\infty} \pi\Big(\Big\{\omega \in \Omega : N^{-1}\mathbf{E}_\omega\left(|Err(Nt)|^2\right) > \delta\Big\}\Big) = 0.$$

Theorem 2. [quenched CLT π-a.s. w.r.t. the envi.] [BT (2018)]
Assume (UPPER) & (ELLIP) & (STATI-STREAM) & (H-1+ε/ELLIP)

$$h_{k,l} \in \mathcal{L}^{2+\varepsilon} \qquad \text{(H-1+}\varepsilon\text{/ELLIP)}$$

Then the statement of Theorem 1 holds with

$$\lim_{N\to\infty} N^{-1/2}|Err(Nt)| = 0 \quad \text{a.s.}$$

Comments.

∘ Theorem 1: CLT in probability w.r.t. ω ("semi-quenched").
Theorem 2: CLT for π-a.a. ω. ("quenched")

∘ Proof of Theorem 1: martingale approx. + functional analysis.
Proof of Theorem 2: extra ingredients: Nash-type arguments (therefore (ELLIP) needed)

Martingale approx, Kipnis-Varadhan theory - in a nutshell: (5 slides)

$t \mapsto \eta_t, \eta^*_t \in \Omega$ stat&erg MP on (Ω, π) and its time-reversal.
Assume suff. regularity. Their semigroups P_t, P^*_t, resolvents R_λ, R^*_λ, infinitesimal generators L, L^* act on $\mathcal{L}^p(\Omega,\pi)$, $p \in [1,\infty]$.

$$P_t f(\omega) := \mathbf{E}_\omega\left(f(\eta_t)\right) \qquad \|P_t\|_{p\to p} = 1$$

$$R_\lambda := \int_0^\infty e^{-\lambda t} P_t\, dt \qquad \|R_\lambda\|_{p\to p} = \lambda^{-1}$$

$$L := \text{st-}\lim_{t\to 0} t^{-1}(P_t - I)$$

On $\mathcal{L}^2(\Omega,\pi)$, assume

$$L = -S + A \qquad S := -(L+L^*)/2 \qquad A := (L-L^*)/2,$$

define $S^{-1/2}$ in terms of the `Spectral Theorem`, and $\mathcal{H}_- := \mathcal{H}_{-1}(S)$:

$$\mathcal{H}_- := \left\{ f \in \mathcal{L}^2 : \ \|f\|^2_- := \lim_{\lambda\to 0} \langle f, (\lambda I + S)^{-1} f\rangle = \|S^{-1/2} f\|^2_2 < \infty \right\}^{\mathtt{cl}-}$$

WANTED: Efficient martingale approx. $t \mapsto \int_0^t \varphi(\eta_s)\, ds$.

Theorem 3. `[non-reversible KV]` [KV (1986)], [T (1986)]

Let $\varphi \in \mathcal{L}^1$ *such that* $\int_\Omega \varphi\, d\pi = 0$ *and for all* $\lambda > 0$, $R_\lambda \varphi \in \mathcal{L}^2$*.*
If the following two conditions hold

$$\text{(A)} \qquad \lim_{\lambda \to 0} \lambda^{1/2} \| R_\lambda \varphi \|_2 = 0,$$

$$\text{(B)} \qquad \lim_{\lambda \to 0} \| S^{1/2} R_\lambda \varphi - v \|_2 = 0, \quad v \in \mathcal{L}^2,$$

then there exists a sq-integrable martingale $t \mapsto Z(t)$ *(adapted to the natural filtration), with stat&erg increments and variance*

$$\mathbf{E}\left(|Z(t)|^2 \right) = 2 \|v\|_2^2\, t,$$

such that

$$\lim_{t \to \infty} t^{-1} \mathbf{E}\left(\left| \int_0^t \varphi(\eta_s)\, ds - Z(t) \right|^2 \right) = 0.$$

Comments, remarks:

- The **self-adjoint** case [Kipnis-Varadhan (1986)], $L = -S$,

$$\{\varphi \in \mathcal{H}_{-}\} \quad \Longleftrightarrow \quad \text{(A)} \quad \Longleftrightarrow \quad \text{(B)}$$

- General, **non-self-adjoint** case:
  Proposition 2. [H_{-1} `rules!`] [Varadhan (1995)]
  $\forall\ \varphi \in \mathcal{H}_{-} \cap \mathcal{L}^1$, $t \in [0, \infty)$

$$\mathbf{Var}\left(\int_0^t \varphi(\eta_s)\, ds\right) \le 2\|\varphi\|_{-}^2 t.$$

  Hwvr, $\varphi \in \mathcal{H}_{-}$ is too little (for CLT) & too much (for $\sigma^2 < \infty$).

- **non-self-adjoint** case: (A) & (B) are too implicit to be checked directly. (Though, exceptions exist.)

- **Sufficient:** *Strong Sector Condition* [Varadhan (1995)]; *Graded Sector Condition* [Sethuraman-Varadhan-Yau (2000)] notoriously technical. Appl. restricted to graded structure . . .

A handy sufficient condition: Let

$$\mathcal{B} := \{ f \in \mathcal{H}_{\_} \cap \mathcal{L}^2 : \; S^{-1/2} f \in \text{Dom}(A) \;\&\; AS^{-1/2} f \in \mathcal{H}_{\_} \cap \mathcal{L}^2 \}$$

$$B : \mathcal{B} \to \mathcal{L}^2, \qquad\qquad Bf := S^{-1/2} A S^{-1/2} f.$$

Comments: $B : \mathcal{B} \to \mathcal{L}^2$ is obviously skew-symmetric. However, $\mathcal{B} \subset \mathcal{L}^2$ could be too thin . . . . Even if $\mathcal{B}$ is dense in $\mathcal{L}^2$ it could still happen that $B \prec \overline{B} \precneqq -B^*$.

Theorem 4. `[relaxed sector condition]` [Horváth-T-Vető (2012)] (streamlined)

Assume that $\mathcal{B}$ is dense in $\mathcal{L}^2$ and the operator $B : \mathcal{B} \to \mathcal{L}^2$ is essentially skew-self-adjoint: $B \prec \overline{B} = -B^*$. Then for any $\varphi \in \mathcal{H}_{\_} \cap \mathcal{L}^1$ the conditions of Theorem KV (and hence the martingale approximation) hold.

Proof sketch. [Details in separate set of notes.]
For simplicity assume $\|L\|_{2\to 2} < \infty$ and let

$$B_\lambda := (\lambda I + S)^{-1/2} A (\lambda I + S)^{-1/2}, \qquad B_\lambda^* = -B_\lambda$$

$$K_\lambda := (I + B_\lambda)^{-1}, \qquad \|K_\lambda\| \le 1.$$

$$R_\lambda = (\lambda I + S)^{-1/2} K_\lambda (\lambda I + S)^{-1/2}$$

If *by some miracle*: $K_\lambda \xrightarrow{\text{s.o.t.}} K$, then, for $\varphi \in \mathcal{H}_-$,

(A) $\lambda^{1/2} R_\lambda \varphi = \lambda^{1/2} (\lambda I + S)^{-1/2} K_\lambda (\lambda I + S)^{-1/2} \varphi \to 0$

(B) $S^{1/2} R_\lambda \varphi = S^{1/2} (\lambda I + S)^{-1/2} K_\lambda (\lambda I + S)^{-1/2} \varphi \to \underbrace{K S^{-1/2} \varphi}_{v}$

Under the conditions of the Thm, by an argument reminiscent of Trotter-Kurtz, $K_\lambda \xrightarrow{\text{s.o.t.}} K$. □

Back to the main issue. Spaces.

$\mathcal{L}$: scalars; $\mathcal{V}$: vectors; $\mathcal{R}$: rot-free vects; $\mathcal{D}$: div-free vects; $\mathcal{K}$: ??.

$$\mathcal{L}^p := \left\{ f : \Omega \to \mathbb{R} : \|f\|_p^p := \int_\Omega |f|^p \, d\pi < \infty, \quad \int_\Omega f \, d\pi = 0 \right\}$$

$$\mathcal{V}^p := \left\{ u : \Omega \to \mathbb{R}^{\mathcal{U}} : u_k \in \mathcal{L}^p, \quad u_k(\omega) + u_{-k}(\tau_k \omega) = 0, \ \|u\|_p^p = \sum_{k \in \mathcal{U}} \|u_k\|_p^p \right\}$$

$$\mathcal{R}^p := \left\{ u \in \mathcal{V}^p : u_k(\omega) + u_l(\tau_k \omega) = u_l(\omega) + u_k(\tau_l \omega) \right\} \qquad [\mathcal{R} \texttt{ for } \underline{\texttt{r}}\texttt{ot-free}]$$

$$\mathcal{D}^p := \left\{ u \in \mathcal{V}^p : \sum_{k \in \mathcal{U}} u_k(\omega) = 0 \right\} \qquad [\mathcal{D} \texttt{ for } \underline{\texttt{d}}\texttt{iv-free}]$$

$$\mathcal{K}^p := \left\{ u \in \mathcal{V}^p : (r_k^{-1} u_k)_{k \in \mathcal{U}} \in \mathcal{R} \right\}^{\texttt{cl}\, p}$$

Helmholtz-Hodge: $\mathcal{V}^2 = \mathcal{R}^2 \oplus \mathcal{D}^2$.

$\mathcal{L}$, $\mathcal{V}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{K}$: same, without integrability conditions.

Back to the main issue. Operators.

$$T_z f(\omega) := f(\tau_z \omega), \qquad \partial_k := T_k - I,$$

$$R_k f(\omega) := r_k(\omega) f(\omega), \qquad H_{k,l} f(\omega) := h_{k,l}(\omega) f(\omega)$$

$$\nabla : \mathcal{L} \to \mathcal{V}, \qquad (\nabla f)_k := \partial_k f$$

$$\nabla^* : \mathcal{V} \to \mathcal{L}, \qquad \nabla^* u := \sum_{k \in \mathcal{U}} u_k = -\frac{1}{2} \sum_{k \in \mathcal{U}} \partial_{-k} u_k$$

$$R : \mathcal{V} \to \mathcal{V}, \qquad (Ru)_k := R_k u_k$$

$$H : \mathcal{V} \to \mathcal{V}, \qquad (Hu)_k := \frac{1}{4} \sum_{l \in \mathcal{U}} (T_{-l} + I)\, H_{k,l}\, (T_k + I) u_l$$

$$S = \nabla^* R^2 \nabla = (\nabla^* R)(R \nabla)$$

$$A = \nabla^* H \nabla = (\nabla^* R)(R^{-1} H R^{-1})(R \nabla)$$

Key facts (concrete - valid under the assumed conditions):

(1) $\mathcal{H}_- := \mathcal{H}_{-1}(S) \subset \mathcal{L}^1$ and, for $f \in \mathcal{L}^1$,

$$\|f\|_-^2 = \sup_{g \in \mathcal{L}^\infty} \left(2\langle f, g \rangle - \|R\nabla g\|_2^2 \right)$$

(2) Hilbert space isometries:

$$(\mathcal{L}^2, \|\cdot\|_2) \xrightarrow{R\nabla S^{-1/2}} (\mathcal{K}^2, \|\cdot\|_2) \xrightarrow{S^{-1/2}\nabla^* R} (\mathcal{L}^2, \|\cdot\|_2),$$

Reminiscent of *Riesz kernels* of harmonic analysis:

$$\mathcal{L}^2(\mathbb{R}^d, \text{Leb}) \xrightarrow{\text{grad}\,|\text{lap}\,|^{-1/2}} \underbrace{\mathcal{R}^2(\mathbb{R}^d, \text{Leb})}_{\text{Ker(rot)}} \xrightarrow{|\text{lap}\,|^{-1/2}\text{div}} \mathcal{L}^2(\mathbb{R}^d, \text{Leb}).$$

Proof of Thm 1. *Step 1:* Prove $B^* = -B$:

$$B = S^{-1/2} A S^{-1/2} = \underbrace{S^{-1/2}\nabla^* R}_{\Lambda^*}\ R^{-1} H R^{-1}\ \underbrace{R \nabla S^{-1/2}}_{\Lambda}$$

Reduces to proving essential skew-self-adjointness (on an appropriate dense subspace) of $D : \mathcal{K}^2 \to \mathcal{K}^2$,

$$D := \Lambda\Lambda^* R^{-1} H R^{-1} \Lambda\Lambda^*, \qquad\qquad \Lambda\Lambda^* : \mathcal{V}^2 \to \mathcal{K}^2 \text{ orth. proj.}$$

However, D is (essentially) a multiplication operator. The proof is (essentially) routine. In this step only (STATI-STREAM) matters, and not the integrability conditions.

Step 2: Prove $\varphi, \psi \in \mathcal{H}_-$ for $\varphi, \psi : \Omega \to \mathbb{R}^d$,

$$\varphi(\omega) := \sum_{k\in\mathcal{U}} k s_k(\omega) = \frac{1}{2} \sum_{k\in\mathcal{U}} k \partial_{-k} s_k(\omega) \qquad \psi(\omega) := \sum_{k\in\mathcal{U}} k b_k(\omega)$$

$$
\begin{aligned}
\|\partial_{-k} s_k\|_-^2 &= \sup_{\chi\in\mathcal{L}^\infty} \left(2\langle\chi, \partial_{-k} s_k\rangle - \|R\nabla\chi\|_2^2\right) \\
&\le \sup_{\chi\in\mathcal{L}^\infty} \left(2\langle\chi, \partial_{-k} s_k\rangle - \|R_k\partial_k\chi\|_2^2\right) \\
&= \sup_{\chi\in\mathcal{L}^\infty} \left(2\langle R_k\partial_k\chi, r_k\rangle - \|R_k\partial_k\chi\|_2^2\right) \\
&\le \sup_{\chi\in\mathcal{L}^2} \left(2\langle\chi, r_k\rangle - \|\chi\|_2^2\right) = \underbrace{\|r_k\|^2 < \infty}_{(\texttt{UPPER})}
\end{aligned}
$$

$$
\begin{aligned}
\|b_k\|_-^2 &= \sup_{\chi\in\mathcal{L}^\infty} \left(\langle\chi, \sum_{l\in\mathcal{U}} \partial_{-l} h_{k,l}\rangle - \|R\nabla\chi\|_2^2\right) \\
&= \sup_{\chi\in\mathcal{L}^\infty} \sum_{l\in\mathcal{U}} \left(\langle R_l\partial_l\chi, r_l^{-1} h_{k,l}\rangle - \|R_l\partial_l\chi\|_2^2\right) \\
&\le \sum_{l\in\mathcal{U}} \sup_{\chi\in\mathcal{L}^2} \left(\langle\chi, r_l^{-1} h_{k,l}\rangle - \|\chi\|_2^2\right) = \underbrace{\sum_{l\in\mathcal{U}} \|r_l^{-1} h_{k,l}\|_2^2 < \infty}_{(\texttt{H-1})}.
\end{aligned}
$$

$\square$Thm1

Harmonic coordinates. [S Kozlov (1985)]: Find a random field $\mathbb{Z}^d \ni x \mapsto V(x,\omega) \in \mathbb{R}^d$ with *stationary increments* (i.e, a *cocycle*)

$$V(y,\omega) - V(x,\omega) = V(y-x, \tau_x\omega) \qquad \text{(COCY)}$$

and $\mathbf{E}(V(x)) \equiv 0$, such that

$$\sum_{k\in\mathcal{U}} \underbrace{p_k(x,\omega)}_{p_k(\tau_x\omega)} \Big(k + \underbrace{V(x+k,\omega) - V(x,\omega)}_{V_k(\tau_x\omega)}\Big) = 0 \qquad \pi\text{-a.s.}$$

Then $t \mapsto Y(t) := X(t) + V(X(t))$ is a quenched martingale (for π-a.a. ω) and it is <u>*plausible to expect*</u> that

$$\lim_{t\to\infty} t^{-1/2} V(X(t)) = 0 \qquad \text{a.s.} \qquad \text{(Q-ERR)}$$

If this is done, the *quenched IP* follows.
The field $\mathbb{Z}^d \ni x \mapsto x + V(x,\omega) \in \mathbb{R}^d$ is called (for obvious reasons) *harmonic coordinates*. Geometric meaning:

Bonus - from the proof of Theorem 1:

Proposition 3. [∃! of harmonic coordinates] [BT (2025)]
Given $\phi \in \mathcal{H}_- \cap \mathcal{L}^1$ *there exists a unique solution* $v \in R^{-1}\mathcal{K}^2 \subset \mathcal{U}^1$ *(*$\mathcal{L}^1$ *rot-free field) of the equation*

$$\sum_{k \in \mathcal{U}} p_k(\omega) v_k(\omega) = \phi(\omega). \qquad \text{(HARM-COORD)}$$

Proof sketch. The solution is

$$v = R^{-1} (I + B)^{-1} S^{-1/2} \phi$$

$$\|v\|_1 \leq \underbrace{\|r^{-1}\|_2 \|\phi\|_- < \infty}_{\text{(LOWER)}} \qquad \square$$

$$\|v\|_2 \leq \underbrace{\|r^{-1}\|_\infty \|\phi\|_- < \infty}_{\text{(ELLIP)}}$$

Proof of Theorem 2:

Step 1: [Harmonic coordinates + error term]

Solve (HARM-COORD) for $\phi := \sum_{k\in\mathcal{U}} kp_k = \varphi+\psi \in \mathcal{H}\_\cap\mathcal{L}^1$ (component-wise) and let $\mathbb{Z}^d \ni x \mapsto V(x,\omega) \in \mathbb{R}^d$ be the (COCY) field defined by the gradients

$$V(0,\omega) = 0 \qquad\qquad V(x+k,\omega) - V(x,\omega) = v_k(\tau_x\omega)$$

Write

$$X(t) = \overbrace{X(t) + V(X(t))}^{Y(t)} - V(X(t))$$

Then, $t \mapsto Y(t)$ is a *quenched* martingale with stat&erg *annealed* increments with covariance computable

$$\mathbf{E}\left(Y_i(t)Y_j(s)\right) = \delta_{i,j}\sigma^2 \min\{t,s\}, \qquad \sigma^2 = ????$$

By the MIP, for π-a.a. ω, $\qquad N^{-1/2}Y(N\cdot) \Rightarrow W_\sigma(\cdot)$
in $D([0,1],\mathbb{R}^d)$, under the *quenched* measure $\mathbf{P}_\omega(\cdot)$.

Intersteps:

It remains to quenched-bound the error term $V(X(t))$:
for π-a.a. ω, and all $\delta > 0$,

$$\lim_{t\to\infty} \mathbf{P}_\omega \left(t^{-1/2} |V(X(t))| > \delta \right) = 0. \qquad \text{(Q-ERR-BOUND)}$$

Philosophy and why it fails . . .

To prove (Q-ERR-BOUND), we'll assume

$$r_k(\omega) \geq r_* > 0. \qquad \text{(ELLIP)}$$

$$h_{k,l} \in \mathcal{L}^{2+\varepsilon} \qquad \text{(H-1+}\varepsilon\text{/ELLIP)}$$

[rather than (LOWER)&(H-1)].

Step 2: [Ergodic theorem for cocycles]

Proposition 4. [an ergodic theorem] [BT (2018)]
Let $\left(\Omega, \pi, (\tau_z : \Omega \to \Omega, z \in \mathbb{Z}^d)\right)$ *be an ergodic* $\mathbb{Z}^d$*-action and* $\mathbb{Z}^d \ni x \mapsto \Psi(x, \omega) \in \mathbb{R}$ *a cocycle (i.e. stationary increments) such that* $\Psi \in \mathcal{L}^1 \log^{d-1} \mathcal{L}$ *and* $\mathbf{E}(\Psi) = 0$*. Then* π*-a.s.*

$$\lim_{N\to\infty} N^{-(d+1)} \sum_{|x|\le N} |\Psi(x)| = 0 \qquad \text{(COCY-ERG)}$$

The proof of Proposition 4 (see later) relies on

Theorem 5. [Multidim. unrestricted erg. thm. [Zygmund (1951)]
Let $\left(\Omega, \pi, \tau\right)$ *be as above, and* $f \in \mathcal{L}^1 \log^{d-1} \mathcal{L}$*. Then*

$$\lim_{N_1,\dots,N_d\to\infty} (N_1 \dots N_d)^{-1} \sum_{z\in[0,N_1-1]\times\cdots\times[0,N_d-1]} f(\tau_z \omega) = \int_\Omega f \, d\pi \quad \text{a.s.}$$

Step 3: ["Nash theory"]

Proposition 5.

(i) [Heat kernel bound] [Nash (1958)]

Assume (ELLIP). *There exists a constant* $C = C(\|r^{-1}\|_\infty)$ *such that for* π*-a.a.* ω,

$$\sup_{\substack{x \in \mathbb{Z}^d \\ 0<t<\infty}} t^{d/2} \mathbf{P}_\omega (X(t) = x) \le C \qquad \text{(NASH-HKB)}$$

(ii) [quenched tightness] [BT (2018)] following [Nash (1958)]

Assume (ELLIP), (UPPER), (H-1+ε/ELLIP). *There exists a constant* $M = M(\|r^{-1}\|_\infty, \|r\|_2, \varepsilon, \|h\|_{2+\varepsilon}) < \infty$ *such that for* π*-a.a.* ω

$$\overline{\lim_{t\to\infty}}\, t^{-1/2} \mathbf{E}_\omega (|X(t)|) \le M \qquad \text{(NASH-MB)}$$

$$\overline{\lim_{t\to\infty}}\, \mathbf{P}_\omega \left(t^{-1/2} |X(t)| > K \right) \le \frac{M}{K} \qquad \text{(Q-TIGHT)}$$

Comments:

- The diagonal heat kernel upper bound (NASH-HKB) is, actually, a deterministic statement about *any* strictly elliptic (ELLIP) divergence-free (DIV-FREE) environment. It is a consequence of *Nash's inequality*. See separate notes.
- Note that from (UPPER), (H-1) we get the annealed bound

$$\overline{\lim_{t\to\infty}} t^{-1}\mathbf{E}\left(|X(t)|^2\right) < \infty.$$

- In [Nash (1958)] the moment bound (NASH-MB) is proved for deterministic strictly elliptic, (ELLIP), divergence-free (DIV-FREE) environments which in addition are *bounded* and have a *bounded* stream tensor. The proof of (NASH-MB)/(Q-TIGHT) as stated relies on not fully straightforward adaptation of ideas from [J Nash (1958)] and on an ergodic theorem due to [RV Chacon, DS Ornstein (1960)].

Step 4. (completion of proof): (COCY-ERG)&(Q-TIGHT)⇒(Q-ERR-BOUND):

$$\mathbf{P}_\omega\left(|V(\omega,X_t)| \geq \delta t^{1/2}\right) \leq$$

$$\stackrel{1}{\leq} \mathbf{P}_\omega\left(\{|V(\omega,X_t)| \geq \delta t^{1/2}\} \wedge \{|X_t| \leq K t^{1/2}\}\right) + \mathbf{P}_\omega\left(|X_t| > K t^{1/2}\right)$$

$$\stackrel{2}{\leq} \delta^{-1} t^{-1/2} \mathbf{E}_\omega\left(|V(\omega,X_t)| \mathbb{1}_{\{|X_t| \leq K t^{1/2}\}}\right) + K^{-1} t^{-1/2} \mathbf{E}_\omega\left(|X_t|\right)$$

$$\stackrel{3}{\leq} C \delta^{-1} t^{-(d+1)/2} \sum_{|x| \leq K t^{1/2}} |\Theta(\omega,x)| \qquad + M(\omega) K^{-1}$$

$$\stackrel{4}{\to} 0 \quad \pi\text{-a.s. as first } t \to \infty, \quad \text{then} \quad K \to \infty$$

1: straightforward decomposition

2: Markov's inequality (x2)

3: Nash's heat kernel bound (**??**) and moment bound (NASH-MB)

4: cocycle ergodic theorem (COCY-ERG) □ Thm 2

Diffusion in div-free random drift field $t \mapsto X(t) \in \mathbb{R}^d$

$$dX(t) = \Big(\frac{1}{2}\nabla \cdot r^2(X(t)) + b(X(t))\Big)\, dt + r(X(t))\, dB(t)$$

with infinitesimal generator

$$L := \frac{1}{2}\nabla \cdot r^2\, \nabla + b \cdot \nabla,$$

where

$$r = r(\omega) : \mathbb{R}^d \to \mathbb{R}_+^{d\times d} \qquad\qquad b = b(\omega) : \mathbb{R}^d \to \mathbb{R}^d$$

sufficiently (locally) regular, space-wise stationary & ergodic.

Conditions analogous to (UPPER), (LOWER)/(ELLIP), (STATI-STREAM), (H-1)/(H-1/ELLIP)/(H-1+ε/ELLIP).

The analogous theorems should hold - though, the details not fully worked out yet.

Historical comments (sketchy, far from complete):

- [SM Kozlov (1979)], [G Papanicolaou, SRS Varadhan (1981)]:
  $s \in \mathcal{L}^\infty$ $v \equiv 0$, self-adjoint, diffusion, initiation of the problem
- [H Osada (1983)], [SM Kozlov (1985)]:
  $s \in \mathcal{L}^\infty$, $h \in \mathcal{L}^\infty$, [O]: quenched diffusion; [K]: annealed walk
- [K Oelschläger (1988)], [A Fannjiang, G Papanicolaou (1996)]:
  $s =$ const., $h \in \mathcal{L}^2$, annealed, diffusion, with some restrictions
- [A Fannjiang, T Komorowski (1997)]:
  $s =$ const., $h \in \mathcal{L}^{d+\varepsilon}$, quenched diffusion.
- [T Komorowski, S Olla (2003)], [J-D Deuschel, H Kösters (2008)]:
  $s \in \mathcal{L}^\infty$, $h \in \mathcal{L}^\infty$, [K,O]: annealed walk, [D,K]: quenched walk
- [T Komorowski, C Landim, S Olla (2012)]:
  $s \in \mathcal{L}^\infty$, $h \in \mathcal{L}^d$, annealed walk, + diffusion with s, h Gaussian

Recall the trichotomy about (STATI-STREAM) (page 10).
Theorems 1&2 provide CLT/IP for case (Y1).
What happens in cases (Y2) and (N)?

When (STATI-STREAM) fails - case (N): expect *superdiffusive* (faster than $t^{1/2}$) large scale behaviour.

Two examples:

Manhattan: (see fig. on page 13) [Ledger-T-Valkó (2018)]

$d = 2: \; t^{5/4} \ll \mathbf{E}\left(|X_t|^2\right) \ll t^{3/2}$ conj: $\mathbf{E}\left(|X_t|^2\right) \asymp t^{4/3}$

$d = 3: \; t \log\log t \ll \mathbf{E}\left(|X_t|^2\right) \ll t \log t$ conj: $\mathbf{E}\left(|X_t|^2\right) \asymp t\sqrt{\log t}$

$d \geq 4:$ (ELLIP)&(H-1/ELLIP) hold Thm: quenched CLT

Diffusion in the curl of GFF $d = 2$: (rel. to 6-vertex on p. 13):

$$H(x,\omega) = \phi * GFF(x) \qquad\qquad b(x,\omega) = (\nabla \times H)(x,\omega)$$

$b : \mathbb{R}^2 \to \mathbb{R}^2$ zero-mean Gaussian vector field with covariances

$$\widehat{c}(p) = \widehat{\phi}(p)^2 \begin{pmatrix} \frac{p_2^2}{p_1^2 + p_2^2} & -\frac{p_1 p_2}{p_1^2 + p_2^2} \\ -\frac{p_1 p_2}{p_1^2 + p_2^2} & \frac{p_1^2}{p_1^2 + p_2^2} \end{pmatrix}$$

The diffusion: $$dX(t) = b(X(t))\,dt + dB(t)$$

The problem has some notoriety in the physics literature, starting with [Bouchaud-Comtet-Georges-LeDoussal (1987)].

Math results: [T-Valkó (2012)]:

$$t \log\log t \ll \mathbf{E}\left(|X_t|^2\right) \ll t \log t \qquad \text{conj: } \mathbf{E}\left(|X_t|^2\right) \asymp t\sqrt{\log t}$$

More recent: [Cannizzaro-Haunschmid·Sibitz-Toninelli (2022)], [Chatzigeorgiou-Morfe-Otto-Wang(2023+)]:

$$\mathbf{E}\left(|X_t|^2\right) \asymp t\sqrt{\log t} \qquad ✓$$

Moreover, [Armstrong-Bou·Rabee-Kuusi (2024+)]: for π-a.a. ω

$$(N\sqrt{\log N})^{-1/2} X(N\cdot) \Rightarrow W_\sigma(\cdot) \qquad ✓$$

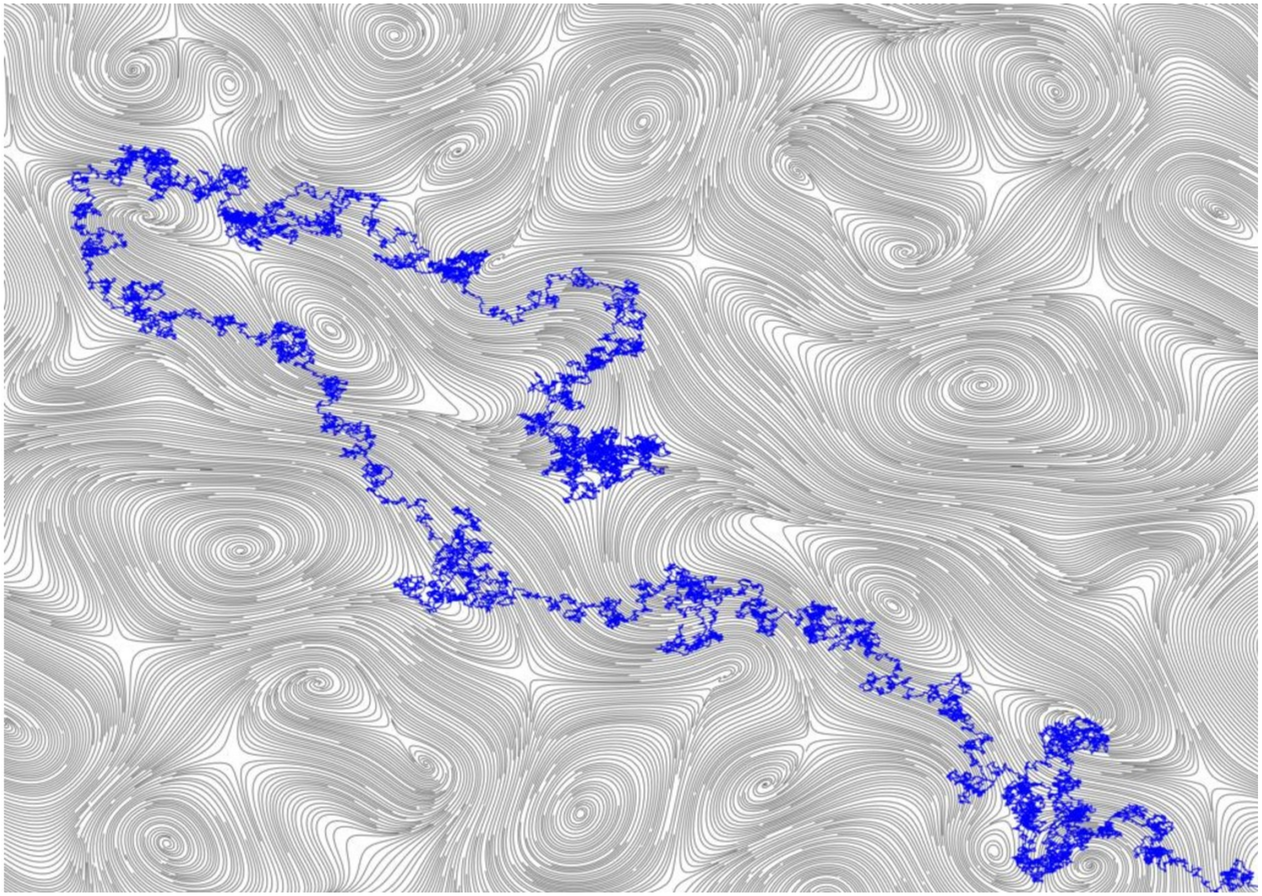
under the quenched measure $\mathbf{P}_\omega(\cdot)$.

Superdiffusive bounds (of [TV (2012)], [CHST (2022)]): by [HT Yau (2000)]'s variational method:

Proposition 6.

$$\langle \varphi, R_\lambda \varphi \rangle = \sup_{\psi \in \mathcal{L}_0^2} \left\{ 2\langle \psi, \varphi \rangle - \langle \psi, (\lambda I + S)\psi \rangle - \langle A\psi, (\lambda I + S)^{-1} A\psi \rangle \right\}.$$

A picture from [Armstrong-Bou·Rabee-Kuusi (2024+)] illustrating convective (super-)diffusion in random incompressible flow:



Proof of Proposition 1. Due to the Spectral Theorem,

$$\| \, |\Delta|^{-1} \partial_m \partial_l \, \|_{2\to 2} < \infty, \qquad\qquad m, l \in \mathcal{U}.$$

For $b \in \mathcal{D}^2$ define

$$g_{m;k,l} := |\Delta|^{-1} \partial_m \Big(\partial_l b_k - \partial_k b_l \Big) \in \mathcal{L}^2.$$

Then, g is rot-free in m, tensor in (k,l), and Helmholtz for $\partial_m b$:

$$g_{m;k,l}(\omega) + g_{n;k,l}(\tau_m \omega) = g_{n;k,l}(\omega) + g_{m;k,l}(\tau_n \omega)$$

$$g_{m;-k,l}(\tau_k \omega) = g_{m;k,-l}(\tau_l \omega) = g_{m;l,k}(\omega) = -g_{m;k,l}(\omega)$$

$$\sum_{l \in \mathcal{U}} g_{m;k,l}(\omega) = b_k(\tau_m \omega) - b_k(\omega)$$

Let $(x,k,l) \mapsto H_{k,l}(x,\omega)$ be defined by its gradients:

$$H_{k,l}(0,\omega) = 0, \quad H_{k,l}(x+m,\omega) - H_{k,l}(x,\omega) = g_{m;k,l}(\tau_x \omega). \quad \checkmark$$

Wanted: $|\Delta|^{-1}\partial_m\partial_l : \mathcal{L}^p \to \mathcal{L}^p$, $p \in [1,2)$?. Formally,

$$|\Delta|^{-1}\partial_m\partial_l f(\omega) = \sum_{z\in\mathbb{Z}^d} \mathcal{K}_{l,m}(z) f(\tau_z\omega)$$

$$\mathcal{K}_{l,m}(z) := (2\pi)^{-d} \int_{[-\pi,\pi]^d} e^{ip\cdot z} \frac{(e^{iz\cdot m} - 1)(e^{iz\cdot l} - 1))}{\sum_{j=1}^d (1 - \cos p_j)} dp$$

Theorem 6. `[Calderón-Zygmund thm,` $\mathbb{Z}^d$`-version]`
Let $\mathcal{K} : \mathbb{Z}^d \to \mathbb{C}$, *and* $\widehat{\mathcal{K}} : [-\pi,\pi]^d \to \mathbb{C}$ *its FT. Assume*

(A) $\|\widehat{\mathcal{K}}\|_\infty < \infty$, (B) $\sup_L \max_{|x|\le L} \sum_{|y|\ge 2L} |\mathcal{K}(x-y) - \mathcal{K}(y)| < \infty$.

Define, for $f \in \ell^1(\mathbb{Z}^d)$, $(Kf)(x) := \sum_{y\in\mathbb{Z}^d} \mathcal{K}(x-y) f(y)$ *Then*

$$\|Kf\|_{1w} := \sup_{0<\lambda<\infty} \lambda |\{x \in \mathbb{Z}^d : |Kf(x)| > \lambda\}| \le C\|f\|_1 \qquad \text{(C-Z)}$$

with some $C < \infty$, *depending on* (A)&(B).

Proposition 7. `[Calderón-Zygmund theorem, ergodic version]`
Let $\big(\Omega, \pi, (\tau_z : \Omega \to \Omega, z \in \mathbb{Z}^d)\big)$ *be an ergodic* $\mathbb{Z}^d$*-action,*
$\mathcal{K} : \mathbb{Z}^d \to \mathbb{C}$ *a kernel like in Thm 6 and, for* $\varphi \in \mathcal{L}^2(\Omega, \pi)$*, let*
$(K\varphi)(\omega) := \sum_{y\in\mathbb{Z}^d} \mathcal{K}(y)\varphi(\tau_y\omega)$*. (Nb.* $\|K\varphi\|_2 \le \|\widehat{\mathcal{K}}\|_\infty \|\varphi\|_2$*.) Then*

$$\|K\varphi\|_{1w} := \sup_{0<\lambda<\infty} \lambda\, \pi(\{\omega \in \Omega : |K\varphi(\omega)| > \lambda\}) \le C\|\varphi\|_1, \quad \text{(C-Z-ERG)}$$

with some $C < \infty$*, depending on* (A)&(B)*. The operator* K *extends to* $K : \mathcal{L}^1(\Omega, \pi) \to \mathcal{L}^{1w}(\Omega, \pi)$*.*

Proof of Proposition 7: Assume first that $\text{supp}(\mathcal{K}) \subset B_L$ and with slight *abuse of notation* denote by *the same symbol* K the operators $K : \ell(\mathbb{Z}^d) \to \ell(\mathbb{Z}^d)$ and $K : \mathcal{L}(\Omega, \pi) \to \mathcal{L}(\Omega, \pi)$,

$$(Kf)(x) := \sum_{y\in\mathbb{Z}^d} \mathcal{K}(x-y)f(y) \qquad (K\varphi)(\omega) := \sum_{y\in\mathbb{Z}^d} \mathcal{K}(y)\varphi(\tau_y\omega)$$

The meaning will be clear from the context.

Let $\varphi \in \mathcal{L}^2(\Omega, \pi)$, and $f_N \in \ell^1(\mathbb{Z}^d)$, $f_N(x) := \varphi(\tau_x \omega)\mathbb{1}_{\{|x| \le N\}}$.

$$|B_N|^{-1}\lambda|\{x \in B_{N-L} : |Kf_N(x)| > \lambda\}|$$

$$\le |B_N|^{-1}\lambda|\{x \in \mathbb{Z}^d : |Kf_N(x)| > \lambda\}| \overset{(\texttt{C-Z})}{\le} C|B_N|^{-1}\|f_N\|_1$$

On the other hand, by the ergodic theorem, π-a.s.

$$\lim_{N\to\infty} |B_N|^{-1}|\{x \in B_{N-L} : |Kf_N(x)| > \lambda\}| = \pi(\{\omega : |K\varphi(\omega)| > \lambda\})$$

$$\lim_{N\to\infty} |B_N|^{-1}\|f_N\|_1 = \|\varphi\|_1$$

Putting these together we obtain exactly the bound (C-Z-ERG). Since the constant C does not depend on L the condition $\text{supp}(\mathcal{K}) \subset B_L$ can be lifted. □Proposition 7

Corollary 1.

$$\|\,|\Delta|^{-1/2}\partial_l\,\|_{1\to 1\mathrm{w}} < \infty, \quad \|\,|\Delta|^{-1}\partial_m\partial_l\,\|_{1\to 1\mathrm{w}} < \infty,$$

$$\|\,|\Delta|^{-1/2}\partial_l\,\|_{p\to p} < \infty, \qquad \Big(\|\,|\Delta|^{-1}\partial_m\partial_l\,\|_{p\to p} < \infty\Big), \quad p \in (1,2]$$

The $p \in (1,2)$ cases follow from Marcinkiewicz interpolation.

Proposition 1 is proved exactly as in the $b \in \mathcal{L}^2$ case. $\square$Prop. 1

Proof of Proposition 4 By induction on d.

- $d = 1$: Birkhoff ✓.
- Notation: $\Lambda_N^d := [0, N-1]^d$; $(\underline{n}, m) \in \Lambda_N^d \times \Lambda_N^1$;
- $L \in \mathbb{N}$ fixed (at the end of the proof $L \to \infty$)

$$\sum_{m \in \Lambda_N^1} \sum_{\underline{n} \in \Lambda_N^d} |\Psi(\underline{n}, m)| \leq \sum_{l=0}^{L-1} \sum_{j=0}^{\lceil N/L \rceil - 1} \sum_{\underline{n} \in \Lambda_N^d} |\Psi(\underline{n}, jL + l)|$$

$$\leq \underbrace{(N+1) \sum_{\underline{n} \in \Lambda_N^d} |\Psi(\underline{n}, 0)|}_{1} + \frac{1}{L} \sum_{l=0}^{L-1} \underbrace{(N+L) \sum_{\underline{n} \in \Lambda_N^d} |\Psi(\underline{n}, l) - \Psi(\underline{n}, 0)|}_{2}$$

$$+ \frac{1}{L} \sum_{l=0}^{L-1} \underbrace{L \sum_{j=1}^{\lceil N/L \rceil - 1} \sum_{i=0}^{j-1} \sum_{\underline{n} \in \Lambda_N^d} |\Psi(\underline{n}, (i+1)L + l) - \Psi(\underline{n}, iL + l)|}_{3}$$

First term:

$$\lim_{N\to\infty} N^{-(d+2)}(N+1) \sum_{\underline{n}\in\Lambda_N^d} |\Psi(\underline{n},0)| = \lim_{N\to\infty} N^{-(d+1)} \sum_{\underline{n}\in\Lambda_N^d} |\Psi(\underline{n},0)| \overset{☺}{=} 0$$

☺: use the induction hypothesis

Second term: use the multidimensional a.s. ergodic theorem ($l \in [0, L-1]$ is fixed)

$$\lim_{N\to\infty} N^{-(d+2)}(N+L) \sum_{\underline{n}\in\Lambda_N^d} |\Psi(\underline{n},l) - \Psi(\underline{n},0)|$$
$$\leq 2 \lim_{N\to\infty} N^{-d-1} \sum_{\underline{n}\in\Lambda_N^d} |\Psi(\underline{n},l) - \Psi(\underline{n},0)| \overset{☺}{=} 0$$

☺: use the multidimensional a.s. ergodic theorem

Third term:

$$\lim_{N\to\infty} N^{-(d+2)} L \sum_{j=1}^{\lceil N/L\rceil-1} \sum_{i=0}^{j-1} \sum_{\underline{n}\in\Lambda_N^d} |\Psi(\underline{n},(i+1)L+l)-\Psi(\underline{n},iL+l)| =$$

$$\lim_{N\to\infty} \frac{L^2}{N^2} \sum_{j=1}^{\lceil N/L\rceil-1} j \; \underbrace{(jN^d)^{-1} \sum_{i=0}^{j-1} \sum_{\underline{n}\in\Lambda_N^d} \frac{|\Psi(\underline{n},(i+1)L+l)-\Psi(\underline{n},iL+l)|}{L}}$$

$$\overset{☺}{=} L^{-1}\mathbf{E}\left(|\Psi(\underline{0},L)-\Psi(\underline{0},0)|\right).$$

☺: multidim. unrestricted erg. thm. cf [Zygmund (1951)].

Finally, letting $L\to\infty$, by the multidimensional version of the mean ($\mathcal{L}^1$) ergodic theorem we obtain (COCY-ERG) in dimension $d+1$. □Prop. 4

Proof of Proposition 5.