Schémas des relations de double mélange et distribution de multizêtas cyclotomiques
(a joint work with Henrik Bachmann arXiv:2411.18952)

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Cyclotomic multiple zeta values

Let $N \in \mathbb{Z}_{>0}$.

Definition

A *Multiple Polylogarithm Value* at N^{th} roots of unity (short N-MPV) is a complex number given by the series

$$\mathrm{Li}_{(k_1,\dots,k_r)}(z_1,\dots,z_r) := \sum_{m_1>\cdots>m_r>0} \frac{z_1^{m_1}\cdots z_r^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}$$

where $r, k_1, \dots, k_r \in \mathbb{Z}_{>0}$ and $z_1, \dots, z_r$ in μ_N the group of complex N^{th} roots of unity, with $(k_1, z_1) \neq (1, 1)$.

Example

If $N = 1$, we have a *Multiple Zeta Value* (short MZV)

$$\mathrm{Li}_{(k_1,\dots,k_r)}(1,\dots,1) = \sum_{m_1>\cdots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} = \zeta(k_1,\dots,k_r).$$

Cyclotomic multiple zeta values

Proposition

N-MPVs can be expressed in terms of iterated integrals

$$\mathrm{Li}_{(k_1,\ldots,k_r)}(z_1,\ldots,z_r) = \int_0^1 \Omega_0^{k_1-1}\Omega_{z_1}\Omega_0^{i_2-1}\Omega_{z_1z_2}\cdots\Omega_0^{k_r-1}\Omega_{z_1\cdots z_r}$$

with $\Omega_0 = \frac{dt}{t}$ and $\Omega_\zeta = \frac{dt}{\zeta^{-1}-t}$ for $\zeta \in \mu_N$.

where “iterated integrals” means:

$$\begin{cases} \displaystyle\int_a^b \omega = \int_a^b f(s)ds & \text{for } \omega = f(s)ds \\ \displaystyle\int_a^b \omega_1\cdots\omega_n = \int_a^b f_1(s)\left(\int_a^s \omega_2\cdots\omega_n\right)ds & \text{for } \omega_j = f_j(s)ds \end{cases}$$

Double shuffle relations of N-MPVs

Fact

A product of N-MPVs can be expressed as a $\mathbb{Q}$-linear combination of N-MPVs.

Example $(z_1, z_2 \in \mu_N \setminus \{1\})$

$$L_1(z_1)L_1(z_2) = \sum_{m_1,m_2>0} \frac{z_1^{m_1} z_2^{m_2}}{m_1 m_2} = \left(\sum_{m_1>m_2>0} + \sum_{m_2>m_1>0} + \sum_{m_1=m_2>0} \right) \frac{z_1^{m_1} z_2^{m_2}}{m_1 m_2}$$
$$= L_{(1,1)}(z_1, z_2) + L_{(1,1)}(z_2, z_1) + L_2(z_1 z_2)$$

$$L_1(z_1)L_1(z_2) = \int_0^1 \Omega_{z_1} \int_0^1 \Omega_{z_2} = \int_0^1 \Omega_{z_1}\Omega_{z_2} + \int_0^1 \Omega_{z_2}\Omega_{z_1}$$
$$= L_{(1,1)}(z_1, z_1^{-1} z_2) + L_{(1,1)}(z_2, z_1 z_2^{-1})$$

where second equality comes from

$$\int_a^b \omega_1 \cdots \omega_n \int_a^b \omega_{n+1} \cdots \omega_{n+m} = \sum_{\sigma \in \mathfrak{S}_{(n,m)}} \int_a^b \omega_{\sigma^{-1}(1)} \cdots \omega_{\sigma^{-1}(n+m)}$$

Double shuffle relations of N-MPVs

Fact

Via iterated sums and iterated integrals, we obtain algebraic relations between N-MPVs called *Double Shuffle Relations*.

Example $(z_1, z_2 \in \mu_N \setminus \{1\})$

From previous example we obtain

$$\begin{aligned} &L_{(1,1)}(z_1, z_2) + L_{(1,1)}(z_2, z_1) + L_2(z_1 z_2) \\ &= L_1(z_1) L_1(z_2) \\ &= L_{(1,1)}(z_1, z_1^{-1} z_2) + L_{(1,1)}(z_2, z_1 z_2^{-1}). \end{aligned}$$

Therefore, we have a new relation

$$L_{(1,1)}(z_1, z_2) + L_{(1,1)}(z_2, z_1) + L_2(z_1 z_2) = L_{(1,1)}(z_1, z_1^{-1} z_2) + L_{(1,1)}(z_2, z_1 z_2^{-1}).$$

Double shuffle relations of N-MPVs

Fact

By formal vanishing of divergences, we obtain *Regularization Relations* (Hoffmann, Ihara-Kaneko-Zagier ($N = 1$), Arakawa-Kaneko, Zhao ($N \geq 1$)).

Example (roughly)

$$\zeta(2,1) + \zeta(1,2) + \zeta(3) \overset{\text{sum}}{=} \zeta(2)\zeta(1) \overset{\text{int}}{=} \zeta(1,2) + 2\zeta(2,1)$$

Therefore,

$$\zeta(3) = \zeta(2,1).$$

Double shuffle relations of N-MPVs

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Consequence

We obtain a Double Shuffle and Regularization system of relations.

Another formulation of the double shuffle relations

Set $\zeta_N := e^{\frac{\mathrm{i}2\pi}{N}} \in \mu_N$. The series

$$\Phi_{\mathrm{KZ},N} := 1 + \sum (-1)^r \mathrm{Li}_{(k_1,\dots,k_r)}(\zeta_N^{l_2-l_1}, \dots, \zeta_N^{l_r-l_{r-1}}, \zeta_N^{-l_r}) x_0^{k_1-1} x_{\zeta_N^{l_1}} \cdots x_0^{k_r-1} x_{\zeta_N^{l_r}} + \text{(regularized terms)} \in \mathbb{C}\langle\langle x_0, (x_\zeta)_{\zeta\in\mu_N}\rangle\rangle$$

is such that

$$\Delta_{\text{ш}}(\Phi_{\mathrm{KZ},N}) = \Phi_{\mathrm{KZ},N} \otimes \Phi_{\mathrm{KZ},N},$$

where $\Delta_{\text{ш}} : \mathbb{C}\langle\langle x_0, (x_\zeta)_{\zeta\in\mu_N}\rangle\rangle \to \mathbb{C}\langle\langle x_0, (x_\zeta)_{\zeta\in\mu_N}\rangle\rangle^{\otimes 2}$ is the algebra morphism given by

$$x_0 \mapsto x_0 \otimes 1 + 1 \otimes x_0 \text{ and } x_\zeta \mapsto x_\zeta \otimes 1 + 1 \otimes x_\zeta$$

Another formulation of the double shuffle relations

Proposition (Racinet 2002)

The series $\Phi_{\mathrm{KZ},N}$ satisfies double shuffle and regularization relations if and only if

$$\Delta_*(\Phi_{\mathrm{KZ},N,*}) = \Phi_{\mathrm{KZ},N,*} \otimes \Phi_{\mathrm{KZ},N,*}.$$

Here $\Delta_* : \mathbb{C}\langle\langle (x_0^{k-1}x_\zeta)_{(k,\zeta)\in\mathbb{Z}_{>0}\times\mu_N}\rangle\rangle \to \mathbb{C}\langle\langle (x_0^{k-1}x_\zeta)_{(k,\zeta)\in\mathbb{Z}_{>0}\times\mu_N}\rangle\rangle^{\otimes 2}$ is the algebra morphism given by

$$\Delta_* : x_0^{k-1}x_\zeta \mapsto x_0^{k-1}x_\zeta \otimes 1 + 1 \otimes x_0^{k-1}x_\zeta + \sum_{\substack{\eta\in\mu_N \\ l=1}}^{k-1} x_0^{l-1}x_\eta \otimes x_0^{k-l-1}x_{\zeta\eta^{-1}}$$

and

$$\Phi_{\mathrm{KZ},N,*} \text{``}=\text{''} \exp\left(\sum_{n\geq 2}\frac{(-1)^{n-1}}{n}(\Phi_{\mathrm{KZ},N}|x_0^{n-1}x_1)x_1^n\right)\cdot \mathrm{twist}(\Phi_{\mathrm{KZ},N}).$$

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Basic ingredients

- G : finite commutative group of order N.
- $\mathbf{k}$: commutative $\mathbb{Q}$-algebra.
- $X_G := \{x_g \,|\, g \in \{0\} \sqcup G\}$.
- $Y_G := \{y_{k,g} \,|\, (k,g) \in \mathbb{Z}_{>0} \times G\}$.

Define the $\mathbf{k}$-module automorphism $\mathbf{q}_G$ of $\mathbf{k}\langle\langle X_G \rangle\rangle$ by
$\mathbf{q}_G(x_0^{k_1-1} x_{g_1} x_0^{k_2-1} x_{g_2} \cdots x_0^{k_r-1} x_{g_r} x_0^{k_{r+1}-1}) =$

$$x_0^{k_1-1} x_{g_1} x_0^{k_2-1} x_{g_1^{-1} g_2} \cdots x_0^{k_r-1} x_{g_{r-1}^{-1} g_r} x_0^{k_{r+1}-1}$$

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Main ingredients of Racinet's formalism

$$\underbrace{\underset{\Delta_*}{\mathbf{k}\langle\langle Y_G\rangle\rangle}}_{\text{Hopf alg}} \simeq \underset{y_{k,g}\mapsto x_0^{k-1}x_g}{\mathbf{k}\oplus\bigoplus_{g\in G}\mathbf{k}\langle\langle X_G\rangle\rangle x_g} \hookrightarrow \underbrace{\underset{\Delta_{ш}}{\mathbf{k}\langle\langle X_G\rangle\rangle}}_{\text{Hopf alg}} \overset{\pi_Y}{\twoheadrightarrow} \mathbf{k}\langle\langle Y_G\rangle\rangle$$

where

- $\Delta_{ш} : x_g \mapsto x_g \otimes 1 + 1 \otimes x_g$, $(g \in \{0\} \sqcup G)$;
- $\Delta_* : y_{k,g} \mapsto y_{k,g} \otimes 1 + 1 \otimes y_{k,g} + \sum_{\substack{h\in G\\ l=1}}^{k-1} y_{l,h} \otimes y_{k-l,gh^{-1}}$,

  $((k,g) \in \mathbb{Z}_{>0} \times G)$.

The scheme DMR

Definition (Racinet, 2002)

Denote by DMR($\mathbf{k}$) the set of elements $\Phi \in \mathbf{k}\langle\langle X_G \rangle\rangle$ such that $(\Phi|1) = 1$ and $(\Phi|x_0) = (\Phi|x_1) = 0$ that satisfy

1. $\Delta_{\text{ш}}(\Phi) = \Phi \otimes \Phi$;
2. $\Delta_*(\Phi_\star) = \Phi_\star \otimes \Phi_\star$;

where

$$\Phi_* := \exp\left(\sum_{n\geq 2} \frac{(-1)^{n-1}}{n} (\Phi|x_0^{n-1}x_1)x_1^n\right) \cdot \mathbf{q} \circ \pi_Y(\Phi) \in \mathbf{k}\langle\langle Y_G \rangle\rangle.$$

Remark

The the double shuffle and regularization relations on N-MPVs are expressed as the statement $\Phi_{\mathrm{KZ},N} \in$ DMR($\mathbb{C}$).

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Main ingredients of Arakawa-Kaneko's formalism

$$\underbrace{\mathbb{Q}\langle Y_G\rangle \simeq}_{\text{Alg}} \quad \mathbb{Q}\oplus\bigoplus_{g\in G}\mathbb{Q}\langle X_G\rangle x_g \hookrightarrow \underbrace{\mathbb{Q}\langle X_G\rangle}_{\text{Alg}}$$

$$* \qquad y_{k,g}\mapsto x_0^{k-1}x_g \qquad ш$$

where the products $*$ and ш are defined inductively as follows:

$$\begin{cases} 1 \,ш\, w = w \,ш\, 1 = w & \text{for } w \text{ a word in } \mathfrak{H}_G \\ uw_1 \,ш\, vw_2 = u(w_1 \,ш\, vw_2) + v(uw_1 \,ш\, w_2) & \text{for } w_1, w_2 \text{ words in } \mathfrak{H}_G \\ & \text{and } u, v \in X_G, \end{cases}$$

$$\begin{cases} 1 * w = w * 1 = w \\ y_{k_1,g_1}w_1 * y_{k_2,g_2}w_2 = \\ y_{k_1,g_1}(w_1 * y_{k_2,g_2}w_2) + y_{k_2,g_2}(y_{k_1,g_1}w_1 * w_2) + y_{k_1+k_2,\ g_1g_2}(w_1 * w_2), \end{cases}$$

for words w, w_1, w_2 in $\mathfrak{H}_G^1$, $(k_1,g_1),(k_2,g_2)\in\mathbb{Z}_{>0}\times G$.

Main ingredients of Arakawa-Kaneko's formalism

Lemma

Denote by

$$\mathfrak{H}_G^0 := \mathbb{Q} + \sum_{g \in G} x_0 \mathbb{Q}\langle X_G \rangle x_g + \sum_{g \in G \setminus \{1\}} x_g \mathbb{Q}\langle Y_G \rangle.$$

Equipped with the $ш$*-product,* $\mathfrak{H}_G^0$ *is a subalgebra of* $\mathbb{Q}\langle X_G \rangle$*.*

Example (1)

Consider the $\mathbb{Q}$-linear map $Z_{\mathbb{C}} : \mathfrak{H}_{\mu_N}^0 \to \mathbb{C}$ given by $1 \mapsto 1$ and

$$x_0^{k_1-1} x_{z_1} \cdots x_0^{k_r-1} x_{z_r} \longmapsto \mathrm{Li}_{(k_1,\ldots,k_r)}(z_1, \ldots, z_r).$$

Main ingredients of Arakawa-Kaneko's formalism

Example (2)

Then the harmonic product

$$y_{1,z_1} * y_{1,z_2} = y_{1,z_1} y_{1,z_2} + y_{1,z_2} y_{1,z_1} + y_{2,z_1 z_2}$$

corresponds to the identity

$$\mathrm{Li}_1(z_1)\mathrm{Li}_1(z_2) = \mathrm{Li}_{(1,1)}(z_1, z_2) + \mathrm{Li}_{(1,1)}(z_2, z_1) + \mathrm{Li}_2(z_1 z_2).$$

The shuffle product

$$x_{z_1} \sqcup\!\sqcup\, x_{z_2} = x_{z_1} x_{z_2} + x_{z_2} x_{z_1}$$

corresponds to the identity

$$\mathrm{Li}_1(z_1)\mathrm{Li}_1(z_2) = \mathrm{Li}_{(1,1)}(z_1, z_1^{-1} z_2) + \mathrm{Li}_{(1,1)}(z_2, z_1 z_2^{-1}).$$

The scheme EDS

Definition (Espie-Novelli-Racinet 2002 ($N = 1$), Bachmann-Y. 2024 ($N \geq 1$))

We define EDS$(G)(\mathbf{k})$ to be the set of elements $Z_{\mathbf{k}} \in \mathrm{Hom}_{\mathbb{Q}}(\mathfrak{H}^0_G, \mathbf{k})$ such that

1. $Z_{\mathbf{k}} : \mathfrak{H}^0_{G,ш} \to \mathbf{k}$ is an algebra morphism;
2. $\rho^{-1}_{Z_{\mathbf{k}}} \circ Z^{ш}_{\mathbf{k}} \circ \mathrm{q}^{-1}_G : (\mathbb{Q}\langle Y_G \rangle, *) \to \mathbf{k}[T]$ is an algebra morphism;

where

- $Z^{ш}_{\mathbf{k}} : (\mathbb{Q}\langle Y_G \rangle, ш) \to \mathbf{k}[T]$ is the algebra morphism which agrees with $Z_{\mathbf{k}}$ on $\mathfrak{H}^0_G$ and maps x_1 to T
- $\rho_{Z_{\mathbf{k}}}$ the linear automorphism of $\mathbf{k}[T]$ given by

$$\rho_{Z_{\mathbf{k}}}(\exp(Tu)) = \exp\left(\sum_{n \geq 2} \frac{(-1)^n}{n} Z_{\mathbf{k}}(x_0^{n-1} x_1) u^n\right) \exp(Tu).$$

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Comparison between DMR($\mathbf{k}$) and EDS($\mathbf{k}$)

Denote by $\mathcal{L} = X_G$ or Y_G and by $\bullet = ш$ or $*$. Define the pairing

$$\begin{aligned} \mathbf{k}\langle\langle\mathcal{L}\rangle\rangle \otimes \mathbb{Q}\langle\mathcal{L}\rangle &\longrightarrow \mathbf{k} \\ \Phi \otimes w &\longmapsto (\Phi \mid w), \end{aligned}$$

Comparison between DMR($\mathbf{k}$) and EDS($\mathbf{k}$)

Denote by $\mathcal{L} = X_G$ or Y_G and by $\bullet = ш$ or $*$. Define the pairing

$$\begin{aligned}\mathbf{k}\langle\langle\mathcal{L}\rangle\rangle \otimes \mathbb{Q}\langle\mathcal{L}\rangle &\longrightarrow \mathbf{k}\\ \Phi \otimes w &\longmapsto (\Phi \mid w),\end{aligned}$$

Lemma

For any $\Phi \in \mathbf{k}\langle\langle\mathcal{L}\rangle\rangle$ we have

$$\Delta_\bullet(\Phi) = \sum_{u,v\in\mathcal{L}^*} (\Phi \mid u \bullet v) u \otimes v.$$

Comparison between DMR($\mathbf{k}$) and EDS($\mathbf{k}$)

Given a $\mathbb{Q}$-linear map $\varphi : \mathbb{Q}\langle\mathcal{L}\rangle \to \mathbf{k}$, denote by

$$\Phi_\varphi := \sum_{w\in\mathcal{L}^*} \varphi(w)w.$$

Proposition (The following two statements are equivalent)

1. *The map $\varphi : (\mathbb{Q}\langle\mathcal{L}\rangle, \bullet) \to \mathbf{k}$ is an $\mathbb{Q}$-algebra homomorphism.*
2. *We have $\Delta(\Phi_\varphi) = \Phi_\varphi \otimes \Phi_\varphi$.*

Proof.

From previous Lemma

$$\Delta_\bullet(\Phi_\varphi) = \sum_{w,v\in\mathcal{L}^*} \underbrace{(\Phi_\varphi | w \bullet v)}_{=\varphi(w\bullet v)} w \otimes v.$$

By direct calculation we have

$$\Phi_\varphi \otimes \Phi_\varphi = \Big(\sum_{w\in\mathcal{L}^*} \varphi(w)w\Big) \otimes \Big(\sum_{v\in\mathcal{L}^*} \varphi(v)v\Big) = \sum_{w,v\in\mathcal{L}^*} \varphi(w)\varphi(v)\, w \otimes v.$$

□

Comparison between DMR($\mathbf{k}$) and EDS($\mathbf{k}$)

Theorem (Espie-Novelli-Racinet 2002 ($N = 1$), Bachmann-Y. 2024 ($N \geq 1$))

Let $Z_{\mathbf{k}} : \mathfrak{H}^0_G \to \mathbf{k}$ be a $\mathbb{Q}$-linear map. We have

$$Z_{\mathbf{k}} \in \mathsf{EDS}(G)(\mathbf{k}) \Longleftrightarrow \Phi_{\mathrm{ev}_0^{\mathbf{k}} \circ \overline{Z}_{\mathbf{k}}^{ш}} \in \mathsf{DMR}(G)(\mathbf{k}),$$

where

- $\mathrm{ev}_0^{\mathbf{k}} : \mathbf{k}[T] \to \mathbf{k}$ *is the evaluation of polynomial at* 0;
- $\overline{Z}_{\mathbf{k}}^{ш} : (\mathbb{Q}\langle X_G \rangle, ш) \to \mathbf{k}[T]$ *is the algebra morphism which agrees with* $Z_{\mathbf{k}}^{ш}$ *on* $\mathbb{Q}\langle Y_G \rangle$ *and maps* x_0 *to* 0.

Proof.

Uses Proposition of previous slide + the identity

$$(\Phi_{\mathrm{ev}_0^{\mathbf{k}} \circ \overline{Z}_{\mathbf{k}}^{ш}})_* = \Phi_{\mathrm{ev}_0^{\mathbf{k}} \circ \rho_{Z_{\mathbf{k}}}^{-1} \circ Z_{\mathbf{k}}^{ш} \circ \mathbf{q}_G^{-1}}.$$

□

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Distribution relations of N-MPVs

From now let, let $N > 1$. For a N-MPV $\mathrm{Li}_{(k_1,\ldots,k_r)}(z_1,\ldots,z_r)$, define

- Depth : r;
- Weight : $\mathrm{wt}(k_1,\ldots,k_r) = k_1 + \cdots + k_r$.

Lemma (Goncharov, 2001)

Distribution relations among N-MPVs are expressed by

$$\mathrm{Li}_{(k_1,\ldots,k_r)}(z_1,\ldots,z_r) = d^{k_1+\cdots+k_r-r} \sum_{\substack{t_i^d = z_i \\ 1\le i\le r}} \mathrm{Li}_{(k_1,\ldots,k_r)}(t_1,\ldots,t_r),$$

for any divisor d of N and $z_1,\ldots,z_r \in \mu_N^d$.

Racinet's formulation of regularized distribution relations

Recall that G is a group of order N. For any $d \mid N$, define

$$G^d = \{g^d \mid g \in G\}.$$

Definition (Racinet 2002)

We define $\mathsf{DMRD}(G)(\mathbf{k})$ to be the set of $\Phi \in \mathsf{DMR}(G)(\mathbf{k})$ such that for every divisor d of N, we have

$$p^d_*(\Phi) = \exp\Big(\sum_{g^d=1} (\Phi|x_g)x_1\Big) i^*_d(\Phi),$$

where $i^*_d : \mathbf{k}\langle\langle X_G\rangle\rangle \to \mathbf{k}\langle\langle X_{G^d}\rangle\rangle$ and $p^d_* : \mathbf{k}\langle\langle X_G\rangle\rangle \to \mathbf{k}\langle\langle X_{G^d}\rangle\rangle$ are the algebra morphisms given by

$$i^*_d : \begin{array}{l} x_0 \mapsto x_0 \\ x_g \mapsto \begin{cases} x_g & \text{if } g \in G^d \\ 0 & \text{otherwise} \end{cases} \end{array} \quad \text{and} \quad p^d_* : \begin{array}{l} x_0 \mapsto d\,x_0 \\ x_g \mapsto x_{g^d} \end{array}$$

Formulation with EDS setting

Definition

Define the algebra morphisms $i_d^{\sharp} : \mathbf{k}\langle X_{G^d}\rangle \to \mathbf{k}\langle G\rangle$ and $p_{\sharp}^d : \mathbf{k}\langle X_{G^d}\rangle \to \mathbf{k}\langle G\rangle$ given by

$$i_d^{\sharp} : \quad \begin{matrix} x_0 \mapsto x_0 \\ x_h \mapsto x_h \end{matrix} \quad \text{and} \quad p_{\sharp}^d : \quad \begin{matrix} x_0 \mapsto d\, x_0 \\ x_h \mapsto \sum_{g^d = h} x_g \end{matrix}$$

Lemma

We have

1. $(i_d^*(S_2), P_1)_{G^d} = (S_2, i_d^{\sharp}(P_1))_G$, *for* $S_2 \in \mathbf{k}\langle\langle X_G\rangle\rangle$ *and* $P_1 \in \mathbf{k}\langle X_{G^d}\rangle$.
2. $(p_*^d(S_1), P_2)_{G^d} = (S_1, p_{\sharp}^d(P_2))_G$, *for* $S_1 \in \mathbf{k}\langle\langle X_G\rangle\rangle$ *and* $P_2 \in \mathbf{k}\langle X_{G^d}\rangle$.

Formulation with EDS setting

Definition

A $\mathbb{Q}$-linear map $Z_{\mathbf{k}} : \mathfrak{H}^0_G \to \mathbf{k}$ satisfies *distribution relations* if for every divisor d of the order of G, we have (equality of $\mathbb{Q}$-linear maps $\mathfrak{H}^0_{G^d} \to \mathbf{k}$)

$$Z_{\mathbf{k}} \circ p^d_\sharp = Z_{\mathbf{k}} \circ i^\sharp_d.$$

Example

Recall the $\mathbb{Q}$-linear map $Z_{\mathbb{C}} : \mathfrak{H}^0_{\mu_N} \to \mathbb{C}$. For any divisor d of N and any word $x_0^{k_1-1}x_{z_1}\cdots x_0^{k_r-1}x_{z_r}$ of $\mathfrak{H}^0_{\mu_N^d}$, we have

$$Z_{\mathbb{C}} \circ p^d_\sharp(x_0^{k_1-1}x_{z_1}\cdots x_0^{k_r-1}x_{z_r}) = d^{k_1+\cdots+k_r-r} \sum_{\substack{t_i^d = z_i \\ 1\leq i\leq r}} Z_{\mathbb{C}}(x_0^{k_1-1}x_{t_1}\cdots x_0^{k_r-1}x_{t_r}).$$

On the other hand,

$$Z_{\mathbb{C}} \circ i^\sharp_d(x_0^{k_1-1}x_{z_1}\cdots x_0^{k_r-1}x_{z_r}) = Z_{\mathbb{C}}(x_0^{k_1-1}x_{z_1}\cdots x_0^{k_r-1}x_{z_r}).$$

Formulation with EDS setting

Definition

We define $\mathsf{EDSD}(G)(\mathbf{k})$ to be the set of elements $Z_{\mathbf{k}} \in \mathsf{EDS}(G)(\mathbf{k})$ such that for every divisor d of N, we have we have an equality of $\mathbb{Q}$-linear maps $\mathbf{k}\langle X_{G^d}\rangle \to \mathbf{k}$:

$$\overline{Z}^{ш}_{\mathbf{k}} \circ p^d_\sharp = \sigma_{Z_{\mathbf{k}}} \circ \overline{Z}^{ш}_{\mathbf{k}} \circ i^\sharp_d,$$

where $\sigma_{Z_{\mathbf{k}}}$ is the $\mathbf{k}$-module automorphism of $\mathbf{k}[T]$ such that

$$\sigma_{Z_{\mathbf{k}}}(\exp(Tu)) = \exp\Big(\sum_{\substack{g^d=1 \\ g\neq 1}} Z_{\mathbf{k}}(x_g)u\Big)\exp(Tu),$$

Theorem (Bachmann-Y. 2024)

Let $Z_{\mathbf{k}} : \mathfrak{H}^0_G \to \mathbf{k}$ be a $\mathbb{Q}$-linear map. We have

$$Z_{\mathbf{k}} \in \mathsf{EDSD}(G)(\mathbf{k}) \iff \Phi_{\mathrm{ev}^{\mathbf{k}}_0 \circ \overline{Z}^{ш}_{\mathbf{k}}} \in \mathsf{DMRD}(G)(\mathbf{k}).$$

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A conjecture of Zhao

Conjecture (Zhao 2010)

In weight two, all regularized distribution relations are consequences of the extended double shuffle relations, distribution relations of weight one and distribution relations of depth two.

This conjecture is equivalent to:

Theorem (Bachmann-Y. 2024)

Let $Z_{\mathbf{k}} : \mathfrak{H}_G^0 \to \mathbf{k}$ *be a* $\mathbb{Q}$*-linear map and* $d \mid N$ *such that:*

1. $Z_{\mathbf{k}} \in \mathsf{EDS}(G)(\mathbf{k})$;
2. $Z_{\mathbf{k}} \circ p_\sharp^d(x_h) = Z_{\mathbf{k}} \circ i_d^\sharp(x_h)$, *for any* $h \in G^d \smallsetminus \{1\}$;
3. $Z_{\mathbf{k}} \circ p_\sharp^d(x_{h_1} x_{h_2}) = Z_{\mathbf{k}} \circ i_d^\sharp(x_{h_1} x_{h_2})$, *for any* $h_1 \in G^d \smallsetminus \{1\}, h_2 \in G^d$;

where (2) *and* (3) *are equalities in* $\mathbf{k}$. *Then (equality in* $\mathbf{k}[T]$*)*

$$\overline{Z}_{\mathbf{k}}^{ш} \circ p_\sharp^d(x_{h_1} x_{h_2}) = \sigma_{Z_{\mathbf{k}}} \circ \overline{Z}_{\mathbf{k}}^{ш} \circ i_d^\sharp(x_{h_1} x_{h_2}),$$

for any $h_1, h_2 \in G^d \sqcup \{0\}$.

A conjecture of Zhao

Idea of the proof

Depending of the values of h_1 and h_2, the proof is an evaluation at all possible cases displayed in the following table:

h_1 \ h_2	0	1	$G^d \setminus \{1\}$
0	Case 3	**Case 4**	**Case 5**
1	Case 6	Case 2	Case 7
$G^d \setminus \{1\}$	Case 6	Case 1	Case 1

In **Case 4** and **Case 5**, we define

$$\mathrm{DT}_{d,1}(h) := d \sum_{g^d = h} x_0 x_g - x_0 x_h,$$

$$\mathrm{DT}_{d,2}(h_1, h_2) := \sum_{g_1^d = h_1} \sum_{g_2^d = h_2} x_{g_1} x_{g_2} - x_{h_1} x_{h_2},$$

A conjecture of Zhao

Idea of the proof.

$$\mathrm{DS}(g_1, g_2) := x_0 x_{g_1 g_2} + x_{g_1} x_{g_1 g_2} + x_{g_2} x_{g_1 g_2} - x_{g_1} x_{g_2} - x_{g_2} x_{g_1},$$
$$\mathrm{RDS}(g) := x_0 x_g + x_g x_g - x_g x_1.$$

We then have

$$\mathrm{DT}_{d,1}(h) = \begin{cases} \displaystyle\sum_{g_1, g_2 \in K_d \smallsetminus \{1\}} \mathrm{DS}(g_1, g_2) + 2 \sum_{g \in K_d \smallsetminus \{1\}} \mathrm{RDS}(g) & \text{if } h = 1 \\ \displaystyle\sum_{\substack{g_1^d = h \\ g_2 \in K_d \smallsetminus \{1\}}} \mathrm{DS}(g_1, g_2) + \sum_{g^d = h} \mathrm{RDS}(g) - \mathrm{RDS}(h) & \text{otherwise} \\ + \mathrm{DT}_{d,2}(h, 1) - \mathrm{DT}_{d,2}(h, h) & \end{cases}$$

where $K_d = \{g \in G \mid g^d = 1\}$. □

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