Introduction to the spectral theory of Toeplitz operators

Eugene Shargorodsky

King's College London

Fredholm operators

For Banach spaces X and Y, let $\mathcal{B}(X, Y)$ and $\mathcal{K}(X, Y)$ denote the sets of bounded linear and compact linear operators from X to Y, respectively.

For $A \in \mathcal{B}(X, Y)$, let

$$\operatorname{Ker} A := \{x \in X |\ Ax = 0\}, \quad \operatorname{Ran} A := \{Ax |\ x \in X\}.$$

The operator A is called Fredholm if

$$\dim \operatorname{Ker} A < +\infty, \ \ \dim (Y/\operatorname{Ran} A) < +\infty,$$

in which case $\operatorname{Ran} A$ is closed, and the index of A is finite

$$-\infty < \operatorname{Ind} A := \dim \operatorname{Ker} A - \dim (Y/\operatorname{Ran} A) < +\infty.$$

We will denote by $\Phi(X, Y)$ the set of all Fredholm operators in $\mathcal{B}(X, Y)$.

For any $A \in \Phi(X, Y)$, there exists $\varepsilon > 0$ such that $A + C + K \in \Phi(X, Y)$ for any $C \in \mathcal{B}(X, Y)$ with $\|C\| < \varepsilon$ and any $K \in \mathcal{K}(X, Y)$. Moreover

$$\operatorname{Ind}(A + C + K) = \operatorname{Ind} A.$$

Hence, for any continuous mapping

$$[0, 1] \ni t \mapsto A_t \in \Phi(X, Y),$$

one has $\operatorname{Ind} A_0 = \operatorname{Ind} A_1$.

For any $A \in \Phi(X, Y)$ and $B \in \Phi(Y, Z)$, one has $BA \in \Phi(X, Z)$ and

$$\operatorname{Ind}(BA) = \operatorname{Ind} B + \operatorname{Ind} A.$$

Suppose $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, Z)$ are such that $BA \in \Phi(X, Z)$. If one of A and B is Fredholm, then the other one is also Fredholm.

$A \in \mathcal{B}(X, Y)$ is Fredholm if and only if there exists $R \in \mathcal{B}(Y, X)$, usually called a regulariser of A, such that

$$RA - I_X \in \mathcal{K}(X) := \mathcal{K}(X, X), \quad AR - I_Y \in \mathcal{K}(Y).$$

So, $A \in \mathcal{B}(X) := \mathcal{B}(X, X)$ is Fredholm: $A \in \Phi(X) := \Phi(X, X)$, if and only if the corresponding equivalence class $[A]$ is invertible in the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$.

$A \in \mathcal{B}(X, Y)$ is Fredholm if and only if $A^* \in \mathcal{B}(Y^*, X^*)$ is Fredholm. In this case,

$$\dim \operatorname{Ker} A^* = \dim (Y/\operatorname{Ran} A), \quad \dim (X^*/\operatorname{Ran} A^*) = \dim \operatorname{Ker} A,$$

and hence $\operatorname{Ind} A^* = -\operatorname{Ind} A$.

The spectrum and the essential spectrum

The spectrum of $A \in \mathcal{B}(X)$ is the set

$$\mathrm{Spec}(A) := \{\lambda \in \mathbb{C} : \ A - \lambda I \ \text{ is not invertible}\}.$$

The essential spectrum of $A \in \mathcal{B}(X)$ is the set

$$\mathrm{Spec}_{\mathrm{e}}(A) := \{\lambda \in \mathbb{C} : \ A - \lambda I \ \text{ is not Fredholm}\}.$$

$\mathrm{Spec}_{\mathrm{e}}(A)$ is the spectrum of the equivalence class $[A]$ in $\mathcal{B}(X)/\mathcal{K}(X)$, and

$$\mathrm{Spec}_{\mathrm{e}}(A + T) = \mathrm{Spec}_{\mathrm{e}}(A), \quad \forall T \in \mathcal{K}(X).$$

Lebesgue and Hardy spaces

For $1 \leq p \leq \infty$, let $L^p = L^p(\mathbb{T})$ be the standard Lebesgue space on the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$, equipped with the norm

$$\|f\|_p := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta \right)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\|f\|_\infty := \operatorname*{ess\,sup}_{\theta \in [0,2\pi]} |f(e^{i\theta})|.$$

For $f \in L^1$, let

$$\widehat{f}_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} \, d\theta, \quad n \in \mathbb{Z},$$

be the Fourier coefficients of f.

For $1 \leq p \leq \infty$, the classical Hardy spaces $H^p = H^p(\mathbb{T})$ are defined by

$$H^p(\mathbb{T}) := \left\{ f \in L^p(\mathbb{T}) \ : \ \widehat{f}_n = 0 \quad \text{for all} \quad n < 0 \right\}.$$

Riesz projection

The Cauchy singular integral operator $\mathcal{C}$ and the Riesz projection P are defined as follows

$$(\mathcal{C}f)(t) := \frac{1}{\pi i}\,\text{p.v.}\int_{\mathbb{T}} \frac{f(\tau)}{\tau - t}\,d\tau, \quad P := \frac{1}{2}(I + \mathcal{C}),$$

where $f \in L^1$ and the integral is understood in the Cauchy principal value sense.

$$P\left(\sum_{n=-\infty}^{\infty} \widehat{f}_n t^n\right) = \sum_{n=0}^{\infty} \widehat{f}_n t^n.$$

P is bounded on L^p iff $1 < p < \infty$. In this case, P is a projection of L^p onto H^p.

Toeplitz operators

Toeplitz operator with symbol $a \in L^\infty(\mathbb{T})$:

$$T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T}), \ \ 1 < p < \infty,$$
$$T(a)f = P(af), \ \ f \in H^p(\mathbb{T}).$$

Clearly, $T(a) - \lambda I = T(a - \lambda)$ for all $\lambda \in \mathbb{C}$.

$T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ has the same invertibility / Fredholm properties as the singular integral operator

$$aP + Q : L^p(\mathbb{T}) \to L^p(\mathbb{T}),$$

where $Q := I - P$ is the complementary projection.

Let X be a Banach space, $P \in \mathcal{B}(X)$ be a projection (i.e. $P^2 = P$), and $Q := I - P$ be the complementary projection.

Let $X_+ := \mathrm{Ran}(P) = PX$, $X_- := \mathrm{Ran}(Q) = QX$. Then $X = X_+ \oplus X_-$.

For any $A \in \mathcal{B}(X)$, we have

$$\begin{aligned} AP + Q : X \to X \ &= \ AP + Q : X_+ \oplus X_- \to X_+ \oplus X_- \\ &= \begin{pmatrix} P(AP+Q)|_{X_+} & P(AP+Q)|_{X_-} \\ Q(AP+Q)|_{X_+} & Q(AP+Q)|_{X_-} \end{pmatrix} : \begin{matrix} X_+ \\ \oplus \\ X_- \end{matrix} \to \begin{matrix} X_+ \\ \oplus \\ X_- \end{matrix} \\ &= \begin{pmatrix} PA|_{X_+} & 0 \\ QA|_{X_+} & I|_{X_-} \end{pmatrix} : \begin{matrix} X_+ \\ \oplus \\ X_- \end{matrix} \to \begin{matrix} X_+ \\ \oplus \\ X_- \end{matrix} . \end{aligned}$$

So,

$$\begin{aligned} \mathrm{Ker}(AP+Q) &= \{x - QAx \mid x \in \mathrm{Ker}(PA|_{X_+})\}, \\ \mathrm{Ran}(AP+Q) &= \mathrm{Ran}(PA|_{X_+}) \oplus X_- . \end{aligned}$$

It is natural to consider more general singular integral operators

$$aP + bQ : L^p(\mathbb{T}) \to L^p(\mathbb{T}), \qquad (*)$$

where $a, b \in L^\infty(\mathbb{T})$.

Since $P + Q = I$, we have $aP + bQ - \lambda I = (a - \lambda)P + (b - \lambda)Q$ for all $\lambda \in \mathbb{C}$.

If $(*)$ is Fredholm, then $\frac{1}{a}, \frac{1}{b} \in L^\infty(\mathbb{T})$ (see, e.g., A. Böttcher and B. Silbermann, Remark 3 to Theorem 2.30, or I. Gohberg and N. Krupnik, Vol. II, Ch. 7, Sect. 4, Theorem 4.1).

If $\frac{1}{b} \in L^\infty(\mathbb{T})$, then the multiplication operator $bI : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ is invertible, $(bI)^{-1} = \frac{1}{b} I$, and

$$aP + bQ = b\left(\frac{a}{b}\, P + Q\right).$$

So, for many purposes, it is sufficient to consider operators $cP + Q : L^p(\mathbb{T}) \to L^p(\mathbb{T})$, $c \in L^\infty(\mathbb{T})$, i.e. Toeplitz operators $T(c) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$.

Main books

A. Böttcher and B. Silbermann, *Analysis of Toeplitz operators,* Springer-Verlag, Berlin, 2006.

I. Gohberg and N. Krupnik, *One-dimensional linear singular integral equations,* Vol. I & II, Birkhäuser Verlag, Basel, 1992.

Toeplitz operators are usually defined over the unit circle $\mathbb{T}$ as above or over the real line $\mathbb{R}$, while one-dimensional singular integral operators are often considered over more general curves $\Gamma \subset \mathbb{C}$: $aP_\Gamma + bQ_\Gamma$,

$$P_\Gamma = \frac{1}{2}(I + \mathcal{C}_\Gamma), \quad Q_\Gamma = I - P_\Gamma = \frac{1}{2}(I - \mathcal{C}_\Gamma),$$
$$(\mathcal{C}_\Gamma f)(t) := \frac{1}{\pi i}\,\text{p.v.} \int_\Gamma \frac{f(\tau)}{\tau - t}\, d\tau, \quad t \in \Gamma,$$

where $f \in L^1(\Gamma)$ and the integral is understood in the Cauchy principal value sense.

If Γ is a smooth Jordan curve, then spectral / Fredholm properties of singular integral operators over Γ are the same as those of their counterparts over $\mathbb{T}$. If Γ is less regular, then the essential spectrum of $aP_\Gamma + Q_\Gamma$ can depend on the the properties of Γ.

A. Böttcher and Y.I. Karlovich, *Carleson curves, Muckenhoupt weights, and Toeplitz operators,* Birkhäuser Verlag, Basel, 1997.

We will not discuss this part of the theory.

Let Γ be a piecewise smooth Jordan curve and D^+ (D^-) be the bounded (the unbounded) component of $\mathbb{C} \setminus \Gamma$.

The Riemann–Hilbert problem (the linear conjugation problem): given functions $b, g : \Gamma \to \mathbb{C}$, find analytic functions $\Phi^\pm : D^\pm \to \mathbb{C}$ whose (appropriately understood) boundary values at Γ satisfy

$$\Phi^+(t) = b(t)\Phi^-(t) + g(t), \quad t \in \Gamma, \tag{R-H}$$

and

$$\Phi^-(\infty) := \lim_{|z|\to\infty} \Phi^-(z) = 0. \tag{∞}$$

According to the Cauchy integral formula,

$$\frac{1}{2\pi i}\int_\Gamma \frac{\Phi^+(\tau)}{\tau - z}\,d\tau = \begin{cases}\Phi^+(z), & z \in D^+,\\ 0, & z \in D^-,\end{cases}$$

$$\frac{1}{2\pi i}\int_\Gamma \frac{\Phi^-(\tau)}{\tau - z}\,d\tau = \begin{cases}0, & z \in D^+,\\ -\Phi^-(z), & z \in D^-.\end{cases}$$

Let $\varphi(\tau) := \Phi^+(\tau) - \Phi^-(\tau)$, $\tau \in \Gamma$. Then

$$\frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - z}\,d\tau = \begin{cases}\Phi^+(z), & z \in D^+,\\ \Phi^-(z), & z \in D^-.\end{cases}$$

Substituting the above formula into (R-H) and using the Sokhotski–Plemelj theorem (Polish: Sochocki):

$$\lim_{D^{\pm} \ni z \to t} \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - z}\, d\tau = \frac{1}{2}(\pm\varphi(t) + (\mathcal{C}_{\Gamma}\varphi)(t)), \quad t \in \Gamma,$$

we reduce the Riemann–Hilbert problem to the singular integral equation

$$(P_{\Gamma} + bQ_{\Gamma})\varphi = g.$$

The above reduction can be made rigorous for functions in suitable function spaces (e.g., Hardy–Smirnov spaces).

The Riemann–Hilbert problem and Toeplitz operators are closely related to each other.

Back to Toeplitz operators

Spectral / Fredholm properties of Toeplitz operators depend on the underlying Hardy space, and going beyond the Hilbert space setting of H^2 has its advantages.

Example: Let $a(e^{i\theta}) = \pm 1$ for $\pm\theta \in (0, \pi)$. We will see later that the only value of $p \in (1, \infty)$ for which $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ is not Fredholm is $p = 2$.

Weighted spaces

Toeplitz operators are often considered on weighted Hardy spaces.

For $1 \leq p \leq \infty$ and a measurable function $w \geq 0$, let

$$L^p(w) := \{f : \ wf \in L^p\} \quad \text{and} \quad \|f\|_{L^p(w)} := \|wf\|_{L^p}.$$

Suppose $w \in L^p$ and $1/w \in L^{p'}$, $p' = \frac{p}{p-1}$. Then

$$L^\infty \subseteq L^p(w) \subseteq L^1.$$

The weighted Hardy space:

$$H^p(w) := \left\{f \in L^p(w) \ : \ \widehat{f}_n = 0 \quad \text{for all} \quad n < 0\right\}.$$

Muckenhoupt weights

Let $1 < p < \infty$. The Riesz projection is bounded on $L^p(w)$ if and only if $w \in A_p$, i.e.

$$\sup_I \left(\frac{1}{|I|} \int_I w^p(t)|dt| \right)^{\frac{1}{p}} \left(\frac{1}{|I|} \int_I w^{-p'}(t)|dt| \right)^{\frac{1}{p'}} < \infty,$$

where $I \subset \mathbb{T}$ is an arbitrary arc, $|I|$ denotes its length (R. Hunt, B. Muckenhoupt, and R. Wheeden, 1973).

Khvedelidze weights: Let $w(t) := \prod_{j=1}^{n} |t - t_j|^{\mu_j}$, $t \in \mathbb{T}$, where $t_1, \ldots, t_n \in \mathbb{T}$ are pairwise distinct, and $\mu_1, \ldots, \mu_n \in \mathbb{R}$. Then $w \in A_p$ if and only if $-\frac{1}{p} < \mu_j < \frac{1}{p'}$ for $j = 1, \ldots, n$.

$w \in A_2$ if and only if $w = e^{u+\widetilde{v}}$, where u and v are real-valued functions in L^∞, and $\|v\|_\infty < \pi/4$ (H. Helson and G. Szegő, 1960).

Here, $\widetilde{v}$ denotes the Hilbert transform (the harmonic conjugate) of v:

$$\widetilde{v}\left(e^{i\vartheta}\right) = (\mathcal{H}v)\left(e^{i\vartheta}\right) := \frac{1}{2\pi}\,\text{p.v.}\int_{-\pi}^{\pi} v\left(e^{i\theta}\right)\cot\frac{\vartheta-\theta}{2}\,d\theta, \quad \vartheta \in [-\pi,\pi],$$

$$\mathcal{H}v = -i\,\mathcal{C}v + i\,\widehat{v}_0.$$

$w \in A_p \cap A_{p'}$ if (I.B. Simonenko, 1964) and only if (N.Ya. Krupnik, 1978) $w = e^{u+\widetilde{v}}$, where u and v are real-valued functions in L^∞, and $\|v\|_\infty < \pi/(2\max\{p,p'\})$.

Toeplitz operators have been studied on Hardy spaces built upon a variety of function spaces, including Orlicz, Lorentz, Marcinkiewicz, Morrey, and variable Lebesgue spaces.

It seems natural to try to develop a unified theory of Toeplitz operators on Hardy spaces built upon general Banach function spaces.

Banach function norms

L^0_+ – the set of measurable functions on $\mathbb{T}$ whose values lie in $[0, \infty]$.

A mapping $\rho : L^0_+ \to [0, \infty]$ is called a Banach function norm if, for all functions $f, g, f_n \in L^0_+$ with $n \in \mathbb{N}$, and for all constants $a \geq 0$, the following properties hold:

$$
\begin{aligned}
&\text{(A1)} \quad \rho(f) = 0 \Leftrightarrow f = 0 \text{ a.e.}, \ \rho(af) = a\,\rho(f), \ \rho(f+g) \leq \rho(f) + \rho(g),\\
&\text{(A2)} \quad 0 \leq g \leq f \text{ a.e.} \ \Rightarrow \ \rho(g) \leq \rho(f) \quad \text{(the lattice property)},\\
&\text{(A3)} \quad 0 \leq f_n \uparrow f \text{ a.e.} \ \Rightarrow \ \rho(f_n) \uparrow \rho(f) \quad \text{(the Fatou property)},\\
&\text{(A4)} \quad \rho(1) < \infty,\\
&\text{(A5)} \quad \int_{-\pi}^{\pi} f(e^{i\theta})\, d\theta \leq C\rho(f)
\end{aligned}
$$

with a constant $C \in (0, \infty)$ that may depend on ρ, but is independent of f.

Banach function spaces

Identifying functions that are equal a.e., one gets a Banach function space

$$X := \left\{ f : \mathbb{T} \to \mathbb{C} \text{ measurable} \;\middle|\; \|f\|_X := \rho(|f|) < \infty \right\}.$$

$$\text{(A4), (A5)} \quad \Longrightarrow \quad L^\infty(\mathbb{T}) \subseteq X \subseteq L^1(\mathbb{T}).$$

The norm

$$\|g\|_{X'} := \sup \left\{ \left| \int_{-\pi}^{\pi} g(e^{i\theta}) f(e^{i\theta}) \, d\theta \right| : \; \|f\|_X \le 1 \right\}$$

generates the associate (Köthe dual) Banach function space X'. The associate space X' can be viewed as a subspace of the Banach dual space X^*.

- $X' = X^*$ if and only if X is separable. X is reflexive if and only if both X and X' are separable.
- $X'' = X$.
- $(L^\infty(\mathbb{T}))' = L^1(\mathbb{T})$.

Abstract Hardy space built upon the space X

$$H[X] = H[X](\mathbb{T}) := \left\{ g \in X \ : \ \widehat{g}(n) = 0 \quad \text{for all} \quad n < 0 \right\}.$$

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$, $\mathbb{D} := \{\zeta \in \mathbb{C} | \ |\zeta| < 1\}$. Let $H[X](\mathbb{D})$ be the Hardy space over $\mathbb{D}$, which consist of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{HX(\mathbb{D})} := \sup_{0<r<1} \|f(r\,\cdot)\|_{X(\mathbb{T})} < \infty.$$

Since $X \subseteq L^1$, each $f \in H[X](\mathbb{D}) \subseteq H^1$ has a nontangential limit at almost every $e^{i\vartheta} \in \mathbb{T}$. Let's denote this limit by $f^*(e^{i\vartheta})$. Then $f^* \in H[X](\mathbb{T})$, $\|f^*\|_{X(\mathbb{T})} \leq \|f\|_{HX(\mathbb{D})}$, and the mapping

$$H[X](\mathbb{D}) \ni f \mapsto f^* \in H[X](\mathbb{T}) \tag{BV}$$

is injective.

If $X = L^p$ or, more generally, $X = L^p(w)$ with $w \in A_p$, then the mapping (BV) is surjective, and one can identify $H[X](\mathbb{D})$ with $H[X](\mathbb{T})$. However, there are weights w for which (BV) with $X = L^p(w)$ is not surjective (M. Rosenblum, 1962).

Toeplitz operators on abstract Hardy spaces

From now on, we will aways assume that the Riesz projection P is bounded on X.

In this case, P projects X onto $H[X]$, and one can define the Toeplitz operator $T(a) \in \mathcal{B}(H[X])$ with symbol $a \in L^\infty$ as above:

$$T(a)f = P(af), \quad f \in H[X].$$

Clearly,

$$\|T(a)\|_{\mathcal{B}(H[X])} \le \|P\|_{\mathcal{B}(X)} \|a\|_{L^\infty}.$$

For $f \in X$ and $g \in X'$, put

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta.$$

If P is bounded on X, it is bounded on X', $\|P\|_{\mathcal{B}(H[X'])} = \|P\|_{\mathcal{B}(H[X])}$, and, for all $f \in X$ and $h \in X'$,

$$\langle Pf, h \rangle = \langle Pf, Ph \rangle = \langle f, Ph \rangle,$$
$$\langle Pf, Qh \rangle = 0 = \langle Qf, Ph \rangle$$

(O. Karlovych & ES, 2023).

Let $\mathbf{e}_k(z) := z^k,\ z \in \mathbb{C},\ k \in \mathbb{Z}$.

Let $a \in L^\infty$. Then

$$\begin{aligned}\langle T(a)\mathbf{e}_j, \mathbf{e}_k\rangle &= \langle P(a\mathbf{e}_j), \mathbf{e}_k\rangle = \widehat{P(a\mathbf{e}_j)}_k = \widehat{(a\mathbf{e}_j)}_k \\ &= \widehat{a}_{k-j} \quad \text{for all} \quad j, k \geq 0.\end{aligned}$$

So, $T(a)$ has the following matrix representation

$$\begin{pmatrix} \widehat{a}_0 & \widehat{a}_{-1} & \widehat{a}_{-2} & \widehat{a}_{-3} & \cdots \\ \widehat{a}_1 & \widehat{a}_0 & \widehat{a}_{-1} & \widehat{a}_{-2} & \cdots \\ \widehat{a}_2 & \widehat{a}_1 & \widehat{a}_0 & \widehat{a}_{-1} & \cdots \\ \widehat{a}_3 & \widehat{a}_2 & \widehat{a}_1 & \widehat{a}_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Brown-Halmos Theorem

Theorem (A. Karlovich & ES, 2019)

Let X be a separable Banach function space and let the Riesz projection P be bounded on X. If $A \in \mathcal{B}(H[X])$ and there is a sequence $(a_n)_{n\in\mathbb{Z}}$ of complex numbers such that

$$\langle A\mathbf{e}_j, \mathbf{e}_k\rangle = a_{k-j} \quad \text{for all} \quad j, k \geq 0,$$

then there exists a function $a \in L^\infty$ such that $A = T(a)$ and $\widehat{a}_n = a_n$ for all $n \in \mathbb{Z}$. Moreover,

$$\|a\|_{L^\infty} \leq \|T(a)\|_{\mathcal{B}(H[X])} \leq \|P\|_{\mathcal{B}(X)}\|a\|_{L^\infty}.$$

A very short sketch of the proof.

Let $b_n := \mathbf{e}_{-n} A \mathbf{e}_n \in X$, $n \geq 0$. Using the Banach-Alaoglu theorem, one can prove the existence of a subsequence such that, for some $a \in X$, one has

$$\lim_{k\to\infty} \langle b_{n_k}, \mathbf{e}_j\rangle = \langle a, \mathbf{e}_j\rangle \quad \text{for all} \quad j \in \mathbb{Z}.$$

On the other hand, for $n_k + j \geq 0$,

$$\langle b_{n_k}, \mathbf{e}_j\rangle = \langle \mathbf{e}_{-n_k} A \mathbf{e}_{n_k}, \mathbf{e}_j\rangle = \langle A\mathbf{e}_{n_k}, \mathbf{e}_{n_k+j}\rangle = a_j.$$

Then one shows that $\|aI\|_{\mathcal{B}(X)} \leq \|A\|_{\mathcal{B}(H[X])} < \infty$, and hence $a \in L^\infty$.

Finally,

$$\widehat{(a\,\mathbf{e}_j)}_k = \langle a\,\mathbf{e}_j, \mathbf{e}_k\rangle = \langle a, \mathbf{e}_{k-j}\rangle = a_{k-j} = \langle A\,\mathbf{e}_j, \mathbf{e}_k\rangle = \widehat{(A\,\mathbf{e}_j)}_k \quad \text{for all} \quad j, k \geq 0$$

implies, by the uniqueness theorem for Fourier series, that $P(a\,\mathbf{e}_j) = A\,\mathbf{e}_j$ for all $j \geq 0$. Since X is separable, the closed linear span of $\{\mathbf{e}_j\}_{j\geq 0}$ is equal to $H[X]$, and one concludes that $T(a) = A$. □

Discrete Toeplitz operators

Let $(a_n)_{n\in\mathbb{Z}}$ be sequence of complex numbers. For a Banach sequence space $X(\mathbb{Z}_+)$ (e.g., for ℓ^p or for a weighted $\ell^p(w)$), one can consider the discrete Toeplitz operator

$$A = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} : X(\mathbb{Z}_+) \to X(\mathbb{Z}_+),$$

$$\left(A(f_j)_{j\in\mathbb{Z}_+}\right)_k = \sum_{j=0}^{\infty} a_{k-j}\, f_j \quad \text{for all} \quad k \geq 0.$$

We will not discuss this part of the theory.

Let $a, b \in L^\infty(\mathbb{T})$. In general, $T(a)T(b) \neq T(ab)$, but if $\overline{a} \in H^\infty(\mathbb{T})$ or $b \in H^\infty(\mathbb{T})$, then the operators $T(a), T(b) \in \mathcal{B}(H[X])$ satisfy the equality

$$T(a)T(b) = T(ab).$$

For all $u \in H[X]$ and $v \in H[X']$,

$$\langle T(a)u, v\rangle = \langle u, T(\overline{a})\, v\rangle$$

(O. Karlovych & ES, 2023).

If X is separable, then $X' = X^*$, and $T(\overline{a}) : H[X'] \to H[X']$ can be identified with the adjoint of the operator $T(a) : H[X] \to H[X]$. This is not the case if X is non-separable. Nevertheless, one has the following result.

Theorem (O. Karlovych & ES, 2025?)

$T(a) : H[X] \to H[X]$ is Fredholm if and only if $T(\overline{a}) : H[X'] \to H[X']$ is Fredholm.

Coburn's lemma

Theorem (L.A. Coburn, 1966)

If $T(a)$ is a non-zero Toeplitz operator on H^2, then

$$\operatorname{Ker} T(a) = \{0\} \quad \text{or} \quad \operatorname{Ker} T(\overline{a}) = \{0\}.$$

This result remains true for H^p with $1 < p < \infty$, and it can be rephrased as follows.

Theorem (I.B. Simonenko, 1968)

If $a \in L^\infty \setminus \{0\}$, then the Toeplitz operator $T(a)$ has a trivial kernel or a dense range on each Hardy space H^p with $1 < p < \infty$.

Corollary (Important!)

Let $a \in L^\infty \setminus \{0\}$ and $T(a) : H^p \to H^p$ be Fredholm. Then $T(a)$ has a trivial kernel or its range is equal to H^p. Hence, $T(a)$ is left-invertible if $\operatorname{Ind} T(a) \leq 0$, right-invertible if $\operatorname{Ind} T(a) \geq 0$, and invertible if $\operatorname{Ind} T(a) = 0$.

Coburn's lemma in its original form can be extended to abstract Hardy spaces $H[X]$.

Theorem (O. Karlovych & ES, 2023)

If $a \in L^\infty \setminus \{0\}$, then the kernel of the Toeplitz operator $T(a) : H[X] \to H[X]$ or the kernel of the Toeplitz operator $T(\overline{a}) : H[X'] \to H[X']$ is trivial.

A short sketch of the proof.

Suppose there exist $u \in H[X]$ and $v \in H[X']$ be such that $T(a)u = 0$ and $T(\overline{a})\,v = 0$. Then one can show that all Fourier coefficients of $g := au\overline{v} \in L^1$ are equal to 0. Hence $g = 0$ a.e. Since $a \neq 0$, the product $u\overline{v}$ is equal to 0 on a set of positive measure. Then at least one of the functions $u \in H^1$ and $v \in H^1$ is equal to 0 on a set of positive measure and hence a.e. due to the uniqueness theorem for H^1 functions. □

The second form of Coburn's lemma can be extended to $H[X]$ if X is separable, in which case $T(\overline{a}) : H[X'] \to H[X']$ is the adjoint operator of $T(a) : H[X] \to H[X]$, but it might fail if X is non-separable.

The weak L^p-space (the Marcinkiewicz space) $L^{p,\infty}$, $1 < p < \infty$ consists of all measurable a.e. finite functions $f : \mathbb{T} \to \mathbb{C}$ such that

$$\|f\|_{p,\infty} := \sup_{\lambda>0} \left(\lambda^p \big|\{t \in \mathbb{T} : |f(t)| > \lambda\}\big|\right)^{1/p} < \infty.$$

$\|\cdot\|_{p,\infty}$ is a quasinorm but not a norm. It is equivalent to the norm

$$\|f\|_{(p,\infty)} := \sup_{|E|>0} |E|^{-1+\frac{1}{p}} \int_E |f(t)|\,|dt|.$$

$L^{p,\infty}$ is non-separable, and P is bounded on $L^{p,\infty}$.

It is well known that elements of the kernel of a Toeplitz operator with piecewise continuous symbol may have power singularities at the points of discontinuity of the symbol. For any $\zeta_0 \in \mathbb{T}$, the function $\varphi(\zeta) := |\zeta - \zeta_0|^{-1/p}$ does not belong to $L^p(\mathbb{T})$, $1 < p < \infty$, but it does belong to $L^{p,\infty}(\mathbb{T})$. So, it seems natural to consider Toeplitz operators on the weak Hardy (Hardy-Marcinkiewicz) spaces $H^{p,\infty}(\mathbb{T}) = H[L^{p,\infty}(\mathbb{T})]$.

Theorem (A. Karlovich & ES, 2022)

Let $1 < p < \infty$. Then there exists a function $a \in C(\mathbb{T}) \setminus \{0\}$ depending on p such that $a(-1) = 0$ and the following equalities hold for the kernel and the closure of the range of the Toeplitz operator $T(a)$ acting on $H[L^{p,\infty}]$:

$$\dim(\operatorname{Ker} T(a)) = \infty, \quad \dim\left(H[L^{p,\infty}]/\operatorname{clos}_{H[L^{p,\infty}]}\left(\operatorname{Ran} T(a)\right)\right) = \infty.$$

The range $\operatorname{Ran} T(a)$ of the Toeplitz operator in the previous example is not closed. Unfortunately, we don't know the answer to the following question.

Question

Does there exists a function $a \in L^\infty(\mathbb{T}) \setminus \{0\}$ such that the range of the Toeplitz operator $T(a)$ acting on $H[L^{p,\infty}]$ is closed, and

$$\dim(\operatorname{Ker} T(a)) > 0, \quad \dim\left(H[L^{p,\infty}]/\operatorname{Ran} T(a)\right) > 0?$$

Fortunately, the above difficulty does not arise for Fredholm Toeplitz operators, and we still have the following result.

Theorem (O. Karlovych & ES, 2025?)

Let $a \in L^\infty$ and $T(a) : H[X] \to H[X]$ be Fredholm. Then $T(a)$ has a trivial kernel or its range is equal to $H[X]$. Hence, $T(a)$ is left-invertible if $\operatorname{Ind} T(a) \leq 0$, right-invertible if $\operatorname{Ind} T(a) \geq 0$, and invertible if $\operatorname{Ind} T(a) = 0$.

Let $a \in L^\infty$, $T(a) : H[X] \to H[X]$ be Fredholm and let $n := \operatorname{Ind} T(a)$. Then $T(\mathbf{e}_n a)$ is invertible.

Indeed,

$$T(\mathbf{e}_n a) = \begin{cases} T(a)T(\mathbf{e}_n) & \text{if} \quad n \geq 0, \\ T(\mathbf{e}_n)T(a) & \text{if} \quad n \leq 0. \end{cases}$$

It is easy to see that $\operatorname{Ind}(T(\mathbf{e}_n) : H[X] \to H[X]) = -n$. Hence,

$$\operatorname{Ind} T(\mathbf{e}_n a) = \operatorname{Ind} T(a) + \operatorname{Ind} T(\mathbf{e}_n) = n - n = 0.$$

So, $T(\mathbf{e}_n a)$ is invertible.

Auxiliary results from the theory of H^p spaces

Let $f \in H^p(\mathbb{D}) \setminus \{0\}$, $p > 0$, and let $z_1, z_2, \ldots$ be its zeros repeated according to their multiplicity. Then $\sum_{n=1}^{\infty}(1 - |z_n|) < \infty$.

For any sequence of (not necessarily distinct) points $z_1, z_2, \ldots$ in $\mathbb{D}$ such that $\sum_{n=1}^{\infty}(1 - |z_n|) < \infty$, the Blaschke product

$$B(z) := \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z}, \quad |z| < 1$$

converges uniformly in each disc $|z| \leq r < 1$. Here, we adopt the convention that $\frac{|z_n|}{z_n} = 1$ when $z_n = 0$.

Let $g : \mathbb{T} \to [0, \infty)$ be a measurable function such that $\log g \in L^1$. Consider the outer function

$$G(z) := \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log g(e^{i\theta})\, d\theta\right), \quad z \in \mathbb{D}.$$

The functions $G^{\pm 1}$ are holomorphic in $\mathbb{D}$, their nontangential limits at $\mathbb{T}$ exist a.e. and satisfy $|G|^{\pm 1} = g^{\pm 1}$ a.e. on $\mathbb{T}$.

If $g^{\pm 1} \in L^1$, one has $G^{\pm 1} \in H^1(\mathbb{D})$ and $G^{\pm 1} \in H^1(\mathbb{T})$.

If, additionally, $g \in Y$ (or $g^{-1} \in Y$) for some Banach function space $Y = Y(\mathbb{T})$, then $G \in Y \cap H^1(\mathbb{T}) = H[Y]$ ($G^{-1} \in H[Y]$, respectively).

Canonical factorisation in H^p spaces

If $f \in H^p(\mathbb{D}) \setminus \{0\}$, $p > 0$, then $\log |f| \in L^1$.

Every $f \in H^p(\mathbb{D}) \setminus \{0\}$, $p > 0$ admits a unique factorisation $f = cBSG$, where $c \in \mathbb{C}$, $|c| = 1$, B is a Blaschke product, S is a singular inner function given by the formula

$$\exp\left(-\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta)\right), \quad z \in \mathbb{D},$$

with a finite positive Borel measure μ on $[-\pi, \pi]$, singular with respect to the Lebesgue measure, and G is the outer function with $g = |f|$.

$|B(z)| < 1$, $|S(z)| < 1$ for all $z \in \mathbb{D}$, and $|B(t)| = 1 = |S(t)|$ for almost all $t \in \mathbb{T}$.

A holomorphic function $u : \mathbb{D} \to \mathbb{C}$ is called an inner function if $|u(z)| < 1$ for all $z \in \mathbb{D}$ and $|u(t)| = 1$ for almost all $t \in \mathbb{T}$. Every inner function admits a unique factorisation $f = cBS$, where $c \in \mathbb{C}$, $|c| = 1$, B is a Blaschke product, and S is a singular inner function.

The Hartman-Wintner-Simonenko theorem

Theorem (O. Karlovych & ES, 2023)

If $a \in L^\infty \setminus \{0\}$ and the range $\operatorname{Ran} T(a)$ of the Toeplitz operator $T(a) : H[X] \to H[X]$ is closed, then $\frac{1}{a} \in L^\infty$.

The following corollary has an easier direct proof.

Corollary

If $a \in L^\infty \setminus \{0\}$ and the Toeplitz operator $T(a) : H[X] \to H[X]$ is Fredholm, then $\frac{1}{a} \in L^\infty$.

Proof of the corollary.

Suppose the contrary: $\frac{1}{a} \notin L^\infty$. Then for every $\varepsilon > 0$, there exists a set $E_\varepsilon \subset \mathbb{T}$ of positive measure such that $|a| \le \varepsilon$ a.e. in E_ε. We can assume that $|\mathbb{T} \setminus E_\varepsilon| > 0$. Let $a_\varepsilon := \mathbb{1}_{\mathbb{T}\setminus E_\varepsilon} a$. Then $a_\varepsilon = 0$ a.e. in E_ε, and

$$\|a - a_\varepsilon\|_{L^\infty} = \|\mathbb{1}_{E_\varepsilon} a\|_{L^\infty} \le \varepsilon.$$

Hence

$$\|T(a) - T(a_\varepsilon)\|_{\mathcal{B}(H[X])} \le \|P\|_{\mathcal{B}(X)} \|a - a_\varepsilon\|_{L^\infty} \le \|P\|_{\mathcal{B}(X)} \varepsilon.$$

If ε is sufficiently small, the above estimate implies that $T(a_\varepsilon)$ is Fredholm. So, its range is closed, which contradicts the following lemma and completes the proof. □

Lemma

If $a \in L^\infty \setminus \{0\}$ and $a = 0$ on a set of positive measure, then the range of $T(a) : H[X] \to H[X]$ is not closed.

Sketch of the proof.

Let $E \subset \mathbb{T}$ be such that $|E| > 0$ and $a = 0$ a.e. in E.

Let $u \in H[X]$ be such that $T(a)u = 0$, i.e. $P(au) = 0$. Then $\widehat{(au)}_n = 0$ for $n \geq 0$. Hence, $\widehat{(\overline{au})}_n = 0$ for $n \leq 0$. So, $\overline{au} \in H[X] \subset H^1$. Since $\overline{au} = 0$ a.e. in E, it follows from the uniqueness theorem that $\overline{au} = 0$ a.e. in $\mathbb{T}$. Since $a \neq 0$, the function $u \in H[X] \subset H^1$ has to be equal to 0 on a set of positive measure. Using the uniqueness theorem again, one gets $u = 0$. Hence, $\operatorname{Ker} T(a) = \{0\}$.

We know from the above that there exists $u_n \in H^\infty \subseteq H[X]$ satisfying $|u_n| = \mathbb{1}_E + \frac{1}{n}\mathbb{1}_{\mathbb{T}\setminus E}$ a.e., $n \in \mathbb{N}$. Then

$$\|u_n\|_{H[X]} = \|u_n\|_X \geq \|\mathbb{1}_E\|_X > 0,$$

$$\|T(a)u_n\|_{H[X]} = \|P(au_n)\|_X = \left\|P\left(a\frac{1}{n}\mathbb{1}_{\mathbb{T}\setminus E}\right)\right\|_X \leq \frac{1}{n}\|P\|_{\mathcal{B}(X)}\|a\|_{L^\infty}\|\mathbb{1}_{\mathbb{T}\setminus E}\|_X,$$

$$\frac{\|T(a)u_n\|_{H[X]}}{\|u_n\|_{H[X]}} \leq \frac{1}{n}\,\frac{\|P\|_{\mathcal{B}(X)}\|a\|_{L^\infty}\|\mathbb{1}_{\mathbb{T}\setminus E}\|_X}{\|\mathbb{1}_E\|_X} \to 0 \quad \text{as} \quad n \to \infty.$$

Then the range of $T(a) : H[X] \to H[X]$ is not closed. □

A number $c \in \mathbb{C}$ is called a (left, right) cluster value of a measurable function $a : \mathbb{T} \to \mathbb{C}$ at a point $t \in \mathbb{T}$ if $1/(a - c) \notin L^\infty(W)$ for every neighbourhood (left semi-neighbourhood or right semi-neighbourhood, respectively) $W \subset \mathbb{T}$ of t. Cluster values are invariant under changes of the function on measure zero sets. We denote the set of all left (right) cluster values of a at t by $a(t-0)$ (by $a(t+0)$), and use also the following notation $a(t) = a(t-0) \cup a(t+0)$, $a(E) = \cup_{t \in E} a(t)$ for $E \subseteq \mathbb{T}$. It is easy to see that $a(t-0)$, $a(t+0)$, $a(t)$ and $a(\mathbb{T})$ are closed sets. Hence they are all compact if $a \in L^\infty(\mathbb{T})$.

The above results imply the following theorem.

Theorem

Let $a \in L^\infty$. Then

$$a(\mathbb{T}) \subseteq \mathrm{Spec_e}\left(T(a) : H[X] \to H[X]\right).$$

Theorem

Let $a \in C(\mathbb{T})$. *Then*

$$\operatorname{Spec}_e\big(T(a) : H[X] \to H[X]\big) = a(\mathbb{T}).$$

Proof.

It follows from the previous theorem that we only need to prove the inclusion

$$\operatorname{Spec}_e\big(T(a) : H[X] \to H[X]\big) \subseteq a(\mathbb{T}).$$

Take any $\lambda \notin a(\mathbb{T})$. Then $\frac{1}{a-\lambda} \in C(\mathbb{T})$, and it follows from the lemma below that

$$(T(a) - \lambda I)T\left(\frac{1}{a-\lambda}\right) = T(a-\lambda)T\left(\frac{1}{a-\lambda}\right) = I + K_1, \quad K_1 \in \mathcal{K}(H[X]),$$

$$T\left(\frac{1}{a-\lambda}\right)(T(a) - \lambda I) = I + K_2, \quad K_2 \in \mathcal{K}(H[X]).$$

So, $T\left(\frac{1}{a-\lambda}\right)$ is a regulariser of $T(a) - \lambda I$. Hence $T(a) - \lambda I$ is Fredholm, i.e. $\lambda \notin \operatorname{Spec}_e T(a)$. □

Lemma

Let $b \in C(\mathbb{T})$*. Then*

$$[P, bI] = PbI - bP \in \mathcal{K}(X).$$

Hence,

$$T(bc) - T(b)T(c),\ \ T(bc) - T(c)T(b) \in \mathcal{K}(H[X])$$

for any $c \in L^\infty(\mathbb{T})$*.*

Proof.

Since $P = \frac{1}{2}(I + \mathcal{C})$, it is sufficient to prove that $[\mathcal{C}, bI] \in \mathcal{K}(X)$.
Let $b = \mathbf{e}_k$, $k \in \mathbb{N}$. Then

$$([\mathcal{C}, \mathbf{e}_k I]f)(t) = \frac{1}{\pi i}\,\text{p.v.}\int_{\mathbb{T}} \frac{(\tau^k - t^k)f(\tau)}{\tau - t}\,d\tau = \frac{1}{\pi i}\sum_{j=0}^{k-1} t^j \int_{\mathbb{T}} \tau^{k-1-j} f(\tau)\,d\tau.$$

So, $[\mathcal{C}, \mathbf{e}_k I]$ is a finite rank operator. Similarly,

$$([\mathcal{C}, \mathbf{e}_{-k} I]f)(t) = \frac{1}{\pi i}\,\text{p.v.}\int_{\mathbb{T}} \frac{(t^k - \tau^k)f(\tau)}{\tau^k t^k(\tau - t)}\,d\tau = -\frac{1}{\pi i}\sum_{j=0}^{k-1} t^{j-k} \int_{\mathbb{T}} \tau^{-1-j} f(\tau)\,d\tau,$$

and $[\mathcal{C}, \mathbf{e}_{-k} I]$ is also a finite rank operator. Hence $[\mathcal{C}, bI] \in \mathcal{K}(X)$ for any trigonometric polynomial b. Since trigonometric polynomials are dense in $C(\mathbb{T})$, we conclude that $[\mathcal{C}, bI] \in \mathcal{K}(X)$ for any $b \in C(\mathbb{T})$. □

Notation: For any Banach algebra $\mathfrak{B}$, we will denote the set of all invertible elements by $G\mathfrak{B}$.

Our next aim is to find the Fredholm index of $T(a) : H[X] \to H[X]$ with $a \in GC(\mathbb{T})$.

Let $b \in C([0, 2\pi])$ be defined by $b(x) := a(e^{ix})$. Then b has a continuous argument $\theta \in C([0, 2\pi])$: $\;b = |b|e^{i\theta}$. Since $b(0) = a(1) = b(2\pi)$, one has $\theta(2\pi) = \theta(0) + 2\pi m$, $m \in \mathbb{Z}$. The number

$$\operatorname{ind} a := \frac{\theta(2\pi) - \theta(0)}{2\pi} \in \mathbb{Z}$$

is called the index (winding number) of a.

The index $\operatorname{ind} a$ does not depend on the choice of a continuous branch θ of the argument. If $d \in C(\mathbb{T})$ is such that $|d(t) - a(t)| < |a(t)|$ for all $t \in \mathbb{T}$, then $d \in GC(\mathbb{T})$, and

$$\operatorname{ind} d = \operatorname{ind} a.$$

If $a_1, a_2 \in GC(\mathbb{T})$, then

$$\operatorname{ind}(a_1 a_2) = \operatorname{ind} a_1 + \operatorname{ind} a_2.$$

(For proofs of the above results, see, e.g., R.H. Dyer and D.E. Edmunds, *From Real to Complex Analysis*, Theorems 3.4.18–3.4.20, 3.4.26.)

We say that $a_0, a_1 \in GC(\mathbb{T})$ are homotopic in $GC(\mathbb{T})$ if there exists a continuous mapping

$$[0,1] \ni s \mapsto a_s \in GC(\mathbb{T}).$$

It follows from above that, in this case, $\operatorname{ind} a_1 = \operatorname{ind} a_0$.

Take any $a \in GC(\mathbb{T})$ and set $\kappa := \operatorname{ind} a$. It is easy to see that the mapping

$$[0,1] \ni s \mapsto \frac{a}{|a|^s} \in GC(\mathbb{T})$$

is continuous, so a and $\frac{a}{|a|}$ are homotopic in $GC(\mathbb{T})$. Hence, $\operatorname{ind} \frac{a}{|a|} = \kappa$.

It is also easy to see that $\operatorname{ind} \mathbf{e}_\kappa = \kappa$. Then $\frac{a}{|a|}$ and $\mathbf{e}_\kappa$ are homotopic in $GC(\mathbb{T})$ (see, e.g., Tej Bahadur Singh, *Introduction to topology*, Theorem 14.3.8).

So, a and $\mathbf{e}_\kappa$ are homotopic in $GC(\mathbb{T})$.

Let

$$[0,1] \ni s \mapsto a_s \in GC(\mathbb{T})$$

be a continuous mapping such that $a_0 = a$ and $a_1 = \mathbf{e}_\kappa$. Then

$$[0,1] \ni s \mapsto T(a_s) \in \Phi(H[X])$$

is also continuous, and

$$\operatorname{Ind} T(a) = \operatorname{Ind} T(\mathbf{e}_\kappa).$$

Since $\operatorname{Ind}(T(\mathbf{e}_\kappa) : H[X] \to H[X]) = -\kappa$, one gets

$$\operatorname{Ind}(T(a)) = -\operatorname{ind} a\,.$$

This formula is an ancestor of the celebrated Atiyah–Singer index theorem.

Putting the above results together, we get the following theorem, which goes back to B.V. Khvedelidze (1956, 1958), I. Gohberg (1958, 1964), and I.B. Simonenko (1959).

Theorem

Let $a \in C(\mathbb{T})$. The operator $T(a) : H[X] \to H[X]$ is Fredholm if and only if $a(t) \neq 0$ for all $t \in \mathbb{T}$, in which case

$$\operatorname{Ind}(T(a)) = -\operatorname{ind} a\,.$$

Moreover, $T(a)$ is left-invertible if $\operatorname{Ind} T(a) \leq 0$, right-invertible if $\operatorname{Ind} T(a) \geq 0$, and invertible if $\operatorname{Ind} T(a) = 0$.

Corollary

Let $a \in C(\mathbb{T})$. Then

$$\operatorname{Spec}\big(T(a) : H[X] \to H[X]\big) = a(\mathbb{T}) \cup \{\lambda \in \mathbb{C} \setminus a(\mathbb{T})|\ \operatorname{ind}(a-\lambda) \neq 0\}.$$

It follows from the above that if $a \in C(\mathbb{T})$, then $\operatorname{Spec}_e T(a)$ and $\operatorname{Spec} T(a)$ are connected.

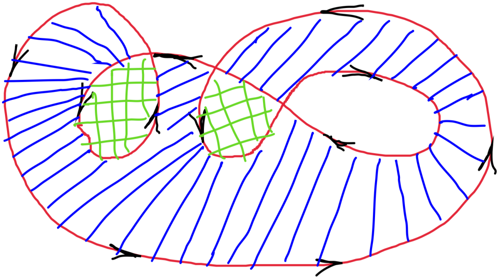

Figure: The red curve is $a(\mathbb{T}) = \operatorname{Spec}_e T(a)$. The rest of $\operatorname{Spec} T(a)$ is coloured in blue, where $\operatorname{Ind}(T(a) - \lambda I) = -1$, and in green, where $\operatorname{Ind}(T(a) - \lambda I) = -2$.

Back to the general case $a \in L^\infty(\mathbb{T})$

P.R. Halmos (1963): Is the spectrum of a Toeplitz operator necessarily connected?

H. Widom (1964 for $p = 2$ and 1966 for any $p \in (1, \infty)$):
$\mathrm{Spec}\,\big(T(a) : H^p \to H^p\big)$ is connected.

R.G. Douglas (1972): $\mathrm{Spec}_{\mathrm{e}}\,\big(T(a) : H^2 \to H^2\big)$ is connected.

A.Yu. Karlovich and I.M. Spitkovsky (2009): Let $1 < p < \infty$ and $w \in A_p$. Then $\mathrm{Spec}_{\mathrm{e}}\,\big(T(a) : H^p(w) \to H^p(w)\big)$ is connected.

We expect $\mathrm{Spec}_{\mathrm{e}}\,\big(T(a) : H[X] \to H[X]\big)$ to be connected in the general case (possibly under additional restrictions on X, like separability or reflexivity). This would imply that $\mathrm{Spec}\,\big(T(a) : H[X] \to H[X]\big)$ is also connected. $\longrightarrow$

We know from the above (Coburn's lemma) that $T(a)$ is invertible if and only if it is Fredholm and its index is equal to 0. Hence, $\operatorname{Spec} T(a)$ is the disjoint union of the closed set $\operatorname{Spec}_{\mathrm{e}} T(a)$ and the set

$$\{\lambda \in \mathbb{C} | \ T(a) - \lambda I \ \text{ is Fredholm and } \ \operatorname{Ind} T(a) \neq 0\}.$$

It follows from the stability of the Fredholm properties of an operator under small perturbations that the latter set is open. So, the boundary of the closed set $\operatorname{Spec} T(a)$ is a subset of $\operatorname{Spec}_{\mathrm{e}} T(a)$. Hence $\operatorname{Spec} T(a)$ is connected if $\operatorname{Spec}_{\mathrm{e}} T(a)$ is. Indeed, any disconnection of $\operatorname{Spec} T(a)$ would have been a disconnection of its boundary and of $\operatorname{Spec}_{\mathrm{e}} T(a)$.

Reminder: Let $a \in L^\infty$. If $T(a) : H[X] \to H[X]$ is Fredholm, then $a \in GL^\infty$.

Theorem

Let $a \in GL^\infty$. Then $T(a) : H[X] \to H[X]$ is invertible (Fredholm) if and only if $T(\frac{a}{|a|}) : H[X] \to H[X]$ is invertible (Fredholm, respectively), and

$$\dim \operatorname{Ker} T(a) = \dim \operatorname{Ker} T\left(\frac{a}{|a|}\right),$$

$$\dim \left(H[X]/\operatorname{Ran} T(a)\right) = \dim \left(H[X]/\operatorname{Ran} T\left(\frac{a}{|a|}\right)\right).$$

Proof.

We know that there exists $G \in H^\infty$ such that $|G| = |a|^{1/2}$ a.e. and $\frac{1}{G} \in H^\infty$. Then $\overline{G}G = |a|$ a.e., and

$$T(a) = T\left(\overline{G}\right) T\left(\frac{a}{|a|}\right) T(G).$$

It is left to note that $T\left(\overline{G}\right)$ and $T(G)$ are invertible:

$$\left(T\left(\overline{G}\right)\right)^{-1} = T\left(\frac{1}{\overline{G}}\right), \quad (T(G))^{-1} = T\left(\frac{1}{G}\right).$$

□

Notation

For a measurable function $w \geq 0$, the weighted space $X(w)$ consists of all measurable functions $f : \mathbb{T} \to \mathbb{C}$ such that $fw \in X$. We equip it with the norm $\|f\|_{X(w)} := \|fw\|_X$. If $w \in X$ and $1/w \in X'$, then $X(w)$ is a Banach function space, whose associate space is $X'(1/w)$.

Let

$$H_-[X] := \{g \in X : \; \widehat{g}(n) = 0 \quad \text{for all} \quad n \geq 0\}$$
$$H_-^0[X] := \{g \in X : \; \widehat{g}(n) = 0 \quad \text{for all} \quad n > 0\} = H_-[X] \oplus \mathbb{C}.$$

Then $Q = I - P$ is a projection of X onto $H_-[X]$, and $P(H_-[X]) = \{0\}$. Also, $g \in H_-^0[X]$ if and only if $\overline{g} \in H[X]$.

Simonenko's factorisation theorem

Definition

A function $a \in GL^\infty(\mathbb{T})$ is said to admit a Wiener-Hopf factorisation in X if it can be represented in the form

$$a = a_- \mathbf{e}_\kappa a_+, \qquad (WHf)$$

where $\kappa \in \mathbb{Z}$,

$$a_- \in H^0_-[X], \quad a_-^{-1} \in H^0_-[X'], \quad a_+ \in H[X'], \quad a_+^{-1} \in H[X],$$

and P is bounded on $X(1/|a_+|)$.

Theorem (O. Karlovych & ES, 2025?)

Let $a \in L^\infty(\mathbb{T})$. Then $T(a) : H[X] \to H[X]$ is Fredholm if and only if $a \in GL^\infty(\mathbb{T})$ and a admits a Wiener-Hopf factorisation in X, in which case $\mathrm{Ind}(T(a)) = -\kappa$ (see (WHf)).

D. E. Sarason (1967): $C + H^\infty$ is a closed subalgebra of L^∞.
The set of functions $\mathbf{e}_{-n}h$ with $h \in H^\infty$ and $n \in \mathbb{Z}_+$ is dense in $C + H^\infty$.

R. G. Douglas (1968): $a \in C + H^\infty$ is invertible in $C + H^\infty$ if and only if there exist $\delta, \varepsilon > 0$ such that

$$\left|\widehat{a}\left(re^{i\theta}\right)\right| > \epsilon \quad \text{for all} \quad r \in (1-\delta, 1),\ \theta \in [-\pi, \pi],$$

where $\widehat{a}$ is the harmonic extension of a to $\mathbb{D}$ given by the Poisson integral:

$$\widehat{a}\left(re^{i\theta}\right) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\, a\left(e^{i\varphi}\right)\, d\varphi.$$

If $a = bg$, where $b \in GC(\mathbb{T})$ and $g \in GH^\infty(\mathbb{T})$, then

$$\operatorname{ind} a_r = \operatorname{ind} b$$

for all sufficiently large $r \in (0,1)$. Here, $a_r\left(e^{i\theta}\right) := \widehat{a}\left(re^{i\theta}\right)$.

Let $d \in L^\infty$, $a \in C + H^\infty$, $a = b + h$, where $b \in C(\mathbb{T})$ and $h \in H^\infty(\mathbb{T})$. Since $[P, bI] \in \mathcal{K}(X)$, one has

$$\begin{aligned} T(d)T(a) &= T(d)(T(b) + T(h)) = T(d)T(b) + T(d)T(h) \\ &= T(db) + K + T(dh) = T(d(b+h)) + K = T(da) + K, \end{aligned}$$

where $K \in \mathcal{K}(H[X])$.

Lemma

If $a \in G(C + H^\infty)$, then $T(a) : H[X] \to H[X]$ is Fredholm.

Proof.

Since $\frac{1}{a} \in C + H^\infty$, it follows from the above that $T\left(\frac{1}{a}\right)$ is a regulariser of $T(a)$. □

Theorem

Let $a \in H^\infty(\mathbb{T}) \setminus \{0\}$. Then the following holds for the operator $T(a) : H[X] \to H[X]$.

1. $\operatorname{Ker} T(a) = \{0\}$.
2. *$T(a)$ is invertible if and only if $a \in GH^\infty(\mathbb{T})$.*[a]
3. *$T(a)$ is Fredholm if and only if $a = bg$, where b is a finite Blaschke product and $g \in GH^\infty(\mathbb{T})$.*

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[a] This result goes back to A. Wintner (1929).

Proof.

1. For any $f \in \operatorname{Ker} T(a)$, one has $af = T(a)f = 0$. Since $a \not\equiv 0$, the function $f \in H[X] \subset H^1$ has to be equal to 0 on a set of positive measure. Then $f = 0$ due to the uniqueness theorem.

2. If $\frac{1}{a} \in H^\infty$, then $T\left(\frac{1}{a}\right)$ is the inverse of $T(a)$. Conversely, if $T(a)$ is invertible, then $\frac{1}{a} \in L^\infty$ (due to the Hartman-Wintner-Simonenko theorem), and there exists $f \in H[X]$ such that $af = T(a)f = \mathbb{1}$. Hence, $\frac{1}{a} = f \in H[X] \cap L^\infty = H^\infty$.

Proof (continued).

3. If $a = bg$, where b is a finite Blaschke product and $g \in GH^\infty(\mathbb{T})$, then $\frac{1}{b} \in C(\mathbb{T})$ and $\frac{1}{a} = \frac{1}{bg} = \frac{1}{b} \cdot \frac{1}{g} \in C + H^\infty$. Hence $T(a)$ is Fredholm due to the previous lemma.

Conversely, suppose $T(a)$ is Fredholm. Then $\frac{1}{a} \in L^\infty$ and $n := \dim (H[X]/ \operatorname{Ran} T(a)) < \infty$. Since the dimension of the space of analytic polynomials of degree not exceeding n is equal to $n+1$, there is a polynomial p that belongs to $\operatorname{Ran} T(a)$. Let $f \in H[X]$ be such that $af = T(a)f = p$.

The canonical factorisation of p has the following form $p = Bg$, where B is a finite Blaschke product (which has the same zeros as p) and $g \in H^\infty$ is an outer function times a unimodular constant.[a] The singular inner factor is trivial, i.e. is equal to $\mathbb{1}$. Let $a = c_1 B_1 S_1 G_1$ and $f = c_2 B_2 S_2 G_2$ be canonical factorisations. Then $p = (c_1 c_2)(B_1 B_2)(S_1 S_2)(G_1 G_2)$, and it follows from the uniqueness of the canonical factorisation that $B_1 B_2 = B$ and $S_1 S_2 = \mathbb{1}$. Hence, both B_1 and B_2 are finite Blaschke products, and $S_1 = \mathbb{1} = S_2$.

So, $a = c_1 B_1 G_1$, where $G_1 \in H^\infty$ is an outer function. Since $\frac{1}{G_1} = \frac{c_1 B_1}{a} \in L^\infty$, one has $\frac{1}{G_1} \in H^\infty$.

□

[a] See, e.g., Section 13.1 in S.R. Garcia, J. Mashreghi, and W.T. Ross, *Finite Blaschke products and their connections*, Springer, Cham, 2018.

The Douglas theorem

Theorem

Let $a \in C + H^\infty$. *Then* $T(a) : H[X] \to H[X]$ *is Fredholm if and only if* $\frac{1}{a} \in C + H^\infty$, *in which case* $\operatorname{Ind}(T(a)) = -\operatorname{ind} a_r$ *for all sufficiently large* $r \in (0, 1)$.

Proof.

We already know that $a \in G(C + H^\infty) \implies T(a) \in \Phi(H[X])$.

Suppose now $T(a) \in \Phi(H[X])$. Then $\frac{1}{a} \in L^\infty$. Take a sequence $a_j = \mathbf{e}_{-n_j} h_j$ with $h_j \in H^\infty$ and $n_j \in \mathbb{Z}_+$ that converges to a in L^∞ as $j \to \infty$. Then $T(a_j) \in \Phi(H[X])$ for all sufficiently large $j \in \mathbb{N}$. Since $T(a_j) = T(\mathbf{e}_{-n_j})T(h_j)$ and $T(\mathbf{e}_{-n_j}) \in \Phi(H[X])$, one gets $T(h_j) \in \Phi(H[X])$. It follows from the previous theorem that $\frac{1}{h_j} \in C + H^\infty$. Then $\frac{1}{a_j} = \frac{1}{\mathbf{e}_{-n_j} h_j} \in C + H^\infty$. On the other hand, $\frac{1}{a_j}$ converges to $\frac{1}{a}$ in L^∞. Since $C + H^\infty$ is a closed subalgebra of L^∞, one concludes that $\frac{1}{a} \in C + H^\infty$.

Proof (continued).

It follows from the previous theorem that $h_j = b_j g_j$, where $b_j \in GC(\mathbb{T})$ and $g_j \in GH^\infty(\mathbb{T})$. Clearly $\mathbf{e}_{-n_j} b_j \in GC(\mathbb{T})$. Since $a_j = \mathbf{e}_{-n_j} b_j g_j$ and $T(g_j)$ is invertible, one gets, for sufficiently large $j \in \mathbb{N}$ and $r \in (0, 1)$,

$$\begin{aligned} \operatorname{Ind}(T(a)) &= \operatorname{Ind}(T(a_j)) = \operatorname{Ind}(T(\mathbf{e}_{-n_j} b_j g_j)) = \operatorname{Ind}(T(\mathbf{e}_{-n_j} b_j) T(g_j)) \\ &= \operatorname{Ind}(T(\mathbf{e}_{-n_j} b_j)) = -\operatorname{ind}(\mathbf{e}_{-n_j} b_j) = -\operatorname{ind}(a_j)_r = -\operatorname{ind} a_r \,. \end{aligned}$$

□

From now on, we will focus on the case where X is a (weighted) L^p space. We will start with $X = L^2$.

A function $a \in L^\infty$ is called sectorial if $a(\mathbb{T})$ lies in an open half-plane whose boundary contains 0. It is easy to see that if a is sectorial, then $a \in GL^\infty$.

Theorem (A. Brown and P.R. Halmos (1964), A. Devinatz (1964))

Let $a \in L^\infty$ be sectorial. Then $T(a) : H^2 \to H^2$ is invertible.

Proof.

Since a is sectorial, there exist $c \in \mathbb{C}$ such that $|c| = 1$ and $ca(\mathbb{T})$ lies in an angle of opening less than π with vertex at 0 and with bisector $(0, +\infty)$. Then it is easy to see that for all sufficiently small $\delta > 0$,

$$\|\mathbb{1} - \delta ca\|_\infty < 1.$$

Since $\|P\|_{\mathcal{B}(L^2)} = 1$, one gets $\|I - \delta c T(a)\|_{\mathcal{B}(L^2)} < 1$. Hence, $\delta c T(a)$ is invertible, and then so is $T(a)$. □

A side remark: $\|P\|_{\mathcal{B}(X)} = 1$ if and only if $X = L^2$ and there exists a constant $C \in (0, \infty)$ such that $\|\cdot\|_X = C\|\cdot\|_{L^2}$ (O. Karlovych & ES, 2023).

Theorem

Let $a \in L^\infty$. Then the following holds for $T(a) : H^2 \to H^2$:

$$a(\mathbb{T}) \subseteq \mathrm{Spec}_\mathrm{e}(T(a)) \subseteq \mathrm{Spec}(T(a)) \subseteq \mathrm{conv}\, a(\mathbb{T}),$$

where conv *stands for the closed convex hull.*

Proof.

We only need to prove the last inclusion, which is due to A. Brown and P.R. Halmos (1964). Take any $\lambda \in \mathbb{C} \setminus \mathrm{conv}\, a(\mathbb{T})$. Then $a - \lambda$ is sectorial. So, $T(a) - \lambda I = T(a - \lambda)$ is invertible according to the previous theorem, i.e. $\lambda \notin \mathrm{Spec}(T(a))$. □

Theorem (P. Hartman and A. Wintner, 1954)

Let $a \in L^\infty$ be real-valued. Then the following holds for $T(a) : H^2 \to H^2$:

$$\mathrm{Spec}_e(T(a)) = \mathrm{Spec}(T(a)) = [\inf a(\mathbb{T}), \sup a(\mathbb{T})].$$

Proof.

Since $[\inf a(\mathbb{T}), \sup a(\mathbb{T})] = \mathrm{conv}\, a(\mathbb{T})$ is the smallest connected set containing $a(\mathbb{T})$, the above equalities follow from the previous theorem and the fact that $\mathrm{Spec}_e(T(a))$ is connected. □

Theorem

Let $a \in GL^\infty$. Then

1. $T(a) : H^2 \to H^2$ *is invertible if and only if* $a = bs$, *where* $b \in GH^\infty$, *and* $s \in GL^\infty$ *is sectorial;*
2. $T(a) : H^2 \to H^2$ *is Fredholm if and only if* $a = bs$, *where* $b \in G(C + H^\infty)$, *and* $s \in GL^\infty$ *is sectorial.*

Sketch of the proof.

2. If a admits the above representation, then

$$T(a) = T(sb) = T(s)T(b) + K, \qquad K \in \mathcal{K}(H^2).$$

Since $T(b)$ is Fredholm and $T(s)$ is invertible, $T(a)$ is Fredholm.

The same proof works for the "if" part of **1**. One only needs to change "Fredholm" for "invertible" (and $K = 0$ in the above formula).

The remaining part of the theorem can be derived from the following result. □

Theorem

Let $a \in L^\infty$ be unimodular, i.e., $|a| = 1$ a.e. on $\mathbb{T}$. Then

1. *$T(a)$ is invertible on H^2 if and only if $\text{dist}_{L^\infty}(a, GH^\infty) < 1$ (H. Widom (1960) and A. Devinatz (1964));*
2. *$T(a)$ is Fredholm on H^2 if and only if $\text{dist}_{L^\infty}(a, G(C + H^\infty)) < 1$ (R.G. Douglas and D. Sarason (1970)).*

Theorem (H. Widom (1960) and A. Devinatz (1964))

Let $a \in GL^\infty$. *Then* $T(a)$ *is invertible on* H^2 *if and only if*

$$\frac{a}{|a|} = e^{i\left(\widetilde{u}+v+c\right)} \quad a.e. \text{ on } \quad \mathbb{T},$$

where $c \in \mathbb{R}$, u *and* v *are real-valued functions in* L^∞*, and* $\|v\|_\infty < \pi/2$. *Here, as above,* $\widetilde{u}$ *denotes the Hilbert transform (the harmonic conjugate) of* u.

Moving to (weighted) H^p spaces

Let $2 \le r < \infty$. We will say that $a \in GL^\infty$ is r-sectorial if $a(\mathbb{T})$ lies in an angle of opening less than $2\pi/r$ with vertex at 0.

Theorem (I.B. Simonenko)

Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $-\frac{1}{p} < \mu < \frac{1}{p'}$, $w(t) := |t - t_0|^\mu$, $t_0 \in \mathbb{T}$,

$$r := \max\left\{p, p', \left(\frac{1}{p} + \mu\right)^{-1}, \left(\frac{1}{p'} - \mu\right)^{-1}\right\}.$$

Let $a \in GL^\infty$ be r-sectorial. Then $T(a) : H^p(w) \to H^p(w)$ is invertible.

Proof.

Let $a_0 := a/|a|$. It is sufficient to prove that $T(a_0) : H^p(w) \to H^p(w)$ is invertible. Clearly, a_0 is r-sectorial. Hence, there exist $c \in \mathbb{C}$ such that $|c| = 1$ and $ca(\mathbb{T}) \subset \left\{e^{i\theta} : -\pi/\rho < \theta < \pi/\rho\right\}$ for some $\rho > r$. Let $\delta := \cos(\pi/\rho)$. Then

$$\begin{aligned}\|\mathbb{1} - \delta c a_0\|_\infty^2 \le \sup_{-\pi/\rho<\theta<\pi/\rho} \left((1-\delta\cos\theta)^2 + \delta^2\sin^2\theta\right) \\ = \sup_{-\pi/\rho<\theta<\pi/\rho} \left(1 - 2\delta\cos\theta + \delta^2\right) = 1 - 2\delta\cos(\pi/\rho) + \delta^2 \\ = 1 - \delta^2 = \sin^2(\pi/\rho) < \sin^2(\pi/r).\end{aligned}$$

Since $\|P\|_{\mathcal{B}(L^p(w))} = \frac{1}{\sin(\pi/r)}$ (B. Hollenbeck and I.E. Verbitsky, 2000), one gets $\|I - \delta c T(a_0)\|_{\mathcal{B}(L^p(w)))} < 1$. Hence, $\delta c T(a_0)$ is invertible, and then so is $T(a_0)$. □

Let $1 < p < \infty$ and w be a Khvedelidze weight: $w(t) := \prod_{j=1}^{n} |t - t_j|^{\mu_j}$, $t \in \mathbb{T}$, where $t_1, \ldots, t_n \in \mathbb{T}$ are pairwise distinct, and $-\frac{1}{p} < \mu_j < \frac{1}{p'}$ for $j = 1, \ldots, n$. Set

$$r_\tau := \begin{cases} \max\{p, p'\} & \text{if} \quad \tau \in \mathbb{T} \setminus \{t_1, \ldots, t_n\}, \\ \max\left\{p, p', \left(\frac{1}{p} + \mu_j\right)^{-1}, \left(\frac{1}{p'} - \mu_j\right)^{-1}\right\} & \text{if} \quad \tau = t_j. \end{cases}$$

We will say that $a \in GL^\infty$ is locally (p, w)-sectorial if for every $\tau \in \mathbb{T}$, there exists an open arc $\ell(\tau) \subset \mathbb{T}$ containing τ and functions g_τ and h_τ such that $\overline{g_\tau}, h_\tau \in G(C + H^\infty)$, and $(g_\tau a h_\tau)(\ell(\tau))$ lies in an angle of opening less than $2\pi/r_\tau$ with vertex at 0.

Using the previous result and localisation techniques, one can prove the following theorem.

Theorem

If $a \in GL^\infty$ is locally (p, w)-sectorial, then $T(a) : H^p(w) \to H^p(w)$ is Fredholm.

Unfortunately, the Widom–Devinatz theorem does not have an H^p-analogue. However, the following holds.

Theorem (N.Ya. Krupnik, 1978)

Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $a \in L^\infty$. Then the following are equivalent:

1. *$T(a)$ is invertible on H^p and $H^{p'}$,*
2. *$T(a)$ is invertible on H^q for all $q \in [\min\{p, p'\}, \max\{p, p'\}]$,*
3. *$a = he^{u+iv}$, where $h \in GH^\infty$, u and v are real-valued functions in L^∞, and $\|v\|_\infty < \pi / \max\{p, p'\}$.*

Theorem (R. Rochberg, 1977)

Let $1 < p < \infty$, w be a Muckenhoupt weight: $w \in A_p$, and let $a \in L^\infty$. Then $T(a)$ is invertible on $H^p(w)$ if and only if $a \in GL^\infty$ and

$$\frac{a}{|a|} = e^{i\left(\widetilde{u}+c\right)},$$

where $c \in \mathbb{R}$ and u is a real-valued function such that $we^{-u/2} \in A_p$.

Our next aim is to consider Toeplitz operators with piecewise continuous symbols. We start with some auxiliary results.

Lemma

Let $1 < p < \infty$, $w \in A_p$, and let $a, b \in L^\infty$ be such that $\operatorname{supp} a \cap \operatorname{supp} b = \emptyset$. Then $T(a)T(b) \in \mathcal{K}(H^p(w))$.

Sketch of the proof.

It is sufficient to prove that $aPbI \in \mathcal{K}(L^p(w))$.

$$\begin{aligned}(aPbf)(t) &= \frac{1}{2}(a(I+\mathcal{C})bf)(t) = \frac{1}{2}(a\mathcal{C}bf)(t) \\ &= \frac{1}{\pi i}\int_{\mathbb{T}} \frac{a(t)b(\tau)f(\tau)}{\tau - t}\, d\tau.\end{aligned}$$

Since $\operatorname{supp} a \cap \operatorname{supp} b = \emptyset$, the above integral operator has a bounded kernel and hence is compact on $L^p(w)$. □

Lemma

Let $1 < p < \infty$, $w \in A_p$, *and suppose the closure of the set of the points of discontinuity of* $a \in L^\infty$ *does not intersect that of* $b \in L^\infty$. *Then* $T(ab) - T(a)T(b) \in \mathcal{K}(H^p(w))$.

Proof.

One can represent a and b as follows: $a = a_1 + a_2$, $b = b_1 + b_2$, where $a_2, b_2 \in C(\mathbb{T})$, and $\operatorname{supp} a_1 \cap \operatorname{supp} b_1 = \emptyset$. Then $a_1 b_1 = 0$ and

$$\begin{aligned} T(a)T(b) &= T(a_1 + a_2)T(b_1 + b_2) \\ &= T(a_1)T(b_1) + T(a_1)T(b_2) + T(a_2)T(b_1) + T(a_2)T(b_2) \\ &= T(a_1 b_2) + T(a_2 b_1) + T(a_2 b_2) + K \\ &= T(a_1 b_2 + a_2 b_1 + a_2 b_2) + K = T(ab) + K, \end{aligned}$$

where $K \in \mathcal{K}(H^p(w))$. □

For $z_1, z_2 \in \mathbb{C}$, $z_1 \neq z_2$, and $r > 1$, let

$$\mathrm{Arc}_r(z_1, z_2) := \left\{ z \in \mathbb{C} \,\middle|\, \arg \frac{z_1 - z}{z_2 - z} \in \frac{2\pi}{r} + 2\pi\mathbb{Z} \right\}.$$

This is the circular arc from the points of which the line segment $[z_1, z_2]$ is seen at the angle $2\pi/r$. For $r = 2$, $\mathrm{Arc}_r(z_1, z_2)$ is the segment $[z_1, z_2]$ itself. For $r' = \frac{r}{r-1}$, the arc $\mathrm{Arc}_{r'}(z_1, z_2)$ is the reflection of $\mathrm{Arc}_r(z_1, z_2)$ through the straight line containing $[z_1, z_2]$.

An elementary argument shows that $\mathrm{Arc}_r(z_1, z_2)$ admits the following parametrisation

$$z(s) = z_1 + (z_2 - z_1)\sigma_r(s), \quad s \in [0, 1],$$

where $\sigma_r(s) = s$ for $r = 2$, and

$$\sigma_r(s) = \frac{\sin(\theta s)}{\sin s}\, e^{i\theta(s-1)}, \qquad \theta = \pi - \frac{2\pi}{r}$$

for $r \neq 2$.

Below[1], the argument $\arg z$ of a number $z \in \mathbb{C} \setminus \{0\}$ will always be chosen so that arg $\arg z \in (-\pi, \pi]$. The logarithm and complex powers will be understood accordingly.

For $\tau \in \mathbb{T}$ and $\beta \in \mathbb{C}$, set

$$\varphi_{\beta,\tau}(t) := e^{i\beta \arg(-t/\tau)}, \qquad t \in \mathbb{T}.$$

Then $\varphi_{\beta,\tau}$ has at most one discontinuity, namely a jump at τ, and $\varphi_{\beta,\tau}(\tau + 0) = e^{-\pi i\beta}$ and $\varphi_{\beta,\tau}(\tau - 0) = e^{\pi i\beta}$.

Let

$$\xi_{\beta,\tau}(t) := \left(1 - \frac{\tau}{t}\right)^{\beta}, \quad \eta_{\beta,\tau}(t) := \left(1 - \frac{t}{\tau}\right)^{\beta}, \qquad t \in \mathbb{T} \setminus \{\tau\}.$$

It is not difficult to see that

$$\varphi_{\beta,\tau}(t) = \xi_{-\beta,\tau}(t)\eta_{\beta,\tau}(t) \quad \text{for all} \quad t \in \mathbb{T} \setminus \{\tau\}.$$

.

[1] Our treatment of Toeplitz operators with piecewise continuous symbols follows closely the presentation in A. Böttcher and B. Silbermann, *Analysis of Toeplitz operators,* Springer-Verlag, Berlin, 2006.

Lemma

Let $\beta \in \mathbb{C}$, $1 < p < \infty$, *and* $w(t) := \prod_{j=1}^{n} |t - t_j|^{\mu_j}$, $t \in \mathbb{T}$, *where* $t_1, \ldots, t_n \in \mathbb{T}$ *are pairwise distinct, and* $-\frac{1}{p} < \mu_j < \frac{1}{p'}$ *for* $j = 1, \ldots, n$. *Then the following are equivalent*

1. $T(\varphi_{\beta,t_1}) \in \Phi(H^p(w))$ *and* $\operatorname{Ind} T(\varphi_{\beta,t_1}) = -\kappa$,
2. $\kappa - \frac{1}{p'} < \operatorname{Re}\beta - \mu_1 < \kappa + \frac{1}{p}$,
3. $0 \notin \operatorname{Arc}_r\left(\varphi_{\beta,t_1}(t_1 - 0), \varphi_{\beta,t_1}(t_1 + 0)\right)$, *where* $r = (1/p + \mu_1)^{-1}$, *and the index of the closed continuous and naturally oriented curve obtained from* $\varphi_{\beta,t_1}(\mathbb{T})$ *by filling in the arc* $\operatorname{Arc}_r\left(\varphi_{\beta,t_1}(t_1 - 0), \varphi_{\beta,t_1}(t_1 + 0)\right)$ *is equal to* κ.

A short sketch of the proof.

2 $\iff$ **3** is straightforward.

2 $\implies$ **1**. Put $\gamma = \beta - \kappa$. Then $-\frac{1}{p'} < \operatorname{Re}\gamma - \mu_1 < \frac{1}{p}$, and it is not difficult to see that $\varphi_{\gamma,t_1} = \xi_{-\gamma,t_1}\eta_{\gamma,t_1}$ is a Wiener-Hopf factorisation of φ_{γ,t_1} in $L^p(w)$. Hence, $T(\varphi_{\gamma,t_1})$ is invertible on $H^p(w)$. Since $\varphi_{\beta,t_1} = \mathbf{e}_\kappa\varphi_{\gamma,t_1}$, the operator $T(\varphi_{\beta,t_1})$ is equal to $T(\varphi_{\gamma,t_1})T(\mathbf{e}_\kappa)$ or to $T(\mathbf{e}_\kappa)T(\varphi_{\gamma,t_1})$, depending on the sign of κ. So, $T(\varphi_{\beta,t_1})$ is Fredholm on $H^p(w)$, and

$$\operatorname{Ind} T(\varphi_{\beta,t_1}) = \operatorname{Ind} T(\varphi_{\gamma,t_1}) + T(\mathbf{e}_\kappa) = 0 - \kappa = -\kappa.$$

1 $\implies$ **2**. There exists $k \in \mathbb{Z}$ such that $k - \frac{1}{p'} < \operatorname{Re}\beta - \mu_1 \leq k + \frac{1}{p}$. If $\operatorname{Re}\beta - \mu_1 < k + \frac{1}{p}$, then $k = \kappa$ by the above, and we are done. So, suppose $\operatorname{Re}\beta - \mu_1 = k + \frac{1}{p}$. It follows from **1** and stability of Fredholm properties that $T(\varphi_{\beta_\pm,t_1})$ are Fredholm of index κ, provided $\beta_\pm \in \mathbb{C}$ are sufficiently close to β. We can choose them in such a way that

$$k - \frac{1}{p'} < \operatorname{Re}\beta_- - \mu_1 < k + \frac{1}{p} < \operatorname{Re}\beta_+ - \mu_1 < k + 1 + \frac{1}{p}.$$

Then it follows from the above that $\operatorname{Ind} T(\varphi_{\beta_-,t_1}) = -k$ and $\operatorname{Ind} T(\varphi_{\beta_+,t_1}) = -k - 1$, which is a contradiction.

$\square$

One can apply the previous lemma to $T(\varphi_{\beta,t_0})$, where $t_0 \in \mathbb{T} \setminus \{t_1, \dots, t_n\}$. Indeed, $w(t) = \prod_{j=0}^{n} |t - t_j|^{\mu_j}$, where $\mu_0 = 0$. So, one only needs to change μ_1 for $\mu_0 = 0$ in the lemma.

Let PC_0 denote the set of all piecewise continuous functions on $\mathbb{T}$ that have at most finitely many jumps. The closure of PC_0 in L^∞ is denoted by PC. A function $a \in PC$ has finite right and left limits $a(t \pm 0)$ at every point $t \in \mathbb{T}$, and there at most countably many t such that $a(t - 0) \neq a(t + 0)$.

Definition

Let $1 < p < \infty$, and $w(t) := \prod_{j=1}^{n} |t - t_j|^{\mu_j}$, $t \in \mathbb{T}$, where $t_1, \ldots, t_n \in \mathbb{T}$ are pairwise distinct, and $-\frac{1}{p} < \mu_j < \frac{1}{p'}$ for $j = 1, \ldots, n$. For $a \in PC$, define $a_{p,w} : \mathbb{T} \times [0,1] \to \mathbb{C}$ by

$$a_{p,w}(t,s) := (1 - \sigma_{r(t)}(s))a(t-0) + \sigma_{r(t)}(s)a(t+0), \quad (t,s) \in \mathbb{T} \times [0,1],$$

where σ_r is the function introduced above, $r(t) = p$ for $t \in \mathbb{T} \setminus \{t_1, \ldots, t_n\}$, and $r(t) = (1/p + \mu_j)^{-1}$ for $t = t_j$.

If $a(t-0) = a(t+0) = a(t)$, then $a_{p,w}(t,s) = a(t)$ for all $s \in [0,1]$.

The range of $a_{p,w}$ is a continuous closed and naturally oriented curve obtained from $a(\mathbb{T})$ by filling in the arcs $\mathrm{Arc}_{r(t)}(a(t-0), a(t+0))$ for each $t \in \mathbb{T}$ at which a has a jump. If the curve does not pass through 0, its winding number with respect to 0 will be denoted by $\mathrm{ind}\, a_{p,w}$.

Lemma

If $a, b \in PC_0$ *have no common points of discontinuity and* $a_{p,w}, b_{p,w}$ *have no zeros in* $\mathbb{T} \times [0,1]$, *then* $(ab)_{p,w} = a_{p,w} b_{p,w}$, *and* $\operatorname{ind}(ab)_{p,w} = \operatorname{ind} a_{p,w} + \operatorname{ind} b_{p,w}$.

Theorem (I. Gohberg and N.Ya. Krupnik, 1968-1969; H. Widom, 1960)

Let $1 < p < \infty$, *and* $w(t) := \prod_{j=1}^{n} |t - t_j|^{\mu_j}$, $t \in \mathbb{T}$, *where* $t_1, \dots, t_n \in \mathbb{T}$ *are pairwise distinct, and* $-\frac{1}{p} < \mu_j < \frac{1}{p'}$ *for* $j = 1, \dots, n$. *Let* $a \in PC_0$. *Then*

$$T(a) \in \Phi(H^p(w)) \iff a_{p,w}(t,s) \neq 0 \quad \text{for all} \quad (t,s) \in \mathbb{T} \times [0,1],$$

in which case $\operatorname{Ind} T(a) = -\operatorname{ind} a_{p,w}$.

A short sketch of the proof.

We can assume that $a \in GL^\infty$. Let $\tau_1, \dots, \tau_m \in \mathbb{T}$ be the points of discontinuity of a. Then a admits a representation

$$a = b\varphi_{\beta_1,\tau_1} \cdots \varphi_{\beta_m,\tau_m},$$

where $b \in GC(\mathbb{T})$ and

$$e^{2\pi i\beta_l} = \frac{a(\tau_l - 0)}{a(\tau_l + 0)}, \quad l = 1, \dots, m.$$

Since the above functions do not have common points of discontinuity,

$$T(a) - T(b)T(\varphi_{\beta_1,\tau_1}) \cdots T(\varphi_{\beta_m,\tau_m}) \in \mathcal{K}(H^p(w)).$$

If $a_{p,w}(t,s) \neq 0$ for all $(t,s) \in \mathbb{T} \times [0,1]$, then each operator in the above product is Fredholm. So, $T(a) \in \Phi(H^p(w))$ and

$$\begin{aligned} \operatorname{Ind} T(a) &= \operatorname{Ind} T(b) + \operatorname{Ind} T(\varphi_{\beta_1,t_1}) + \cdots + \operatorname{Ind} T(\varphi_{\beta_m,t_m}) \\ &= -\operatorname{ind} b - \operatorname{ind}(\varphi_{\beta_1,t_1})_{p,w} - \cdots - \operatorname{ind}(\varphi_{\beta_m,t_m})_{p,w} \\ &= -\operatorname{ind}(b\varphi_{\beta_1,t_1} \cdots \varphi_{\beta_m,t_m})_{p,w} = -\operatorname{ind} a_{p,w}. \end{aligned}$$

Proof (continued).

Once the index formula has been proved, the same perturbation argument as above shows that $a_{p,w}(t,s) \neq 0$ for all $(t,s) \in \mathbb{T} \times [0,1]$ if $T(a) \in \Phi(H^p(w))$. $\square$

Corollary

Under the conditions of the above theorem,

$$\mathrm{Spec}_e(T(a) : H^p(w) \to H^p(w)) = \left(\bigcup_{t\in\mathbb{T}} \{a(t \pm 0)\} \right) \cup \left(\bigcup_{a(t-0)\neq a(t+0)} \mathrm{Arc}_{r(t)}(a(t-0), a(t+0)) \right).$$

The above results can be extended to symbols $a \in PC$.

I.M. Spitkovsky (1992): in the case of general Muckenhoupt weights $w \in A_p$, the discontinuities of $a \in PC$ contribute not just circular arcs but two-dimensional sets, so-called horns, to $\mathrm{Spec}_e(T(a) : H^p(w) \to H^p(w))$.

For a definitive exposition of the theory of singular integral operators on $L^p(w)$ with general Muckenhoupt weights over general Carleson (Ahlfors regular) curves, see A. Böttcher and Y.I. Karlovich, *Carleson curves, Muckenhoupt weights, and Toeplitz operators,* Birkhäuser Verlag, Basel, 1997.

A. Karlovich (2002) extended the Böttcher–Karlovich theory to the case of rearrangement-invariant spaces with Muckenhoupt weights.

V. Kokilashvili and S. Samko (2003): singular integral operators with piecewise continuous coefficients on variable Lebesgue spaces.

Suppose that for each $t \in \mathbb{T}$, the set $a(t)$ consists of two points

$$a_1(t), a_2(t) \in \mathbb{C}$$

(which may coincide). We say that $t \in \mathbb{T}_{\rm I}$ if $a_1(t) \neq a_2(t)$ and each of the sets $a(t-0)$ and $a(t+0)$ consists of one point, i.e. if a has a left and a right limits at t and they do not coincide. We say that $t \in \mathbb{T}_{\rm II}$ if at least one of the sets $a(t-0)$, $a(t+0)$ consists of two points, i.e. if a does not have a left or a right limit at t.

Let

$$\mathcal{R}_p(a;t) := \left\{ \zeta \in \mathbb{C} \middle| \ \frac{2\pi}{\max\{p,p'\}} \leq \arg \frac{a_1(t)-\zeta}{a_2(t)-\zeta} \leq \frac{2\pi}{\min\{p,p'\}} \right\}, \quad 1 < p < \infty.$$

$\mathcal{R}_p(a;t)$ is a two-dimensional (lenticular) set bounded by the circular arcs $\mathrm{Arc}_p(a_1(t), a_2(t))$ and $\mathrm{Arc}_{p'}(a_1(t), a_2(t))$. $\mathbb{C} \setminus \mathcal{R}_p(a;t)$ is the set of points from which the line segment $[a_1(t), a_2(t)]$ is seen at an angle less than $\frac{2\pi}{\max\{p,p'\}}$.

Theorem (K.F. Clancey, 1976 and 1984; I.M. Spitkovsky, 1983)

Let $1 < p < \infty$ *and* $a \in L^\infty(\mathbb{T})$*. Suppose that, for each* $t \in \mathbb{T}$*, the set* $a(t)$ *consists of at most two points. Then*

$$\mathrm{Spec}_e(T(a) : H^p \to H^p) = a(\mathbb{T}) \bigcup \left(\bigcup_{t \in \mathbb{T}_{\mathrm{I}}} \mathrm{Arc}_p(a(t-0), a(t+0)) \right) \bigcup \left(\bigcup_{t \in \mathbb{T}_{\mathrm{II}}} \mathcal{R}_p(a; t) \right).$$

A complete description of the (essential) spectrum of $T(a)$ in terms of $a(t \pm 0)$, $t \in \mathbb{T}$ is no longer possible if $a(t)$ is allowed to contain more than two points.

Suppose $a \in L^\infty(\mathbb{T})$ is such that $a(1)$ consists of three points, $a(1 \pm 0) = a(\mathbb{T}) = \{c_1, c_2, c_3\} \subset \mathbb{C}$ and the closed triangle $\triangle(c_1, c_2, c_3)$ with the vertices c_1, c_2, c_3 is non-degenerate. Then the (essential) spectrum of $T(a) : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ is a connected set that contains $\{c_1, c_2, c_3\}$ and is contained in $\triangle(c_1, c_2, c_3)$. It turns out however that this set is not determined solely by c_1, c_2, c_3.

A. Böttcher (1986) constructed examples where the spectrum of $T(a) : H^2(\mathbb{T}) \to H^2(\mathbb{T})$

(i) does not contain any points of the boundary of the triangle $\triangle(c_1, c_2, c_3)$ other than c_1, c_2, c_3;

(ii) contains a side of $\triangle(c_1, c_2, c_3)$ and no other point of the boundary apart from c_1, c_2, c_3;

(iii) coincides with the union of two sides of $\triangle(c_1, c_2, c_3)$;

(iv) coincides with the boundary of $\triangle(c_1, c_2, c_3)$;

(v) coincides with $\triangle(c_1, c_2, c_3)$

These striking examples imply that if $a(t)$ is not required to contain at most two points for every $t \in \mathbb{T}$, then it is no longer possible to describe the (essential) spectrum of $T(a)$ in terms of the cluster values of a. In other words, it is no longer sufficient to know the values of a, it is important to know "how these values are attained" by a.

Let $K \subset \mathbb{C}$ be an arbitrary compact set and $\lambda \in \mathbb{C} \setminus K$. Then the set

$$\sigma(K;\lambda) = \left\{ \frac{w-\lambda}{|w-\lambda|} \;\middle|\; w \in K \right\} \subseteq \mathbb{T}$$

is compact as a continuous image of a compact set. Hence the set $\Delta_\lambda(K) := \mathbb{T} \setminus \sigma(K;\lambda)$ is open in $\mathbb{T}$. So, $\Delta_\lambda(K)$ is the union of an at most countable family of open arcs.
We call an open arc of $\mathbb{T}$ *p–large* if its length is greater than or equal to $2\pi/\max\{p,p'\}$, $1 < p < \infty$.

We know that $a(\mathbb{T}) \subseteq \mathrm{Spec}_e(T(a))$. Böttcher's examples show that no point in $\mathbb{C} \setminus a(\mathbb{T})$ will always belong to the (essential) spectrum of $T(a) : H^2(\mathbb{T}) \to H^2(\mathbb{T})$, unless $a(\mathbb{T})$ lies on a straight line. The following result shows that the situation is somewhat different for $p \neq 2$.

Theorem (ES, 2007)

Let $1 < p < \infty$, $a \in L^\infty(\mathbb{T})$, $\lambda \in \mathbb{C} \setminus a(\mathbb{T})$ and suppose that, for some $t \in \mathbb{T}$,
(i) $\Delta_\lambda(a(t-0))$ (or $\Delta_\lambda(a(t+0))$) contains at least two p–large arcs,
(ii) $\Delta_\lambda(a(t+0))$ (or $\Delta_\lambda(a(t-0))$, respectively) contains at least one p–large arc.
Then λ belongs to the essential spectrum of $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$.

Suppose $a(t)$ consists of two points. Then condition *(ii)* in the above theorem is automatically satisfied, while condition *(i)* means that a does not have a left limit at t (or a right limit at t, respectively) and that λ belongs to $\mathcal{R}_p(a;t)$. Hence, the above theorem is in a sense an extension of the Clancey–Spitkovsky theorem.

Condition *(i)* is optimal in the following sense.

Theorem (ES, 1994)

Let $t \in \mathbb{T}$, $K \subset \mathbb{C}$ be a compact set, $\lambda \in \mathbb{C} \setminus K$, and suppose $\Delta_\lambda(K)$ contains at most one p–large arc. Then there exists $a \in L_\infty(\mathbb{T})$ such that

$$a(t \pm 0) = a(t) = a(\mathbb{T}) = K$$

and

$$T(a) - \lambda I : H^r(\mathbb{T}) \to H^r(\mathbb{T})$$

is invertible for any $r \in [\min\{p, p'\}, \max\{p, p'\}]$.

While condition *(i)* is the main reason why λ belongs to $\mathrm{Spec}_{\mathrm{e}}(T(a))$, the rôle of *(ii)* is to make sure that the behaviour of $a(\tau)$ as τ approaches t from the other side does not counterbalance the effect of *(i)*. It turns out that condition *(ii)* cannot be dropped.

Theorem (S.M. Grudsky and ES, 2008)

There exists $a \in L^\infty(\mathbb{T})$ *such that* $a(1-0) = \{\pm 1\}$, $|a| \equiv 1$, $T(a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ *is invertible for any* $p \in (1,2)$, *and* $T(1/a) : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ *is invertible for any* $p \in (2,+\infty)$.

The main conclusion of the course.

We seem to be very far from being able to find the (essential) spectrum of $T(a)$ for an arbitrary $a \in L^\infty$.

Unfortunately, we have not discussed many important topics, including:

- Banach algebras in the theory of Toeplitz operators;
- localisation (the Allan-Douglas, Gohberg-Krupnik, and Simonenko local principles);
- the case of matrix-valued symbols $a \in L^\infty(\mathbb{T}, \mathbb{C}^{N\times N})$.

Many of the above results can be generalised to the matrix-valued case. However, there are important differences.

Consider, for example, the matrix-valued symbol

$$a_0 := \begin{pmatrix} \mathbf{e}_1 & 0 \\ 0 & \mathbf{e}_{-1} \end{pmatrix}.$$

It is easy to see that

$$\dim(\operatorname{Ker} T(a_0)) = 1, \quad \dim\left(H^p(\mathbb{T}, \mathbb{C}^2)/\operatorname{Ran} T(a_0)\right) = 1.$$

So, there is no analogue of Coburn's lemma in the matrix-valued case, and a Fredholm Toeplitz operator of index zero does not have to be invertible. Also, while the index of a Fredholm operator is stable under small perturbations, $\dim(\operatorname{Ker} T(a))$ and $\dim\left(H^p(\mathbb{T}, \mathbb{C}^N)/\operatorname{Ran} T(a)\right)$ are, in general, unstable.

G.S. Litvinchuk and I.M. Spitkovsky, *Factorization of measurable matrix functions*, Birkhäuser Verlag, Basel, 1987.

... and I nearly forgot to mention one of my favourite books:

N.Ya. Krupnik, *Banach algebras with symbol and singular integral operators*, Springer, 1987.