Exercise 1 (Affine Scaling Terminates only when Path is Linear) Let $z(\mu_0) = (x(\mu_0), s(\mu_0), y(\mu_0)) \in$ CP, $\mu_0 > 0$, and let $\Delta z^p = (\Delta x^p, \Delta s^p, \Delta z^p)$ denote the affine scaling step at z. Assume that $\bar{z} := (\bar{x}, \bar{s}, \bar{y}) := z + \Delta z^p \in \mathcal{P} \times \mathcal{D}$. Prove the following:

1. Prove the $\bar{x}_i \bar{s}_i = 0$, for all $i \in [n]$. Conclude that $\bar{z}$ is an optimal primal-dual pair.

2. Prove exists a partition $B \cup N = [n]$ such that

$$(\Delta x^p_B, \Delta x^p_N) = (0_B, -x_N(\mu_0)),$$
$$(\Delta s^p_B, \Delta s^p_N) = (-s_B(\mu_0), 0_N).$$

   (Hint: Use part 1 together with $s(\mu_0)\Delta x^p + x(\mu_0)\Delta s^p = -x(\mu_0)s(\mu_0)$.)

3. Conclude that $z(\mu) = z(\mu_0) + (1 - \frac{\mu}{\mu_0})\Delta z^p$ for $\mu \geq 0$.
   (Hint: Show that it satisfies the central path equations at μ.)

Exercise 2 (Stepping to the End of the Path) Examine the simple primal-dual LPs $\min x_1, x_1 + x_2 = 1, x_1, x_2 \geq 0$ and $\max -y, s_1 = 1 + y, s_2 = y, s_1, s_2 \geq 0$. Define $\bar{z}(y) := (\bar{x}(y), \bar{s}(y), y)$, where $\bar{x}(y) = (\frac{y}{1+2y}, \frac{1+y}{1+2y})$ and $\bar{s}(y) = (1 + y, y)$. It can be verified that the central path closure satisfies $\overline{\text{CP}} := \{\bar{z}(y) : y \geq 0\}$ where $\bar{z}(y)$, $y \geq 0$, is the central path point at parameter $\mu(y) := \frac{y(1+y)}{1+2y}$, and that $\bar{z}(0) = ((0, 1), (1, 0), 0)$ is the optimal solution.

For $\beta \in (0, 1)$, prove that $(\Delta z, \nu) := (\bar{z}(0) - \bar{z}(y), 0)$ is a feasible solution to the trust-region TR(β) at $\bar{z}(y)$ precisely when $y \in (0, \frac{\beta}{1-\beta}]$.
(Hint: You may assume that the polarization partition in the trust-region program is $(B, N) = (\{2\}, \{1\})$.)