Exercise 1 (Greedy and Staircase Approximation) Let $h : [0, T] \to \mathbb{R}_+$, $T > 0$, be a concave and non-decreasing function and let $\eta \in (0, 1]$. For $0 \leq a < b \leq T$, define the linear interpolation $\ell^h_{a,b} : [a, b] \to \mathbb{R}_+$ of h by $\ell^h_{a,b}(g) = \frac{b-g}{b-a} h(a) + \frac{g-a}{b-a} h(b)$. Define the staircase function $s^h_b(g) := h(b) \min\{\frac{g}{b}, 1\}$ for $g \in [0, T]$.

1. Let $f : \mathbb{R}_+ \to [0, t]$ be the piecewise linear function with the minimum number of pieces which satisfies $\eta h(g) \leq f(g) \leq h(g)$, $\forall g \in [0, T]$. Prove that f can be constructed using the following greedy construction:

   Construct $0 = g_0 < g_1 < \cdots < g_k = T$, where g_i, $i \in \{1, \ldots, k\}$, is chosen to be the maximum value in $[g_{i-1}, T]$ subject to $\eta h(g) \leq \ell^h_{g_{i-1}, g_i}(g) \leq h(g)$, for all $g \in [g_{i-1}, g_i]$. Then, $f(g)$, $g \in [0, T]$, is defined by $\ell^h_{g_{i-1}, g_i}(g)$ for $i \in [k]$ satisfying $g \in [g_{i-1}, g_i]$. Note that f has k pieces in this construction.

   (Hint: Let $0 = b_0 < b_1 < \cdots < b_r = T$ denote the breakpoints a piecewise linear approximation $p : [0, T] \to \mathbb{R}_+$ of h satisfying $\eta h \leq p \leq h$. By breakpoints, we mean here that for $g \in [b_{i-1}, b_i]$, $i \in [r]$, that $p(g) = \ell^p_{b_{i-1}, b_i}(g)$. Prove by induction on $i \in \{0, \ldots, k\}$ that $i \leq r$ and $b_i \leq g_i$.)

2. Given $g_1, \ldots, g_k$ as above, define the staircase approximation

   $$f^s(g) := \max_{i \in [k]} s^h_{g_i}(g), g \in [0, T].$$

   Prove that $\frac{1}{2} f(g) \leq f^s(g) \leq f(g)$, for $g \in [0, T]$. Conclude that up to factor 2 in the number of pieces and the approximation factor, one can achieve $\mathrm{SLC}_\eta(h)$ using a staircase approximation.

Exercise 2 (The Maximum Central Path of Maximum Flow) Let $G := (V, E)$ denote a flow network with source s and sink t in V, and let $u \in \mathbb{R}^E_+$ denote non-negative capacities. For $v \in V$, we use $\delta^+(v)$, $\delta^-(v)$ to denote the outgoing and incoming edges incident to v. Examine the maximum flow problem induced by (G, u, s, t):

$$\begin{aligned} &\nu^* := \max \sum_{e \in \delta^+(s)} f_e \\ &\sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e = 0, \forall \quad v \in V \setminus \{s, t\}, \qquad \text{(Maximum Flow)} \\ &0 \leq f \leq u, \end{aligned}$$

Let $\mathcal{P} \subseteq \mathbb{R}^E_+$ denote the feasible region of maximum flow problem above. Given an edge $e \in E$, recall that maximum central path on e is defined by

$$f^{\mathrm{m}}_e(g) := \max\{f_e : \sum_{e \in \delta^+(s)} f_e \geq \nu^* - g, f \in \mathcal{P}\}.$$

Prove that f^{m}_e is a piecewise linear function with at most 2 pieces. That is, $\mathrm{SLC}_1(f^m_e) \leq 2$. (Hint: You may use that adjacent vertices $f^1, f^2 \in P$ of the flow polytope, where $\sum_{e \in \delta^+(s)} f^1_e > \sum_{e \in \delta^+(s)} f^2_e$, satisfy that $f^2 - f^1$ corresponds to a path P from t to s in the residual network $G[f^1] := \{e \in E : f_e < u_e\} \cup \{\overleftarrow{e} \in E : f_e > 0\}$. P uses a reverse edge $\overleftarrow{e}$ here to indicate that it sends flow backwards on e (i.e, $f^2_e - f^1_e < 0$).)