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LARGE-SCALE BEHAVIOUR OF RANDOM MOTIONS WITH LONG MEMORY

-3-
(A) KIPNIS-VARADHAN THEORY [details]
(B) NASH THEORY [details]

PDE AND PROBABILITY - SUMMER SCHOOL
Sorbonne Université, Paris, June 16-25, 2025

(A) Martingale approx. - Kipnis-Varadhan theory:

$t \mapsto \eta_t, \eta_t^*$ stationary and ergodic continuous-time Markov Chain and its time reversal, on $(\Omega, \mathcal{F}, \pi)$. Assume sufficient regularity.

Its semigroup, resolvent, and infinitesimal generator, acting on $\mathcal{L}^p(\Omega, \pi)$, $p \in [1, \infty]$:

$$P_t f(\omega) := \mathbf{E}_\omega (f(\eta_t)) \qquad \|P_t\|_{p\to p} \le 1$$

$$R_\lambda := \int_0^\infty e^{-\lambda t} P_t \, dt \qquad \|R_\lambda\|_{p\to p} \le \lambda^{-1}$$

$$L := \text{st-}\lim_{t\to 0} t^{-1}(P_t - I) \qquad P_t = e^{tL} \qquad R_\lambda = (\lambda I - L)^{-1}$$

Recall `Hille-Yosida theory` of strongly continuous contraction semigroups on Banach spaces. We assume that on $\mathcal{L}^2$

$$L = (-S + A)/2 \qquad L^* = (-S - A)/2$$

That is: L and L^* have a common core of definition.

Given $\varphi \in \mathcal{L}_0^2$ we want to understand **CLT for** $t^{-1/2}\int_0^t \varphi(\eta_s)\,ds$.
[Recall Doeblin's Thm for $\#\Omega < \infty$.]

The variance: Denote

$$\begin{aligned}
c(t) &:= \mathbf{E}\left(\varphi(\eta_{t_0})\varphi(\eta_{t_0+t})\right) && = \langle\, \varphi\,,\, P_t\varphi\,\rangle \\
d(t) &:= \mathbf{Var}\left(t^{-1/2}\int_0^t \varphi(\eta_s))ds\right) && = 2\int_0^t \frac{t-s}{t}c(s)ds \qquad \text{(HW)} \\
\widehat{d}(\lambda) &:= 2\int_0^\infty e^{-\lambda s}c(s) && = 2\langle\, \varphi\,,\, R_\lambda\varphi\,\rangle
\end{aligned}$$

(c for covariance, d for diffusivity). Note that (HW)

$$\widehat{d}(\lambda) = \lambda^2\int_0^\infty e^{-\lambda s} s d(s) ds,$$

and hence (HW)

$$0 \le \varliminf_{t\to\infty} d(t) \le \varliminf_{\lambda\to 0} \widehat{d}(\lambda) \le \varlimsup_{\lambda\to 0} \widehat{d}(\lambda) \le \varlimsup_{t\to\infty} d(t) \le \infty \qquad (1)$$

We are primarily interested in $\sigma^2 := \lim_{t\to\infty} d(t)$ (assuming the limit exists) but we can better juggle with $\widehat{d}(\lambda)$.

If $t \mapsto \eta_t$ is **reversible**, $L = L^* = -S$, then $c(s) = \langle P_{s/2}\varphi\, ,\, P_{s/2}\varphi \rangle \geq 0$, and $t \mapsto d(t)$ is monotone increasing (HW). Thus

$$\sigma^2 := \lim_{t\to\infty} d(t) \in (0,\infty]$$

exists. Otherwise, anything in (1) could happen.

Proposition 1. [Variational Formula] [HT Yau (2000)]
The following variational formula holds, for $\varphi \in \mathcal{L}^2$,

$$\langle \varphi\, ,\, R_\lambda \varphi \rangle = \sup_{\psi\in\mathcal{L}^2_0} \Big\{ 2\langle \psi\, ,\, \varphi \rangle - \langle \psi\, ,\, (\lambda I + S)\psi \rangle - \langle A\psi\, ,\, (\lambda I + S)^{-1} A\psi \rangle \Big\}. \tag{2}$$

Proof of Proposition 1:

$$\langle \varphi , R_\lambda \varphi \rangle = \langle R_\lambda \varphi , (\lambda I - L^*) R_\lambda \varphi \rangle = \langle R_\lambda \varphi , (\lambda I - L) R_\lambda \varphi \rangle$$

$$= \langle R_\lambda \varphi , (\lambda I + S) R_\lambda \varphi \rangle$$

$$= \sup_{\psi \in \mathcal{L}^2_0} \left\{ 2 \langle \psi , R_\lambda \varphi \rangle - \langle \psi , (\lambda I + S)^{-1} \psi \rangle \right\} \quad \text{(selfadj. var. form}$$

$$= \sup_{\psi \in \mathcal{L}^2_0} \left\{ 2 \langle \psi , \varphi \rangle - \langle (\lambda I - L^*) \psi , (\lambda I + S)^{-1} (\lambda I - L^*) \psi \rangle \right\}$$

$$= \sup_{\psi \in \mathcal{L}^2_0} \left\{ 2 \langle \psi , \varphi \rangle - \langle \psi , (\lambda I + S) \psi \rangle - \langle A\psi , (\lambda I + S)^{-1} A\psi \rangle \right\}$$

□Proposition 1

In particular

$$\langle \varphi , (\lambda I - L)^{-1} \varphi \rangle \leq \langle \varphi , (\lambda I + S)^{-1} \varphi \rangle \qquad (3)$$

Probabilistic meaning: $-S$ is the IG of the $\mathcal{L}^2$-semigroup of the "symmetrized" (and thus reversible) MP ξ_t. (3) means that

$$\int_0^\infty e^{-\lambda t}\mathbf{Var}\left(\int_0^t \varphi(\eta_s)ds\right)dt \le \int_0^\infty e^{-\lambda t}\mathbf{Var}\left(\int_0^t \varphi(\xi_s)ds\right)dt$$

The subspace $\mathcal{H}_{-1}$

$$\mathcal{H}_{-1} := \{\varphi \in \mathcal{L}^2_0 : \|\varphi\|^2_{-1} := \lim_{\lambda\to 0}\langle\, \varphi\, , \, (\lambda I + S)^{-1}\varphi\,\rangle = \|S^{-1/2}\varphi\|^2 < \infty\}$$

$$= \mathrm{Dom}(S^{-1/2}) = \mathrm{Ran}(S^{1/2})$$

where $S^{\pm 1/2}$ are defined in terms of the *spectral theorem*.

Remark: Unboundedness of L, L^*, S, A is a nuisance – but not an essential problem. **The main issue is unboundedness of $S^{-1/2}$.**

Proposition 2. `[`H_{-1} `rules!]` [Varadhan (1995)]
If $\varphi \in \mathcal{H}_{-1}$ *then for all* $t \in [0, \infty)$

$$\mathbf{Var}\left(\int_0^t \varphi(\eta_s)ds\right) \le 2\|S^{-1/2}\varphi\|^2 t.$$

Proof of Proposition 2: Let

$$g_\lambda := (\lambda I + S)^{-1}\varphi$$

and for $t \in [0, T]$ ($T < \infty$ fixed)

$$M_t^\lambda := g_\lambda(\eta_t) - g_\lambda(\eta_0) - \int_0^t Lg_\lambda(\eta_s)ds,$$

$$M_t^{*\lambda} := g_\lambda(\eta_t) - g_\lambda(\eta_T) - \int_t^T L^* g_\lambda(\eta_s)ds$$

Then M_t^λ and $M_t^{*\lambda}$ are forward, respectively, backward (Dynkin) martingales and for $0 \le s \le t \le T$.

$$M_t^\lambda - M_s^\lambda = g_\lambda(\eta_t) - g_\lambda(\eta_s) - \int_s^t Lg_\lambda(\eta_r)dr,$$

$$M_s^{*\lambda} - M_t^{*\lambda} = g_\lambda(\eta_s) - g_\lambda(\eta_t) - \int_s^t L^* g_\lambda(\eta_r)dr,$$

Adding the two eqs we get

$$2\int_s^t Sg_\lambda(\eta_r)dr = \left(M_t^\lambda - M_s^\lambda\right) + \left(M_s^{*\lambda} - M_t^{*\lambda}\right).$$

By Schwarz

$$2\mathbf{Var}\left(\int_s^t Sg_\lambda(\eta_r)dr\right) \le \mathbf{Var}\left(M_t^\lambda - M_s^\lambda\right) + \mathbf{Var}\left(M_s^{*\lambda} - M_t^{*\lambda}\right) \overset{\text{HW}}{=} 4\langle\, g_\lambda\,,\, Sg_\lambda\,\rangle(t-s)$$

By the Spectral Theorem:

$$\lim_{\lambda\to 0} Sg_\lambda = \varphi \qquad\qquad \lim_{\lambda\to 0}\langle\, g_\lambda\,,\, Sg_\lambda\,\rangle = \|S^{-1/2}\varphi\|^2$$

□Proposition 2

Reversible setting: $L = L^* = -S \leq 0$**,** $A = 0$

What is $\mathcal{H}_{-1}$? 1. in terms of spectral measure: Denote by ν_φ the spectral measure (for the operator $S = S^* \geq 0$) of φ. Then

$$\left\{\varphi \in \mathcal{H}_{-1}\right\} \Leftrightarrow \left\{ \int_0^\infty x^{-1} d\nu_\varphi(x) < \infty \right\}$$

(only the singularity at 0 matters). Except for very special cases (e.g. RW in random scenery) the spectral integral is not explicitly computable, but the spectral integrals are very useful for qualitative arguments.

What is $\mathcal{H}_{-1}$? 2. in terms of variance: This is the main point!

$$\left\{\varphi \in \mathcal{H}_{-1}\right\} \Leftrightarrow \left\{ \lim_{t\to\infty} \mathbf{Var}\left(t^{-1/2} \int_0^t \varphi(\eta_s) ds \right) < \infty \right\}$$

Theorem 1. [C Kipnis, SRS Varadhan (1986)]
(i) **Discrete time.** *Assume* $P = P^*$ *(that is:* $n \mapsto \eta_n$ *is reversible). If* $\varphi \in \mathcal{H}_{-1}$ *then there exists an* $\mathcal{L}^2$*-martingale* $n \mapsto M_n$*, with stationary and ergodic increments, adapted to the filtration* $\mathcal{F}_n = \sigma(\eta_m : 0 \le m \le n)$*, with variance*

$$\mathbf{Var}\,(M_n) = \sigma^2 n, \qquad \sigma^2 = 2\|(I-P)^{-1/2}\varphi\|^2 - \|\varphi\|^2,$$

and such that

$$\lim_{N\to\infty} N^{-1}\mathbf{Var}\left(\sum_{k=0}^{N-1}\varphi(\eta_k) - M_N\right) = 0.$$

In particular, the finite dimensional marginal distributions of the scaled process [turn page]

$$t \mapsto N^{-1/2}\sum_{k=0}^{[Nt]}\varphi(\eta_k) \tag{4}$$

Theorem. 1 − ctd
converge to those of a BM of variance σ^2 − *in the following sense (written for 1-d marginals): Let* $F : \mathbb{R} \to \mathbb{R}$ *be bdd and continuous, and* $\xi \sim \mathcal{N}(0, 1)$, *then*

$$\lim_{T\to\infty} \int_\Omega |\mathbf{E}_\omega \left(F(N^{-1/2} \sum_{k=0}^{[Nt]} \varphi(\eta_k)) \right) - \mathbf{E} \left(F(\sigma\sqrt{t}\xi) \right)| \, d\pi(\omega) = 0 \quad (5)$$

(ii) **Continuous time.** *Assume* $L = L^* = -S$ *(that is:* $t \mapsto \eta_t$ *reversible). If* $\varphi \in \mathcal{H}_{-1}$ *then there exists an* $\mathcal{L}^2$*-martingale* $t \mapsto M_t$, *with stationary and ergodic increments, adapted to the filtration* $\mathcal{F}_t = \sigma(\eta_s : 0 \le s \le t)$, *with variance*

$$\mathbf{Var}\,(M_t) = \sigma^2 t, \qquad \sigma^2 = 2\|S^{-1/2}\varphi\|^2,$$

and such that [turn page]

Theorem. 1 − ctd

$$\lim_{T\to\infty} T^{-1}\mathbf{Var}\left(\int_0^T \varphi(\eta_s)ds - M_T\right) = 0.$$

In particular, the finite dim. marginals of the scaled process

$$t \mapsto T^{-1/2}\int_0^{Tt} \varphi(\eta_s)ds \qquad (6)$$

converge to those of a BM of variance σ^2*, in the sense of* (5).

Remarks/Comments:

1. Comment on (5): Here are three ways to formulate the CLT for (4)/(6) – in ascending order of strength

$$\int_\Omega \left(\mathbf{E}_\omega\left(F(T^{-1/2}\int_0^{Tt}\varphi(\eta_s)\,ds)\right) - \mathbf{E}\left(F(\sigma\sqrt{t}\xi)\right)\right)d\pi(\omega) \overset{T\to\infty}{\rightarrow} 0 \quad (7)$$

$$\int_\Omega \left|\mathbf{E}_\omega\left(F(T^{-1/2}\int_0^{Tt}\varphi(\eta_s)\,ds)\right) - \mathbf{E}\left(F(\sigma\sqrt{t}\xi)\right)\right|d\pi(\omega) \overset{T\to\infty}{\rightarrow} 0 \quad (8)$$

$$\left|\mathbf{E}_\omega\left(F(T^{-1/2}\int_0^{Tt}\varphi(\eta_s)\,ds)\right) - \mathbf{E}\left(F(\sigma\sqrt{t}\xi)\right)\right| \quad \pi-\text{a.s.} \quad (9)$$

Wording:

(7) = averaged/annealed wrt the initial condition

(8) = in probability wrt the initial condition

(9) = almost sure/quenched wrt the initial condition.

Theorem 1 states the CLT in the sense of (8).

2. Tightness in $\mathbb{D}(\mathbb{R})$ of the scaled processes (4), respectively, (6) is also proved in the reversible case. Thus, full *invariance principle* holds. I will not cover the tightness part.

3. Theorem 1 is *optimal* for the reversible setting in the sense that it proves the CLT under the minimal condition of *finite asymptotic variance*. **Check finite asymptotic variance, get CLT!** (However, checking the finiteness of the asymptotic variance in concrete cases may be tricky.)

4. Applications: Whenever the underlying MP is reversible. (However, checking finiteness of the asymptotic variance may be tricky!) E.g. RW in random scenery, RW among random conductances, tagged particle diffusion in SEP (and other symmetric interacting particle systems), traditional MCMC.

6. Proof: tour-de-force of Hilbert sp. calculus & SpecThm.

General - non-reversible setting:

Theorem 2. [B Tóth (1986)], following [K&V (1986)] **Continuous time.** *Let* $\varphi \in \mathcal{L}^2_0$. *If the following conditions hold*

$$\text{(A)} \qquad \lim_{\lambda \to 0} \lambda^{1/2} \| R_\lambda \varphi \| = 0 \tag{10}$$

$$\text{(B)} \qquad \lim_{\lambda \to 0} S^{1/2} R_\lambda \varphi =: v \in \mathcal{L}^2_0 \quad \textit{exists,} \tag{11}$$

then

$$\text{(C)} \qquad \sigma^2 := 2 \lim_{\lambda \to 0} (\varphi, R_\lambda \varphi) = 2 \|v\|^2 \in (0, \infty) \quad \textit{exists,} \tag{12}$$

and there exists an $\mathcal{L}^2$*-martingale* $t \mapsto M_t$, *with stationary and ergodic increments, adapted to the filtration* $\mathcal{F}_t = \sigma(\eta_s : 0 \le s \le t)$, *with variance*

$$\mathbf{Var}\,(M_t) = \sigma^2 t,$$

and such that

[turn page]

Theorem. 2 – ctd.

$$\lim_{T\to\infty} T^{-1}\mathbf{Var}\left(\int_0^T \varphi(\eta_s)ds - M_T\right) = 0. \tag{13}$$

In particular, the finite dim. marginals of the scaled process in (6) *converge to those of a BM of variance* σ^2 *– in the sense* (5).

Remarks/Comments:

1. From general "abstract nonsense" it follows that

$$\lim_{\lambda\to 0} \lambda\|R_\lambda\varphi\| = 0. \qquad \text{(HW)}$$

Compare this with condition (A) in (10).

2. $\lambda\|R_\lambda\varphi\|^2 + \|S^{1/2}R_\lambda\varphi\|^2 \overset{\text{(HW)}}{=} \langle\, \varphi\,,\, R_\lambda\varphi \,\rangle \overset{(3)}{\leq} \langle\, \varphi\,,\, (\lambda I + S)^{-1}\varphi \,\rangle$

and hence

$$\text{(A \& B)} \;\Rightarrow\; \text{(C)} \;\Rightarrow\; \underbrace{\overline{\lim_{\lambda\to 0}}\langle\, \varphi\,,\, R_\lambda\varphi \,\rangle < \infty}_{\text{(C}'\text{)}} \;\Leftarrow\; \{\varphi \in \mathcal{H}_{-1}\}.$$

3. In [T (1986)] the following condition is set

$$\text{(D)} \qquad \lim_{\lambda,\mu\to 0}(\lambda+\mu)\langle\, R_\lambda\varphi\,,\, R_\mu\varphi \,\rangle = 0. \tag{14}$$

It is, however, straightforward that

$$(\lambda+\mu)\langle\, R_\lambda\varphi\,,\, R_\mu\varphi \,\rangle \overset{\text{(HW)}}{=} \|S^{1/2}(R_\lambda\varphi - R_\mu\varphi)\|^2 + \lambda\|R_\lambda\varphi\|^2 + \mu\|R_\mu\varphi\|^2.$$

and thus,

$$\text{(A \& B)} \;\Leftrightarrow\; \text{(D)}$$

4. In the reversible setting $L = L^* = -S$, due to the Spectral Theorem, we have (HW)

$$\{\varphi \in \mathcal{H}_{-1}\} \;\Leftrightarrow\; \text{(C)} \;\Leftrightarrow\; \text{(C}'\text{)} \;\Leftrightarrow\; \text{(B)} \;\Rightarrow\; \text{(A)}$$

5. Conditions (A) and (B) (in the non-reversible cases) are subtle and difficult to check. We'll see computationally friendlier sufficient conditions.

Proof of Theorem 2. (3.5 slides)
The proof in [T (1986)] follows the main steps of [K&V (1986)]. Replacing, however, spectral calculus (not available in non-reversible cases) with resolvent calculus and modifying appropriately the conditions.

Denote

$$u_\lambda := R_\lambda \varphi.$$

To prove (12) note that

$$\begin{aligned}
\lim_{\lambda\to 0} \langle\, \varphi \,,\, u_\lambda \,\rangle &= \lim_{\lambda\to 0} \langle\, (\lambda I - L) u_\lambda \,,\, u_\lambda \,\rangle \\
&= \lim_{\lambda\to 0} \lambda \|u_\lambda\|^2 + \lim_{\lambda\to 0} \|S^{1/2} u_\lambda\|^2 \\
&= \|v\|^2.
\end{aligned}$$

Let

$$M_t^\lambda := u_\lambda(\eta_t) - u_\lambda(\eta_0) - \int_0^t L u_\lambda(\eta_s) ds,$$

with

$$\mathbf{E}\left(M_t^\lambda\right) = 0, \qquad \mathbf{Var}\left(M_t^\lambda\right) \overset{\text{(HW)}}{=} 2\|S^{1/2} u_\lambda\|^2 t.$$

Then, by Doob's inequality

$$\mathbf{E}\left(\sup_{0\le s\le t}|M_s^\lambda - M_s^\mu|^2\right) \le 2\mathbf{E}\left(|M_t^\lambda - M_t^\mu|^2\right) = 4\|S^{1/2}(u_\lambda - u_\mu)\|^2 t$$

and hence, due to (B) in (11) there exists an $\mathcal{L}^2$-martingale $t \mapsto M_t$ (adapted to the same filtration) such that

$$\mathbf{E}\left(\sup_{0\le s\le t}|M_s^\lambda - M_s|^2\right) \le 4\|S^{1/2}u_\lambda - v\|^2 t.$$

Furthermore,

$$\mathbf{E}\,(M_t) = 0, \qquad \mathbf{Var}\,(M_t) = \lim_{\lambda\to 0}\mathbf{Var}\left(M_t^\lambda\right) = 2\|v\|^2 t.$$

We write

$$\int_0^t \varphi(\eta_s)ds = M_t + (M_t^\lambda - M_t) - u_\lambda(\eta_t) + u_\lambda(\eta_0) + \int_0^t \lambda u_\lambda(\eta_s)ds$$

and bound the error terms. In turn we get

$$\mathbf{E}\left(\sup_{0\le s\le t}|M_s^\lambda - M_s|^2\right) \overset{\text{Doob}}{\le} 4t\|S^{1/2}u_\lambda - v\|^2$$

$$\mathbf{E}\left(|u_\lambda(\eta_0)|^2\right) = \mathbf{E}\left(|u_\lambda(\eta_t)|^2\right) \overset{\text{stat.}}{=} t(t\lambda)^{-1}\lambda\|u_\lambda\|^2$$

$$\mathbf{E}\left(\sup_{0\le s\le t}|\int_0^s \lambda u_\lambda(\eta_r)dr|^2\right) \overset{\text{Schwarz}}{\le} t\mathbf{E}\left(\int_0^t \lambda^2|u_\lambda(\eta_r)|^2dr\right)$$

$$= t(t\lambda)\lambda\|u_\lambda\|^2$$

Choosing $\lambda = t^{-1}$ and letting $t \to \infty$, due to conditions (A) and (B) in (10), respectively, (11), we readily get (13).

□Theorem 2

A direct application. The conditions (A & B) of Theorem 2 are essentially optimal, but hard to check directly. We'll see sufficient conditions soon. However, here is a RWRE model where the theorem applies directly.

RW among Random Scatterers in $\mathbb{Z}^d$, or randomized Lorentz gas.

$((\gamma_{u,v}(x))_{u,v\in\mathcal{U}})_{x\in\mathbb{Z}^d}$ random scattering matrices, spatially ergodic. Assume

$$\mathtt{BISTOCH}: \sum_{v\in\mathcal{U}} \gamma_{u,v}(x) = 1 = \sum_{u\in\mathcal{U}} \gamma_{u,v}(x) \qquad \mathtt{ELLIPT}: \gamma_{u,v}(x) \geq a > 0$$

The walk:

$$\mathbf{P}_\omega\left(X_{n+1} = x + v | X_n = x, X_{n-1} = x - u\right) = \gamma_{u,v}(x)$$

[BT (1986)]: $n^{-1/2} X_n \Rightarrow \mathcal{N}(0, \sigma^2)$, with $0 < \sigma < \infty$.

The Strong Sector Condition (SSC) [Varadhan (1996)]
There exists $C < \infty$, such that for any $f, g \in \mathcal{L}^2_0$:

$$|\langle f\,,\,Ag\,\rangle|^2 \le C^2 \langle f\,,\,Sf\,\rangle \langle g\,,\,Sg\,\rangle \qquad (\texttt{SSC})$$

or, equivalently

$$\|S^{-1/2} A S^{-1/2}\| \le C. \qquad (\texttt{SSC})$$

Theorem 3. [Varadhan (1996)] *Assume* (SSC) *and* $\varphi \in \mathcal{H}_{-1}$. *Then conditions* (A & B) *of Theorem 2 hold.*

Applications:

1. Tagged particle diffusion in 0-mean ASEP [Varadhan (1996)]
2. RW/diffusion in divergence-free drift field with *bounded stream tensor* (e.g. finite loops on $\mathbb{Z}^d$) [SM Kozlov (1985)], [T Komorowski, S Olla (2003)]

The Graded Sector Condition (GSC):

[S Sethuraman, SRS Varadhan, H-T Yau (2000)]

Structural assumption: grading

$$\mathcal{L}^2 = \overline{\bigoplus_{n=0}^{\infty} \mathcal{H}_n}, \qquad L = \sum_{\substack{m,n\geq 0\\ |m-n|\leq r}} L_{m,n}, \qquad L_{m,n} : \mathcal{H}_m \to \mathcal{H}_n$$

$$L_{m,n} = -S_{m,n} + A_{m,n}, \qquad S^*_{m,n} = S_{n,m}, \qquad A^*_{m,n} = -A_{n,m}$$

The GSC: There exists a *diagonal minoriser* $D : \mathcal{L}^2 \to \mathcal{L}^2$

$$D = D^* = \sum_{n\geq 0} D_n, \qquad D_n : \mathcal{H}_n \to \mathcal{H}_n, \qquad 0 \leq D \leq S$$

such that the following bounds hold

[turn page]

There exist $C < \infty$, $\beta < 1$ such that for all $f, g \in \mathcal{L}^2$

$$|\langle f , A_{n,m} g \rangle|^2 \leq C^2 n^{2\beta} \langle f , D_n f \rangle \langle g , D_m g \rangle \qquad \text{(GSC)}$$

or, equivalently

$$\|D_n^{-1/2} A_{n,m} D_m^{-1/2}\| \leq C^2 n^{\beta} \qquad \text{(GSC)}$$

Theorem 4. [S Sethuraman, SRS Varadhan, H-T Yau (2000)] *Assume* (GSC) *and* $\varphi \in \mathcal{H}_{-1}(D)$. *Then* (A & B) *of Thm 2 hold.*

Applications:

1. Tagged particle diffusion in ASEP, $d \geq 3$. [S-V-Y (2000)]
2. Diffusion in div-free *Gaussian* drift field in $\mathbb{R}^d$, $d \geq 3$. [T Komorowski, S Olla (2003)]
3. Self-repelling Brownian polymer and "myopic" self-avoiding random walk in $d \geq 3$ [I Horváth, BT, B Vető (2012)]

Relaxed Sector Condition [I Horváth, BT, B Vető (2012)]

Define the *bounded operators*

$$B_\lambda := (\lambda I + S)^{-1/2} A (\lambda I + S)^{-1/2}, \quad \|B_\lambda\| \leq \lambda^{-1}\|A\|, \quad B_\lambda^* = -B_\lambda$$

$$K_\lambda := (I + B_\lambda)^{-1}, \quad \|K_\lambda\| \leq 1.$$

Then

$$R_\lambda = (\lambda I + S)^{-1/2} K_\lambda (\lambda I + S)^{-1/2}$$

Assume that *by some miracle* ($\stackrel{\text{s.o.t.}}{\longrightarrow}$ = cvg. in strong op. top.)

$$K_\lambda \stackrel{\text{s.o.t.}}{\longrightarrow} K. \tag{15}$$

Then, for $\varphi \in \mathcal{H}_{-1}$, $\varphi = S^{1/2}\psi$

$$u_\lambda = \underbrace{(\lambda I + S)^{-1/2} K_\lambda (\lambda I + S)^{-1/2}}_{R_\lambda} \underbrace{S^{1/2}\psi}_{\varphi},$$

and, since, by the Spectral Theorem

$$\lambda^{1/2}(\lambda I + S)^{-1/2} \xrightarrow{\text{s.o.t.}} 0, \qquad S^{1/2}(\lambda I + S)^{-1/2} \xrightarrow{\text{s.o.t.}} I,$$

we readily get

(A): $\lambda^{1/2}u_\lambda = \lambda^{1/2}(\lambda I + S)^{-1/2}K_\lambda(\lambda I + S)^{-1/2}S^{1/2}\psi \to 0$

(B): $S^{1/2}u_\lambda = S^{1/2}(\lambda I + S)^{-1/2}K_\lambda(\lambda I + S)^{-1/2}S^{1/2}\psi \to K\psi =: v$

So, we look after sufficient conditions of (15).

The Relaxed Sector Condition Assume that there exists an (possibly unbounded!) *essentially skew-self-adjoint* $B : \mathcal{C} \to \mathcal{L}_0^2$ such that

$$
\begin{gathered}
(\forall f \in \mathcal{C}), \ (\forall \lambda > 0), \ (\exists f_\lambda \in \mathcal{L}_0^2) : \\
\lim_{\lambda \to 0} \|f_\lambda - f\| = 0, \qquad \lim_{\lambda \to 0} \|B_\lambda f_\lambda - Bf\| = 0
\end{gathered}
\tag{RSC}
$$

In plain words: there exists a *skew-self-adjoint* operator $B :=$ " $S^{-1/2} A S^{-1/2}$ " which is the graph-limit, as $\lambda \to 0$, of the sequence $B_\lambda := (\lambda I + S)^{-1/2} A (\lambda I + S)^{-1/2}$.

Theorem 5. [I Horváth, BT, B Vető (2012)]
Assume that the Relaxed Sector Condition (RSC) *holds* **and** $\varphi \in \mathcal{H}_{-1}$. *Then* (A & B), *and thus the efficient martingale approximation in Theorem 2 hold.*

Remarks:

1. In application one can naturally define $B :=$ " $S^{-1/2} A S^{-1/2}$ " as a skew-Hermitian (skew-symmetric) operator on a dense subspace $\mathcal{C}$. The point (and difficulty!) is to prove that its closure is indeed skew-self-adjoint. One must, e.g., check **von Neumann's criterion**

$$\overline{\mathrm{Ran}(I \pm B)} = \mathcal{L}^2$$

2. $$\texttt{(SSC)} \underbrace{\Longrightarrow}_{\text{straightforward}} \texttt{(GSC)} \underbrace{\Longrightarrow}_{\text{[HTV (2012)]}} \texttt{(RSC)}$$

Applications:

1. Conceptual: Streamlined proof of [SVY (2000)]'s Thm. 4
2. RW/Diffusion in div-free random drift: ("weak") [Kozma, T (2017)], [T (2025)], (quenched) [T (2018)]

Proof of Theorem 5:

Lemma 1. (No surprizes.) *Let $\mathcal{H}$ be a separable Hilbert space, $B_n \in \mathcal{B}(\mathcal{H})$, $1 \le n < \infty$, and $B_\infty = B$ densely defined, closed over $\mathcal{H}$ with a core of definition $\mathcal{C}$. Assume*

(i) $\mu \in \mathbb{C}$ is such that $\sup_{1 \le n \le \infty} \|(\mu I - B_n)^{-1}\| < \infty$.

(ii) For all $f \in \mathcal{C}$ there exists a sequence $f_n \in \mathcal{H}$, $1 \le n < \infty$, s.t.

$$\lim_{n \to \infty} \|f_n - f\| = 0, \qquad \text{and} \qquad \|B_n f_n - Bf\| = 0$$

Then

$$(\mu I - B_n)^{-1} \xrightarrow{\text{s.o.t.}} (\mu I - B)^{-1}.$$

Proof of Lemma 1. [Reminiscent of [Trotter-Kurtz] Let

$$\widehat{\mathcal{C}} := \{g = (\mu I - B)f : \; f \in \mathcal{C}\}$$

and note that $\widehat{\mathcal{C}}$ is dense in $\mathcal{H}$. Indeed, since $\mathcal{C}$ is a *core* of B and $\mathrm{Ran}(\mu I - B) = \mathcal{H}$ (due to (i)), for any $g \in \mathcal{H}$ and $\varepsilon > 0$ there exists $f \in \mathcal{C}$ so that $\|g - (\mu I - B)f\| < \varepsilon$.

Let $g \in \widehat{\mathcal{C}}$, $f = (\mu I - B)^{-1} g \in \mathcal{C}$ and $f_n \in \widehat{\mathcal{H}}$ as in (ii). Then

$((\mu I - B_n)^{-1} - (\mu I - B)^{-1})g =$

$(\mu(\mu I - B_n)^{-1} - I)(f - f_n) + (\mu I - B_n)^{-1}(B_n f_n - Bf) \to 0,$ as $n \to \infty$

Since $\widehat{\mathcal{C}}$ is dense in $\mathcal{H}$, this extends to all $g \in \mathcal{H}$. $\square$Lemma 1

Apply Lemma 1 with B_λ, $\lambda \to 0$, and $\mu = \pm 1$ to get (15)

$\square$Thm 5

Take home message:

If $B := S_0^{-1/2} A S_0^{-1/2}$ makes sense as a well-defined skew-self-adjoint (not merely densely defined skew-symmetric) operator, (checked through *von Neumann's criterion* $\overline{\mathrm{Ran}(B \pm I)} = \mathcal{L}^2$), then conditions (A) and (B) hold for $\varphi \in \mathcal{H}_{-1}$.

Three theorems from [JF Nash (1958)]

In the forthcoming section

$$\ell^p := \{f : \mathbb{Z}^d \to \mathbb{R} : \|f\|_p := \Big(\sum_{x\in\mathbb{Z}^d} |f(x)|^p\Big)^{1/p} < \infty\}, \quad p \in [1,\infty]$$

$$\langle\, f \,,\, g \,\rangle := \sum_{x\in\mathbb{Z}^d} f(x)g(x), \qquad f \in \ell^p, g \in \ell^q, \quad \frac{1}{p}+\frac{1}{q}=1$$

$$\widehat{f}(\theta) := \sum_{x\in\mathbb{Z}^d} e^{ix\cdot\theta} f(x), \qquad \theta \in [-\pi,\pi]^d$$

$$\Delta : \ell^p \to \ell^p, \qquad \Delta f(x) := \frac{1}{2}\sum_{k\in\mathcal{U}} (f(x+k) - f(x)),$$

$$\widehat{\Delta f}(\theta) = -\widehat{D}(\theta)\widehat{f}(\theta) \qquad \widehat{D}(\theta) = \sum_{i=1}^{d} (1-\cos(\theta_i))$$

I will use basic facts about the B-spaces ℓ^p and about the FT.

Theorem 6. [Nash Inequality] [J Nash (1958)]
There exists a constant $c = c(d) > 0$ *such that:*

$$\langle f \, , -\Delta f \, \rangle \geq c \, \|f\|_2^{2(d+2)/d} \, \|f\|_1^{-4/d}. \qquad \text{(NASH-INEQ)}$$

Theorem 7. [Nash Heat Kernel Bound] [J Nash (1958)]
Let $\big((p_k(x))_{k\in\mathcal{U}}\big)_{x\in\mathbb{Z}^d}$ *be doubly stochastic* (DIV-FREE) *and strongly elliptic* (ELLIPT) *n.n. rw jump rates on* $\mathbb{Z}^d$*:*

$$\sum_{l\in\mathcal{U}} p_l(x) = \sum_{l\in\mathcal{U}} p_{-l}(x+l) \qquad \text{(DIV-FREE)}$$

$$p_k(x) + p_{-k}(x+k) \geq 2s_* > 0, \qquad \text{(ELLIPT)}$$

and X_t *the rw on* $\mathbb{Z}^d$ *with these jump rates. Then, there exists a constant* $C = C(d, s_*) < \infty$ *such that* $\forall t \in (0, \infty)$, $\forall x, y \in \mathbb{Z}^d$

$$\mathbf{P}\left(X_t = y | X_0 = x\right) < C t^{-\frac{d}{2}}. \qquad \text{(NASH-HKB)}$$

Proof of Theorem 7. Due to (DIV-FREE) the counting measure on $\mathbb{Z}^d$ is stationary for X_t. Denote $P_t : \ell^p \to \ell^p$ the semigroup

$$P_t f(x) := \mathbf{E}\left(f(X_t) | X_0 = x\right)$$

and note that (due to stationarity of the counting measure) $\|P_t\|_{p\to p} \leq 1$ (HW). Then

$$\dot{P}_t = LP_t = P_t L \quad \text{with} \quad Lf(x) := \sum_{k\in\mathcal{U}} p_k(x)(f(x+k) - f(x))$$

Denote its self-adjoint and anti-self-adjoint parts

$$S := -(L + L^*)/2 \qquad A := (L - L^*)/2$$

$$Sf(x) \overset{\text{(HW)}}{=} -\sum_{k\in\mathcal{U}} s_k(x)(f(x+k) - f(x))$$

$$Af(x) \overset{\text{(HW)}}{=} \sum_{k\in\mathcal{U}} v_k(x)(f(x+k) - f(x))$$

Lemma 2. *There exists* $C = C(d, s_*) < \infty$ *such that*

$$\|P_t\|_{1\to 2} \le Ct^{-d/4} \qquad \|P_t^*\|_{1\to 2} \le Ct^{-d/4} \tag{16}$$

Proof of Lemma 2. Fix $f : \mathbb{Z}^d \to \mathbb{R}$ and let $u : [0, \infty) \to \mathbb{R}_+$

$$u(t) := \|P_t f\|_2^2 = \langle\, P_t f \,,\, P_t f \,\rangle.$$

Then

$$-\dot{u}(t) = 2\langle\, P_t f \,,\, S P_t f \,\rangle \overset{(\text{ELLIPT})}{\ge} 4 s_* \langle\, P_t f \,,\, -\Delta P_t f \,\rangle$$

$$\overset{(\text{NASH-INEQ})}{\ge} c \underbrace{\|P_t f\|_2^{2(d+2)/d}}_{= u(t)^{(d+2)/d}} \underbrace{\|P_t f\|_1^{-4/d}}_{\ge \|f\|_1^{-4/d}} \ge c\, u(t)^{(d+2)/d} \|f\|_1^{-4/d}$$

Thus

$$-u(t)^{-(d+2)/d} \dot{u}(t) \ge c\|f\|_1^{-4/d}.$$

Integrate both sides to get:

$$u(t)^{-2/d} \geq c\|f\|_1^{-4/d}\ t \qquad \text{or} \qquad u(t) \leq C\|f\|_1^2\ t^{-d/2}$$

We have proved:

$$\forall f : \mathbb{Z}^d \to \mathbb{R} : \quad \|P_t f\|_2^2 \leq C\|f\|_1^2 t^{-d/2}.$$

This is exactly (16). The same argument holds for the adjoint semigroup P_t^*. □Lemma 2

To conclude (NASH-HKB) note that

$$\|P_t\|_{1\to\infty} \leq \|P_{t/2}\|_{1\to 2}\,\|P_{t/2}\|_{2\to\infty} = \|P_{t/2}\|_{1\to 2}\,\|P_{t/2}^*\|_{1\to 2} \leq Ct^{-d/2}$$

and

$$\mathbf{P}\,(X_t = y | X_0 = x) = (P_t\delta_y)(x), \qquad \|\delta_y\|_1 = 1.$$

□Theorem 7

Proof of Theorem 6. Notation: $⨍ \dots d\theta := (2\pi)^{-d} \int_{[-\pi,\pi]^d} \dots d\theta$

For $\beta \in (0, \pi]$ fixed (Using Parceval's identity) we can write

$$\|f\|_2^2 = ⨍ |\widehat{f}(\theta)|^2 \mathbb{1}_{\{|\theta|_\infty \le \beta\}} \, d\theta + ⨍ |\widehat{f}(\theta)|^2 \mathbb{1}_{\{|\theta|_\infty > \beta\}} \, d\theta$$

$$\le \frac{(2\beta)^d}{(2\pi)^d} \underbrace{\|\widehat{f}\|_\infty^2}_{\le \|f\|_1^2} + \frac{C}{\beta^2} ⨍ |\widehat{f}(\theta)|^2 \widehat{D}(\theta) \, d\theta \le \frac{(2\beta)^d}{(2\pi)^d} \|f\|_1^2 + \frac{C}{\beta^2} \langle f , -\Delta f \rangle$$

where

$$C = C(d) := \sup_\theta \frac{|\theta|^2}{\widehat{D}(\theta)} < \infty$$

Hence, with some constant $C = C(d) < \infty$

$$\|f\|_2^2 \le C \Big(\|f\|_1^2 \beta^d + \langle f , -\Delta f \rangle \beta^{-2} \Big). \tag{17}$$

Now, optimize for β.

$$\beta^* = \left(\frac{2\langle\, f\,,\, -\Delta f\,\rangle}{d\|f\|_1^2}\right)^{1/(d+2)}$$

If $\beta^* < \pi$, then insert in (17) and get

$$\|f\|_2^2 \leq C\langle\, f\,,\, -\Delta f\,\rangle^{d/(d+2)}\|f\|_1^{4/(d+2)} \qquad \checkmark$$

If $\beta^* > \pi$ then

$$\frac{d}{d\beta}\{\text{RHS of (17)}\}|_{\beta=\pi} < 0$$

and hence

$$\|f\|_2^2 \leq \|f\|_1^2 \leq C\langle\, f\,,\, -\Delta f\,\rangle \qquad \checkmark$$

$\square$ Theorem 6

Remarks and comments on Theorems 6 and 7:

∘ Theorem 6 is equally valid in $\mathcal{L}^p(\mathbb{R}, dx)$ and Δ the $\mathbb{R}^d$-Laplacian. (Actually, the original formulation is like that.)

∘ Theorem 7 is equally valid in $\mathbb{R}^d$ for the diffusion

$$dX(t) = a(X(t))^{1/2}dB(t) + (\frac{1}{2}\nabla \cdot a(X(t)) + v(X(t)))dt$$

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} a_{i,j}(x) \frac{\partial}{\partial x_j} f(x) + \sum_{i=1}^{d} v_i(x) \frac{\partial}{\partial x_i} f(x)$$

where

$$\forall x \in \mathbb{R}^d : \qquad \underbrace{0 < a_* I \leq a(x)}_{(\texttt{ELLIPT})} = a(x)^\dagger, \qquad \underbrace{\nabla \cdot v(x) \equiv 0}_{(\texttt{DIV-FREE})}$$

with *no more regularity assumed* than necessary to ensure strong solution to the SDE.

- Note that in Theorem 6 only the conditions (DIV-FREE) and (ELLIPT) are assumed. No boundedness, no more regularity.
- Relaxing (somewhat) ELL is very important and very subtle. See, e.g., [J-C Mourrat, F Otto (2016)]

$* * *$

Theorem 8. [Nash Moment Bound] [J Nash (1958)] *Let* $\big((s_k(x) = s_{-k}(x+k))_{k\in\mathcal{U}}\big)_{x\in\mathbb{Z}^d}$ *be conductances which are strongly elliptic & bounded*

$$0 < s_* \le s_k(x) < s^* < \infty, \qquad \text{(ELL \& BDD)}$$

and $t \mapsto X_t$ *the the n.n. random walk on* $\mathbb{Z}^d$ *with jump rates* $p_k(x) = s_k(x)$*. Then, there exists a constant* $C = C(d, s_*, s^*) < \infty$ *such that* $\forall t \in (0,\infty)$*,*

$$\mathbf{E}\left(|X_t|\,|X_0 = 0\right) \le C\sqrt{t}. \qquad \text{(NASH-MB)}$$

Proof of Theorem 8. Notation throughout the proof:

$$
\begin{aligned}
q_t(x) &:= \mathbf{P}\left(X_t = x | X_0 = 0\right) \\
M_t &:= \mathbf{E}\left(|X_t| | X_0 = 0\right) \\
H_t &:= -\sum_{x \in \mathbb{Z}^d} q_t(x) \log q_t(x) \\
&= H(q_t) = \text{entropy of the distribution } q_t
\end{aligned}
$$

Lemma 3. *There exists $c = c(d) > 0$ such that for all $t > 0$, if $M_t > 1$ then*

$$M_t \geq c\, e^{H_t/d} \tag{18}$$

This statement follows from the *entropy inequality* and it is valid for any probability distribution q on $\mathbb{Z}^d$. See proof below.

Lemma 4. *There exists* $C = C(d, s_*) < \infty$ *such that for all* $t > 0$

$$\frac{H_t}{d} \geq \frac{1}{2} \log t - C \tag{19}$$

The lower bound (19) follows from (NASH-HKB). See proof below.

Denote $$G_t := \frac{H_t}{d} - \frac{1}{2} \log t + C > 0.$$

Then (18) of Lemma 3 reads: with some $c = c(d, s_*) > 0$,

$$M(t) \geq 1 \quad \Rightarrow \quad t^{-1/2} M_t \geq c e^{G(t)} \tag{20}$$

Lemma 5. *There exists* $C = C(d, s_*, s^*)$ *such that for all* $t > 0$

$$t^{-1/2} M_t \leq C(G(t) + 1) \tag{21}$$

This is somewhat more tricky. See proof below.
The bounds (20) and (21) imply (NASH-MB). □Theorem 8

Proof of Lemma 3. Let V be a *countable* set (e.g. $\mathbb{Z}^d$) and q a probability distribution on it (e.g. q_t). Define the *entropy* of the distribution q as

$$H(q) := -\sum_{x \in V} q(x) \log q(x).$$

The variational formula (22), called the *entropy inequality*, holds:

$$\text{(HW)}: \quad H(q) := \inf_{f \geq 0} \Big(\log(\sum_{x \in V} e^{-f(x)}) + \sum_{x \in V} q(x) f(x) \Big) \qquad (22)$$

The infimum is realised by $f(x) = -\log q(x) + C$.

Apply the entropy inequality with $V = \mathbb{Z}^d$, q arbitrary, $f(x) = s|x|$ with some $s \in (0, 1)$. Note that

$$\sum_{x \in \mathbb{Z}^d} e^{-s|x|} \leq C s^{-d}$$

Get: There exists $C = C(d) < \infty$, such that for any $s \in (0,1)$

$$sM \geq H + d \log s - C$$

If $H \leq 2C$ then

$$M > 1 > \underbrace{e^{-2C/d}}_{=c} e^{H/d}.$$

If $H \geq 2C$ choose $s = \exp(-(H-2C)/d)$ and get

$$M \geq \underbrace{C\, e^{-2C/d}}_{=c} e^{H/d}.$$

□Lemma 3

Proof of Lemma 4. This is immediate consequence of (NASH-HKB):

$$H_t = -\sum_{x \in \mathbb{Z}^d} q_t(x) \log \underbrace{q_t(x)}_{\leq Ct^{-d/2}} \geq \sum_{x \in \mathbb{Z}^d} q_t(x)(\frac{d}{2} \log t - C) = \frac{d}{2} \log t - C$$

□Lemma 4

Proof of Lemma 5 (somewhat computational ☹)

$$\dot{H}_t = -\sum_x \dot{q}_t(x) \log q_t(x) = -\sum_{x,k} s_k(x)(q_t(x+k) - q_t(x)) \log q_t(x)$$

$$= \frac{1}{2} \sum_{x,k} s_k(x)(q_t(x+k) - q_t(x))(\log q_t(x+k) - \log q_t(x))$$

$$\geq \frac{s_*}{2} \sum_{x,k} \frac{(q_t(x+k) - q_t(x))^2}{q_t(x+k) + q_t(x)} = s_* \sum_{x,k} \frac{(q_t(x+k) - q_t(x))^2}{(q_t(x+k) + q_t(x))^2} q_t(x)$$

$$\dot{M}_t = \sum_x \dot{q}_t(x)|x| = \sum_{x,k} s_k(x)(q_t(x+k) - q_t(x))|x|$$

$$= \frac{1}{2} \sum_{x,k} s_k(x)(q_t(x+k) - q_t(x))(|x| - |x+k|)$$

$$\leq \frac{s^*}{2} \sum_{x,k} |q_t(x+k) - q_t(x)| = s^* \sum_{x,k} \frac{|q_t(x+k) - q_t(x)|}{q_t(x+k) + q_t(x)} q_t(x)$$

Hence, by Schwarz,

$$\dot{M}_t \le \underbrace{\frac{\sqrt{d}s^*}{\sqrt{s_*}}}_{=:\,C} \left(\dot{H}_t/d\right)^{1/2} = C\left(\dot{G}_t + \frac{1}{2t}\right)^{1/2}$$

$$\begin{aligned} M_t &\le C\int_0^t (2u)^{-1/2}\left(1 + 2u\dot{G}_u\right)^{1/2} du \\ &\le C\int_0^t \left((2u)^{-1/2} + (u/2)^{1/2}\dot{G}_u\right) du \\ &= C\left((2t)^{1/2} + (t/2)^{1/2}G(t)\right) - \frac{C}{2}\int_0^t \underbrace{(2u)^{-1/2}G(u)}_{\ge 0} du \\ &\le C\left((2t)^{1/2} + (t/2)^{1/2}G(t)\right) = C\sqrt{t}(G(t)+1). \end{aligned}$$

□ Lemma 5, Theorem 8

Remarks and comments on Theorem 8:

∘ Theorem 8 is equally valid in $\mathbb{R}^d$ for the diffusion

$$dX(t) = a(X(t))^{1/2} dB(t) + \frac{1}{2} \nabla \cdot a(X(t)) dt$$

with infinitesimal generator

$$Lf(x) = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} a_{i,j}(x) \frac{\partial}{\partial x_j} f(x)$$

where

$$\forall x \in \mathbb{R}^d : \qquad \underbrace{0 < a_* I \leq}_{(\texttt{ELL})} a(x) = a(x)^{\dagger} \underbrace{\leq a^* I < \infty}_{(\texttt{BDD})}$$

with *no more regularity assumed than necessary to ensure strong solution to the SDE.* Actually, the original formulation was like that.

- Extension to diffusions with infinitesimal generator *in divergence form* *with bounded stream tensor*

$$dX(t) = a(X(t))^{1/2}dB(t) + \frac{1}{2}\Big(\nabla \cdot a(X(t)) + \underbrace{\nabla \cdot h}_{=:\, v}(X(t))\Big)dt$$

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{i,j}(x) + h_{i,j}(x)\Big)\frac{\partial}{\partial x_j} f(x)$$

$$\underbrace{0 < a_* I \le}_{\texttt{(ELL)}} a(x) = a(x)^{\dagger} \underbrace{\le a^* I < \infty}_{\texttt{(BDD)}}, \qquad h(x) = -h(x)^{\dagger}\,\texttt{(BDD)}$$

is essentially straightforward. h is called the *stream tensor*. Same on the lattice $\mathbb{Z}^d$ needs more notation only . . .

- Extensions to $a = a(\omega)$ and $h = h(\omega)$ stationary, ergodic, $\mathcal{L}^{2+\varepsilon}$-integrable rather than bounded ($\mathcal{L}^{\infty}$) is tricky (see, [BT (2018)]).