Maximal Blaschke Products

By the Riemannian point of view, holomorphic functions are best described by their boundary behavior and their "singularities". The most common choice for the "singularities" are the zeros and the poles. In this course we take a less traveled path by choosing critical points as "singularities". We discuss the basic features of this approach as well as numerous recent developments. Starting point is the connection between uniformization of surfaces and the maximal solution of the Liouville equation

$$\Delta u = 4e^{2u} \, ,$$

which goes back to Schwarz, Picard and Poincare in the 1890s. Here, the so called Liouville correspondence plays a key role as it relates solutions of the Liouville equation with bounded, locally univalent maps. Further, solutions to the PDE $\Delta u = 4e^{2u}$ with singularities correspond to bounded holomorphic functions with critical points and lead naturally to a class of Blaschke products, which are called maximal Blaschke products. As we will see maximal Blaschke products have a number of distinguished properties, which indicate that they constitute an appropriate infinite generalization of the class of finite Blaschke products. For instance we obtain Ahlfors–Nehari–Schwarz lemmas, which extend earlier work of Nehari (1947), Heins (1962), Zakeri (1996), and boundary versions in the spirit of Burns–Krantz. Secondly, maximal Blaschke products behave nicely with respect to composition and forward iteration. Thirdly, maximal Blaschke products maximize the derivative at a fixed point over the set of all bounded analytic functions in the unit disk with prescribed critical points, a property which closely relates them to canonical divisors in Bergman spaces and the Ahlfors' map. In recent work, O. Ivrii and O. Ivrii & A. Nicolau connect maximal Blaschke products with the invariant subspace problem for Bergman spaces as well as with work of Garnett and Papadimitriakis from 1991 on almost isometries of the hyperbolic plane.