Sensitivity Analysis on (excursion) sets based on kernel embedding of random sets

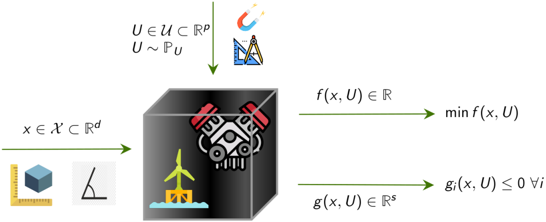
Christophette Blanchet

École Centrale de Lyon & ICJ

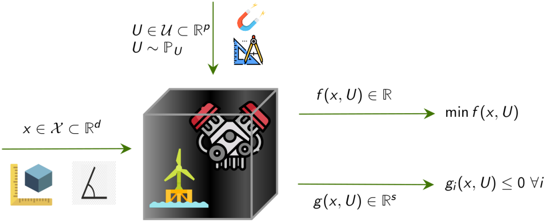
Joint Work with Noé Fellmann& Céline Helbert (ECL)
Adrien Spagnol & Delphine Sinoquet (IFPEN)

Gaussian Processes 7-9/07/25 ANR GAP-Toulouse

Robust conception of an electrical machine



Robust conception of an electrical machine



Robust optimization problem

$$x^* = \arg\min_{\boldsymbol{x} \in \mathcal{K}} \mathbb{E}[f(\boldsymbol{x}, \boldsymbol{U})] \text{ where } \mathcal{K} = \{\boldsymbol{x} \in \mathcal{X},\ \mathbb{P}[g(\boldsymbol{x}, \boldsymbol{U}) \leqslant 0] \geqslant 0.95\}$$

Optimization problem

Robust optimization problem

$$x^* = \arg\min_{\boldsymbol{x} \in \mathcal{K}} \mathbb{E}[f(\boldsymbol{x}, \boldsymbol{U})] \text{ where } \mathcal{K} = \{\boldsymbol{x} \in \mathcal{X},\ \mathbb{P}[g(\boldsymbol{x}, \boldsymbol{U}) \leqslant 0] \geqslant 0.95\}$$

Optimization problem

Robust optimization problem

$$x^* = \arg\min_{\boldsymbol{x} \in \mathcal{K}} \mathbb{E}[f(\boldsymbol{x}, \boldsymbol{U})] \text{ where } \mathcal{K} = \{\boldsymbol{x} \in \mathcal{X},\ \mathbb{P}[g(\boldsymbol{x}, \boldsymbol{U}) \leqslant 0] \geqslant 0.95\}$$

- f and g expensive $\Longrightarrow$ GP processes
- High Dimension for d and $p \Longrightarrow$ Sensitivity analysis to reduce the space.

Optimization problem

Robust optimization problem

$$x^* = \arg\min_{\boldsymbol{x} \in \mathcal{K}} \mathbb{E}[f(\boldsymbol{x}, \boldsymbol{U})] \text{ where } \mathcal{K} = \{\boldsymbol{x} \in \mathcal{X},\ \mathbb{P}[g(\boldsymbol{x}, \boldsymbol{U}) \leqslant 0] \geqslant 0.95\}$$

- f and g expensive $\Longrightarrow$ GP processes
- High Dimension for d and $p \Longrightarrow$ Sensitivity analysis to reduce the space.
- Design variables => Goal Oriented Sensitivity Analysis Spagnol 2020
- Uncertain inputs => How to quantify the impact

Optimization problem

Robust optimization problem

$$x^* = \arg\min_{\boldsymbol{x} \in \mathcal{K}} \mathbb{E}[f(\boldsymbol{x}, \boldsymbol{U})] \text{ where } \mathcal{K} = \{\boldsymbol{x} \in \mathcal{X},\ \mathbb{P}[g(\boldsymbol{x}, \boldsymbol{U}) \leqslant 0] \geqslant 0.95\}$$

- f and g expensive $\Longrightarrow$ GP processes
- High Dimension for d and $p \Longrightarrow$ Sensitivity analysis to reduce the space.
- Design variables => Goal Oriented Sensitivity Analysis Spagnol 2020
- Uncertain inputs => How to quantify the impact

Idea : Quantify the impact of U on excursion set

$$\Gamma_U = \{x \in \mathcal{X}, g(x, U) \leqslant q\},$$

A toy excursion set

Toy function from [El-Amri et al. 2021]

$$\forall(x, u) \in [-5, 5]^4, f(\boldsymbol{x}, \boldsymbol{u}) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1.$$

Excursion sets

New output :

$$\Gamma_U = \{x \in \mathcal{X}, f(x, U) \leqslant 0\}, \quad (1)$$

which is called a random excursion set.

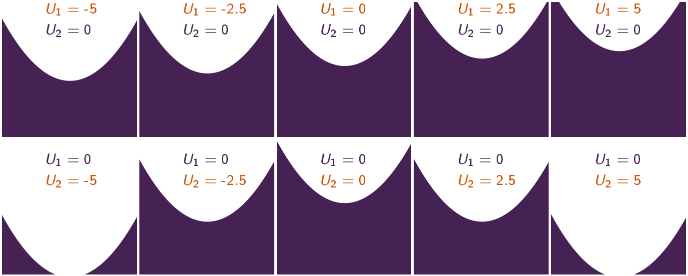


Table of Contents

Introduction

Sensitivity Analysis

Kernel-based Sensitivity Analysis on excursion sets
- Kernel embedding of Probability distribution for Sensitivity Analysis (HSIC)
- Kernel embedding of random sets for Sensitivity Analysis of set-valued models
- Estimation

Numerical tests
- Toy excursion set
- Robust conception of an electrical machine

Sensitivity analysis

Sensitivity analysis (SA)

$$(U_1, ..., U_d) \overset{f}{\mapsto} Y = f(U_1, ..., U_d)$$

How can the uncertainty of Y be divided and allocated to the uncertainty of the inputs U_i?

Sensitivity analysis

Sensitivity analysis (SA)

$$(U_1, ..., U_d) \overset{f}{\mapsto} Y = f(U_1, ..., U_d)$$

How can the uncertainty of Y be divided and allocated to the uncertainty of the inputs U_i?

- Sobol indices: $S_i = \frac{\mathrm{Var}\,\mathbb{E}(Y|U_i)}{\mathrm{Var}\,Y}$
- Dependence measures: $S_i = ||\mathbb{P}_{(U_i,Y)} - \mathbb{P}_{U_i} \otimes \mathbb{P}_Y||$
  - Density-based indices (Borgonovo 2007)
  - Cramer von mises indices (Gamboa, Klein, and Lagnoux 2018)
  - Hilbert Schmidt Independence Criterion : HSIC (Gretton, Bousquet, et al. 2005)

Sensitivity analysis

Sensitivity analysis (SA)

$$(U_1, ..., U_d) \overset{f}{\mapsto} Y = f(U_1, ..., U_d)$$

How can the uncertainty of Y be divided and allocated to the uncertainty of the inputs U_i?

- Sobol indices: $S_i = \frac{\mathrm{Var}\,\mathbb{E}(Y|U_i)}{\mathrm{Var}\,Y}$
- Dependence measures: $S_i = ||\mathbb{P}_{(U_i,Y)} - \mathbb{P}_{U_i} \otimes \mathbb{P}_Y||$
  - Density-based indices (Borgonovo 2007)
  - Cramer von mises indices (Gamboa, Klein, and Lagnoux 2018)
  - Hilbert Schmidt Independence Criterion : HSIC (Gretton, Bousquet, et al. 2005)

Screening: $U_1, ..., U_k$ are influential and $U_{k+1}, ...U_d$ are not influential
Ranking: $U_1 \prec ... \prec U_d$

Sensitivity analysis

Sensitivity analysis (SA)

$$(U_1, \ldots, U_d) \overset{f}{\mapsto} Y = f(U_1, \ldots, U_d)$$

How can the uncertainty of Y be divided and allocated to the uncertainty of the inputs U_i?

- Sobol indices: $S_i = \frac{\text{Var}\,\mathbb{E}(Y|U_i)}{\text{Var}\,Y}$
- Dependence measures: $S_i = ||\mathbb{P}_{(U_i,Y)} - \mathbb{P}_{U_i} \otimes \mathbb{P}_Y||$
  - Density-based indices (Borgonovo 2007)
  - Cramer von mises indices (Gamboa, Klein, and Lagnoux 2018)
  - Hilbert Schmidt Independence Criterion : HSIC (Gretton, Bousquet, et al. 2005)

Screening: $U_1, \ldots, U_k$ are influential and $U_{k+1}, \ldots U_d$ are not influential
Ranking: $U_1 \prec \ldots \prec U_d$

Problem: What if the output Y is set-valued?
Solution : Kernel-based Sensitivity Analysis

Distribution embedding into a RKHS

Dependence measures: $S_i = ||\mathbb{P}_{(U_i,Y)} - \mathbb{P}_{U_i} \otimes \mathbb{P}_Y||$

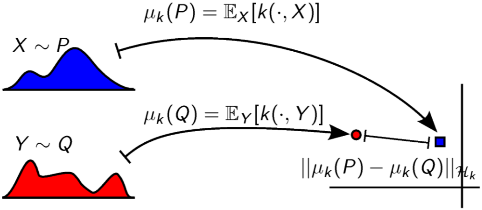


Figure: Kernel mean embedding

with k a (positive definite) kernel $k : (x, x') \in \mathcal{X}^2 \mapsto k(x, x') \in \mathbb{R}$.

Distribution embedding into a RKHS

Dependence measures: $S_i = ||\mathbb{P}_{(U_i,Y)} - \mathbb{P}_{U_i} \otimes \mathbb{P}_Y||$

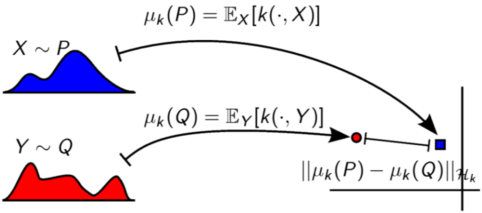


Figure: Kernel mean embedding

with k a (positive definite) kernel $k : (x, x') \in \mathcal{X}^2 \mapsto k(x, x') \in \mathbb{R}$.

Hilbert Schmidt Independence Criterion (HSIC), Gretton, Borgwardt, et al. 2006

With a kernel $K = k_{\mathcal{U}_i} \otimes k_{\mathcal{Y}}$, the HSIC is given by:

$$\text{HSIC}_K(U_i, Y) = ||\mu_K(U_i, Y) - \mu_{k_{\mathcal{U}_i}}(U_i) \otimes \mu_{k_{\mathcal{Y}}}(Y)||^2_{\mathcal{H}_K}$$

HSIC-based indices

- When K is characteristic (injectivity of the mean embedding),

$$\mathrm{HSIC}_K(U_i, Y) = 0 \text{ iif } U_i \perp Y \rightarrow \text{screening}.$$

HSIC-based indices

- When K is characteristic (injectivity of the mean embedding),

$$\mathrm{HSIC}_K(U_i, Y) = 0 \text{ iif } U_i \perp Y \to \text{screening}.$$

- ANOVA-like decomposition (daVeiga 2021) if the inputs are independent and the input kernels are ANOVA:

$$\mathrm{HSIC}(\boldsymbol{U}, Y) = \sum_{A \subseteq \{\mathbf{1},\ldots,d\}} \sum_{B \subseteq A} (-1)^{|A|-|B|} \,\mathrm{HSIC}\,(\boldsymbol{U}_B, Y)$$

$$\left.\begin{aligned} S_i^{\mathrm{HSIC}} &:= \frac{\mathrm{HSIC}(U_i, Y)}{\mathrm{HSIC}(\boldsymbol{U}, Y)} \\ S_{T_i}^{\mathrm{HSIC}} &:= 1 - \frac{\mathrm{HSIC}(\boldsymbol{U}_{-i}, Y)}{\mathrm{HSIC}(\boldsymbol{U}, Y)} \end{aligned}\right\} \to \text{ranking}$$

HSIC-based indices

- When K is characteristic (injectivity of the mean embedding),

$$\mathrm{HSIC}_K(U_i, Y) = 0 \text{ iif } U_i \perp Y \rightarrow \text{screening}.$$

- ANOVA-like decomposition (daVeiga 2021) if the inputs are independent and the input kernels are ANOVA:

$$\mathrm{HSIC}(\boldsymbol{U}, Y) = \sum_{A \subseteq \{1,\ldots,d\}} \sum_{B \subseteq A} (-1)^{|A|-|B|} \mathrm{HSIC}\left(\boldsymbol{U}_B, Y\right)$$

$$\left.\begin{array}{r} S_i^{\mathrm{HSIC}} := \frac{\mathrm{HSIC}(U_i, Y)}{\mathrm{HSIC}(\boldsymbol{U}, Y)} \\ S_{T_i}^{\mathrm{HSIC}} := 1 - \frac{\mathrm{HSIC}(\boldsymbol{U}_{-i}, Y)}{\mathrm{HSIC}(\boldsymbol{U}, Y)} \end{array}\right\} \rightarrow \text{ranking}$$

- Easy to estimate in ANOVA case

$$\mathrm{HSIC}_K(U_i, Y) = \mathbb{E}\left[(k_{\mathcal{U}_i}(U_i, U_i') - 1)k_Y(Y, Y')\right].$$

HSIC-based indices

- When K is characteristic (injectivity of the mean embedding),

$$\mathrm{HSIC}_K(U_i, Y) = 0 \text{ iif } U_i \perp Y \rightarrow \text{screening}.$$

- ANOVA-like decomposition (daVeiga 2021) if the inputs are independent and the input kernels are ANOVA:

$$\mathrm{HSIC}(\boldsymbol{U}, Y) = \sum_{A \subseteq \{1,\dots,d\}} \sum_{B \subseteq A} (-1)^{|A|-|B|} \, \mathrm{HSIC}\left(\boldsymbol{U}_B, Y\right)$$

$$\left.\begin{array}{l} S_i^{\mathrm{HSIC}} := \frac{\mathrm{HSIC}(U_i, Y)}{\mathrm{HSIC}(\boldsymbol{U}, Y)} \\ S_{T_i}^{\mathrm{HSIC}} := 1 - \frac{\mathrm{HSIC}(\boldsymbol{U}_{-i}, Y)}{\mathrm{HSIC}(\boldsymbol{U}, Y)} \end{array}\right\} \rightarrow \text{ranking}$$

- Easy to estimate in ANOVA case

$$\mathrm{HSIC}_K(U_i, Y) = \mathbb{E}\left[(k_{\mathcal{U}_i}(U_i, U_i{}') - 1) k_Y(Y, Y')\right].$$

- Only require (characteristic) kernels on the inputs and on the output (whatever the type of inputs/ouputs you have)

SA on sets: a kernel between sets

With $A \Delta B = A \cup B - B \cap A$ and λ the Lebesgue measure, we define a kernel on the Lebesgue σ-algebra $\mathcal{B}(\mathcal{X})$ by:

$$\forall \Gamma_1, \Gamma_2 \in \mathcal{B}(\mathcal{X}), \;\; k_{set}(\Gamma_1, \Gamma_2) = \exp\left(-\frac{\lambda(\Gamma_1 \Delta \Gamma_2)}{2\sigma^2}\right).$$

SA on sets: a kernel between sets

With $A \Delta B = A \cup B - B \cap A$ and λ the Lebesgue measure, we define a kernel on the Lebesgue σ-algebra $\mathcal{B}(\mathcal{X})$ by:

$$\forall \Gamma_1, \Gamma_2 \in \mathcal{B}(\mathcal{X}), \; k_{set}(\Gamma_1, \Gamma_2) = \exp\left(-\frac{\lambda(\Gamma_1 \Delta \Gamma_2)}{2\sigma^2}\right).$$

Proposition (A kernel between sets)

k_{set} is a kernel [Balança and Herbin 2012] *and is characteristic.*

SA on sets: a kernel between sets

With $A \Delta B = A \cup B - B \cap A$ and λ the Lebesgue measure, we define a kernel on the Lebesgue σ-algebra $\mathcal{B}(\mathcal{X})$ by:

$$\forall \Gamma_1, \Gamma_2 \in \mathcal{B}(\mathcal{X}),\ k_{set}(\Gamma_1, \Gamma_2) = \exp\left(-\frac{\lambda(\Gamma_1 \Delta \Gamma_2)}{2\sigma^2}\right).$$

Proposition (A kernel between sets)

k_{set} is a kernel [Balança and Herbin 2012] *and is characteristic.*

For a given random excursion set $\Gamma_U = \{x \in \mathcal{X}, g(x, U) \leqslant 0\}$, we can define HSIC-based indices on sets:

$$S_i^{\mathrm{H_{set}}} := \frac{\mathrm{HSIC}_{k_{set}}(U_i, \Gamma_U)}{\mathrm{HSIC}_{k_{set}}(\boldsymbol{U}, \Gamma_U)},$$

which quantifies how much U_i impacts the excursion set Γ.

k_{set} is characteristic, sketch of proof

- $\mathcal{B}(\mathcal{X}) \to \mathcal{B} = \mathcal{B}(\mathcal{X})/\sim_\delta$ where δ is the volume of the symmetric difference and $\sim_\delta$ the equivalent relation $A \sim_\delta B$ iif $\delta(A, B) = 0$ i.e. A and B are equal except on a λ-negligible set.
- We show that $(\mathcal{B}, \delta)$ is a Polish space (separable completely metrizable topological space). $(\mathcal{B}, \delta)$ is a metric space so we just need separability and completeness. Separability holds as "it's a subspace of $L_2(\mathcal{X})$" and we show that it's closed.
- We use a Proposition from Ziegel, Ginsbourger, and Dümbgen 2022,

Proposition (Ziegel, Ginsbourger, and Dümbgen 2022)

Let $\mathcal{B}$ be a Polish space, H a separable Hilbert space, T a measurable and injective mapping from $\mathcal{B}$ to H, and $\varphi \in \Phi_\infty^+$. Then, the kernel k on $\mathcal{B}$ defined by

$$k(\gamma, \gamma') := \varphi\left(\| T(\gamma) - T(\gamma')\|_H^2\right), \quad \gamma, \gamma' \in \mathcal{B}$$

is integrally strictly positive definite with respect to $\mathcal{M}(\mathcal{B})$ (which implies that it is characteristic).

with $H = L_2(\mathcal{X})$, $\varphi = \exp(-\frac{\cdot}{2\sigma^2})$ and T defined by $T(\gamma) := x \mapsto \mathbb{1}_\gamma(x)$ for any $\gamma \in \mathcal{B}$ so that $\| T(\gamma) - T(\gamma')\|_H^2 = \lambda(\gamma \Delta \gamma')$.

Estimation with set-valued outputs

In HSIC-based indices expressions, we need to estimate quantities of the form ;

$$S_i^{H_{set}} = \frac{\mathrm{HSIC}(U_i, \Gamma)}{\mathrm{HSIC}(\boldsymbol{U}, \Gamma)} = \frac{\mathbb{E}\left[(k_{\mathcal{U}_i}(U_i, U_i{}') - 1)k_{set}(\Gamma, \Gamma')\right]}{\mathbb{E}\left[(k_{\mathcal{U}}(\boldsymbol{U}, \boldsymbol{U}') - 1)k_{set}(\Gamma, \Gamma')\right]}$$

Estimation with set-valued outputs

In HSIC-based indices expressions, we need to estimate quantities of the form ;

$$S_i^{\mathrm{H}_{set}} = \frac{\mathrm{HSIC}(U_i, \Gamma)}{\mathrm{HSIC}(\boldsymbol{U}, \Gamma)} = \frac{\mathbb{E}\left[(k_{\mathcal{U}_i}(U_i, U_i{}') - 1) k_{set}(\Gamma, \Gamma')\right]}{\mathbb{E}\left[(k_{\mathcal{U}}(\boldsymbol{U}, \boldsymbol{U}') - 1) k_{set}(\Gamma, \Gamma')\right]}$$

Estimation:

$$\widehat{\mathrm{HSIC}}\,(U_i, \Gamma) = \frac{2}{n(n-1)} \sum_{l<j}^{n} \left(k_{\mathcal{U}_i}\left(U_i^{(l)}, U_i^{(j)}\right) - 1 \right) k_{set}\left(\Gamma^{(l)}, \Gamma^{(j)}\right),$$

Estimation with set-valued outputs

In HSIC-based indices expressions, we need to estimate quantities of the form ;

$$S_i^{\mathrm{H}_{set}} = \frac{\mathrm{HSIC}(U_i, \Gamma)}{\mathrm{HSIC}(\boldsymbol{U}, \Gamma)} = \frac{\mathbb{E}\left[(k_{\mathcal{U}_i}(U_i, U_i') - 1)k_{set}(\Gamma, \Gamma')\right]}{\mathbb{E}\left[(k_{\mathcal{U}}(\boldsymbol{U}, \boldsymbol{U}') - 1)k_{set}(\Gamma, \Gamma')\right]}$$

Estimation:

$$\widehat{\mathrm{HSIC}}\,(U_i, \Gamma) = \frac{2}{n(n-1)} \sum_{l<j}^{n} \left(k_{\mathcal{U}_i}\left(U_i^{(l)}, U_i^{(j)}\right) - 1\right) k_{set}\left(\Gamma^{(l)}, \Gamma^{(j)}\right),$$

With sets, $k_{set}(\Gamma_1, \Gamma_2)$ also require an estimation.

$$\widehat{k_{set}}(\Gamma^{(l)}, \Gamma^{(j)}) = \exp\left(-\frac{vol(\mathcal{X})}{2\sigma^2} \frac{1}{m} \sum_{k=1}^{m} 1_{\Gamma^{(l)} \Delta \Gamma^{(j)}}(\boldsymbol{X}^{(k)})\right).$$

To reduce the computational cost we use the same samples of X.

Asymptotic behaviour of the indices on sets

Then we inject it in the indices estimators. $\widehat{\mathrm{HSIC}}(U_i, \Gamma)$: nested Monte Carlo estimator

Proposition (Quadratic risk)

$$\mathbb{E}\left(\widehat{\mathrm{HSIC}}(\boldsymbol{U}_A, \Gamma) - \mathrm{HSIC}(\boldsymbol{U}_A, \Gamma)\right)^2 = \mathcal{O}\left(\frac{1}{n} + \frac{1}{m}\right)$$

In the case of the classic NMC estimator, corresponding here to not reusing the same samples of X, the rate is $\mathcal{O}\left(\frac{1}{n} + \frac{1}{m^2}\right)$.

Toy function 1

From El-Amri et al. 2021,

$$\forall x, u \in [-5,5]^2 \times [-5,5]^3 \quad g(x_1, x_2, u_1, u_2, u_3) = -x_1^2 + 5x_2 - u_1 + u_2^2 - 1$$

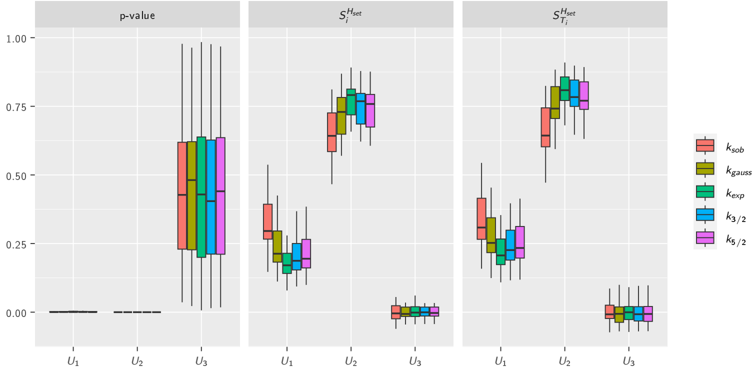


Figure: Estimation of the p-values, $\hat{\hat{S}}_i^{\text{H}set}$ and $\hat{\hat{S}}_{T_i}^{\text{H}set}$ for the excursion set defined by the constraint $g \leqslant 0$ computed for 5 Anova input kernels with $n = 100$, $m = 100$ and repeated 20 times

Convergence Results

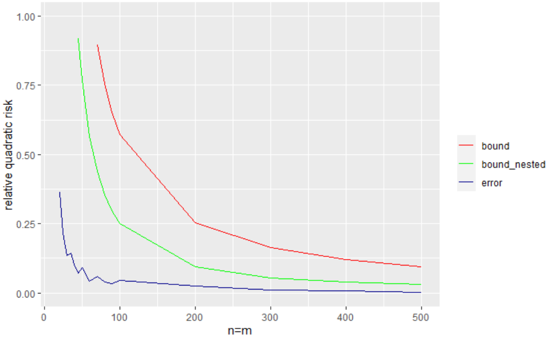
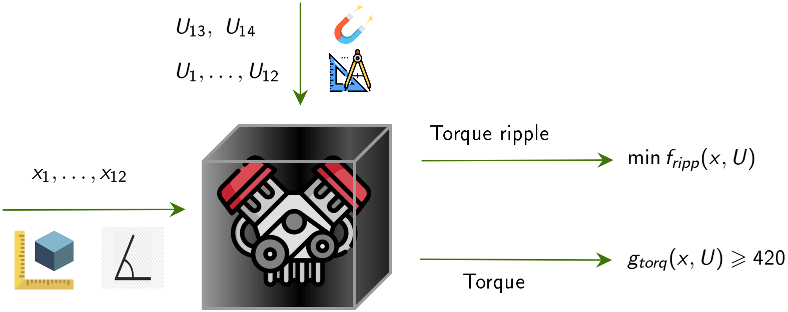


Figure: Evolution of $\mathcal{R}(\widehat{\mathrm{HSIC}}(U_1, \Gamma_g))$ and of the associated upper bounds for the excursion set Γ_g

The "true" value of HSIC, used to compute the quadratic risk, and the constants in the bound are computed for $n = m = 3000$.

Application : excursion sets on the optimisation of the electric machine Meta-models form [Reyes Reyes et al. 2024]



$$x^* = \arg\min_{x \in \mathcal{K}} \mathbb{E}[f_{ripp}(x, U)] \text{ où } \mathcal{K} = \{x \in \mathcal{X}, \; \mathbb{P}[420 - g_{torq}(x, U) \leqslant 0] \geqslant 0.95\}.$$

$$U_1, \ldots, U_{14} \longmapsto \Gamma = \{x \in \mathcal{X}, f_{ripp}(x, U) \leqslant 7, g_{torq}(x, U) \geqslant 420\}$$

What i sthe impact of the input variables U_i on Γ?

Robust conception of an electrical machine

Optimization problem

- f and g are defined on $\mathcal{X} \subset \mathbb{R}^{12}$ and $\mathcal{U} \subset \mathbb{R}^{14}$.
- 12 of the 14 uncertain inputs are manufacturing tolerances on each x,
- U_{13} and U_{14}, describe the magnetic material properties, describe the magnetic material properties and follow uniform distributions on $[-1, 1]$.

Input parameters	**Lower bound** $x_{\min}$	**Upper bound** $x_{\max}$	**Manufacturing Tolerance** U
Slot angle	2.47°	3.27°	±0.1°
β_{L1P1}	27.03°	29.66°	±0.33°
β_{L1P2}	37.03°	39.66°	±0.33°
β_{L2P1}	31.03°	33.66°	±0.33°
β_{L2P2}	47.03°	49.66°	±0.33°
β_{L3P1}	33.7°	37°	±0.33°
β_{L3P2}	59.7°	63°	±0.33°
Airgap	0.55 mm	0.65 mm	±0.03 mm
Bridge_{L1}	2.6 mm	2.98 mm	±0.05 mm
Bridge_{L2}	0.9 mm	1.18 mm	±0.05 mm
Bridge_{L3}	0.5 mm	0.62 mm	±0.03 mm
Bridge_{tang}	0.4 mm	0.6 mm	±0.05 mm

Table: Geometrical variables (see Reyes Reyes et al. (2024) for more details)

Application : sensibility analysis of some excursion sets on the optimisation of the electric machine

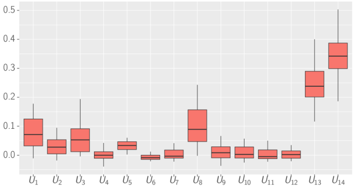


Figure: Estimation des HSIC-ANOVA ($n = m = 100$)

U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8	U_9	U_{10}	U_{11}	U_{12}	U_{13}	U_{14}
0.3	0.6	0.45	0.85	0.65	1	0.9	0.2	0.85	0.8	0.95	0.85	0	0

Table: Acceptance rates (%) over 20 independence tests with a risk of 5% for the excursion

Conclusion

Kernel-based SA on set-valued outputs

- A way to do SA on set-valued outputs
- On excursion sets: An answer to "Which variables is the most influent" => apply on concentration maps of polluant.
- On robust optimization : Use it inside an Bayesian optimization (BO) to reduce the dimension
  - before the BO to reduce the dimension of the meta-model
  - inside the BO to reduce the dimension of acquision function

Thanks for your attention !

El-Amri, Reda et al. (2021). *A sampling criterion for constrained Bayesian optimization with uncertainties*. arXiv: 2103.05706 [stat.ML].

Balança, Paul and Erick Herbin (Jan. 2012). "A set-indexed Ornstein-Uhlenbeck process". In: *Electronic Communications in Probability* 17.none. DOI: 10.1214/ecp.v17-1903. URL: https://doi.org/10.1214%2Fecp.v17-1903.

Borgonovo, E. (2007). "A new uncertainty importance measure". In: *Reliability Engineering and System Safety* 92.6, pp. 771–784. ISSN: 0951-8320. DOI: https://doi.org/10.1016/j.ress.2006.04.015. URL: https://www.sciencedirect.com/science/article/pii/S0951832006000883.

daVeiga, Sébastien (Jan. 2021). "Kernel-based ANOVA decomposition and Shapley effects - Application to global sensitivity analysis". working paper or preprint. URL: https://hal.archives-ouvertes.fr/hal-03108628.

Gamboa, Fabrice, Thierry Klein, and Agnès Lagnoux (2018). "Sensitivity Analysis Based on Cramér–von Mises Distance". In: *SIAM/ASA Journal on Uncertainty Quantification* 6.2, pp. 522–548. DOI: 10.1137/15M1025621. eprint: https://doi.org/10.1137/15M1025621. URL: https://doi.org/10.1137/15M1025621.

Gretton, Arthur, Karsten Borgwardt, et al. (2006). "A Kernel Method for the Two-Sample-Problem". In: *Advances in Neural Information Processing Systems*. Vol. 19. MIT Press. URL: https://proceedings.neurips.cc/paper/2006/hash/e9fb2eda3d9c55a0d89c98d6c54b5b3e-Abstract.html.

Gretton, Arthur, Olivier Bousquet, et al. (2005). “Measuring Statistical Dependence with Hilbert-Schmidt Norms”. In: *Algorithmic Learning Theory*. Ed. by Sanjay Jain, Hans Ulrich Simon, and Etsuji Tomita. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 63–77. ISBN: 978-3-540-31696-1.

Reyes Reyes, Adán et al. (2024). “Study on the impact of uncertain design parameters on the perfomances of a permanent magnet assisted synchronous reluctance motor”. In: *Sci. Tech. Energ. Transition*.

Spagnol, Adrien (July 2020). “Kernel-based sensitivity indices for high-dimensional optimization problems”. Theses. Université de Lyon. URL: https://tel.archives-ouvertes.fr/tel-03173192.

Ziegel, Johanna, David Ginsbourger, and Lutz Dümbgen (2022). *Characteristic kernels on Hilbert spaces, Banach spaces, and on sets of measures*. arXiv: 2206.07588 [stat.ML].