Some recent progress on excited state modular Hamiltonians

Debajyoti Sarkar

IIT Indore

Tours University

Based on 2006.13317, 2103.08636 and ongoing work
with Xiaole Jiang, Dan Kabat and Aakash Marthandan

Set-up

We'll deal with aspects of density matrices and (subregion) modular Hamiltonians in field theory.

For factorizable Hilbert spaces $\mathcal{H}_\Sigma = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ $(\Sigma = A \cup \bar{A})$

$$\begin{aligned} \rho_A &= e^{-H_A} = \mathrm{Tr}_{\bar{A}} \left(|\psi\rangle\langle\psi| \right) \\ \rho_{\bar{A}} &= e^{-H_{\bar{A}}} = \mathrm{Tr}_A \left(|\psi\rangle\langle\psi| \right) \end{aligned}$$

Set-up

We'll deal with aspects of density matrices and (subregion) modular Hamiltonians in field theory.

For factorizable Hilbert spaces $\mathcal{H}_\Sigma = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ $(\Sigma = A \cup \bar{A})$

$$\begin{aligned} \rho_A &= e^{-H_A} = \mathrm{Tr}_{\bar{A}}\left(|\psi\rangle\langle\psi|\right) \\ \rho_{\bar{A}} &= e^{-H_{\bar{A}}} = \mathrm{Tr}_{A}\left(|\psi\rangle\langle\psi|\right) \end{aligned}$$

A better defined quantity in field theory is the extended modular Hamiltonian $\overleftrightarrow{H}$

$$\overleftrightarrow{H} = -\log\Delta = -\log\left(\rho_A \otimes \rho_{\bar{A}}^{-1}\right) = H_A - H_{\bar{A}}$$

Set-up

We'll deal with aspects of density matrices and (subregion) modular Hamiltonians in field theory.

For factorizable Hilbert spaces $\mathcal{H}_\Sigma = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ $(\Sigma = A \cup \bar{A})$

$$\begin{aligned} \rho_A &= e^{-H_A} = \mathrm{Tr}_{\bar{A}} \left(|\psi\rangle\langle\psi| \right) \\ \rho_{\bar{A}} &= e^{-H_{\bar{A}}} = \mathrm{Tr}_{A} \left(|\psi\rangle\langle\psi| \right) \end{aligned}$$

A better defined quantity in field theory is the extended modular Hamiltonian $\overset{\leftrightarrow}{H}$

$$\overset{\leftrightarrow}{H} = -\log \Delta = -\log \left(\rho_A \otimes \rho_{\bar{A}}^{-1} \right) = H_A - H_{\bar{A}}$$

- We compute modular Hamiltonian H_A and the associated $\overset{\leftrightarrow}{H}$ for an excited state obtained via perturbing vacuum state by a unitary operator U.

Set-up

$$|\psi\rangle = U|0\rangle = e^{-i\epsilon G}|0\rangle = \exp\left[-i\epsilon \int_{\Sigma} d^{d-1}x\, f(\mathbf{x})\mathcal{O}(\mathbf{x})\right]|0\rangle$$

Here $\mathcal{O}(\mathbf{x})$ is a Hermitian operator, $f(\mathbf{x})$ is a real-valued function, and ϵ is an expansion parameter. Σ is some spacelike/null hypersurface.

Set-up

$$|\psi\rangle = U|0\rangle = e^{-i\epsilon G}|0\rangle = \exp\left[-i\epsilon \int_\Sigma d^{d-1}x\, f(\mathbf{x})\mathcal{O}(\mathbf{x})\right]|0\rangle$$

Here $\mathcal{O}(\mathbf{x})$ is a Hermitian operator, $f(\mathbf{x})$ is a real-valued function, and ϵ is an expansion parameter. Σ is some spacelike/null hypersurface.

- Our framework is a field theory in arbitrary dimensions d and we compute the change in modular Hamiltonian δH_A to first order in perturbations. For most part $A =$ Rindler half-space.

Set-up

$$|\psi\rangle = U|0\rangle = e^{-i\epsilon G}|0\rangle = \exp\left[-i\epsilon \int_\Sigma d^{d-1}x\, f(\mathbf{x})\mathcal{O}(\mathbf{x})\right]|0\rangle$$

Here $\mathcal{O}(\mathbf{x})$ is a Hermitian operator, $f(\mathbf{x})$ is a real-valued function, and ϵ is an expansion parameter. Σ is some spacelike/null hypersurface.

- Our framework is a field theory in arbitrary dimensions d and we compute the change in modular Hamiltonian δH_A to first order in perturbations. For most part $A =$ Rindler half-space.

- The non-factorizability of Hilbert space in QFT shows up in an interesting way in δH_A or $\delta \overleftrightarrow{H}$.

Set-up

$$|\psi\rangle = U|0\rangle = e^{-i\epsilon G}|0\rangle = \exp\left[-i\epsilon \int_{\Sigma} d^{d-1}x\, f(\mathbf{x})\mathcal{O}(\mathbf{x})\right]|0\rangle$$

Here $\mathcal{O}(\mathbf{x})$ is a Hermitian operator, $f(\mathbf{x})$ is a real-valued function, and ϵ is an expansion parameter. Σ is some spacelike/null hypersurface.

- Our framework is a field theory in arbitrary dimensions d and we compute the change in modular Hamiltonian δH_A to first order in perturbations. For most part $A =$ Rindler half-space.
- The non-factorizability of Hilbert space in QFT shows up in an interesting way in δH_A or $\delta \overleftrightarrow{H}$.
- We provide a general form for $\delta \overleftrightarrow{H}$ for perturbations by $J^{(n)}$, which is a local hermitian operator of modular weight n under vacuum modular flow.

$$J^{(n)}(x)\big|_s = e^{i\overleftrightarrow{H}^{(0)}s/2\pi} J^{(n)}(x) e^{-i\overleftrightarrow{H}^{(0)}s/2\pi} \xrightarrow{\text{Rindler}} e^{ns} J^{(n)}(e^s x^+, e^{-s}x^-, \mathbf{x}_\perp)$$

Earlier works

A good amount of literature already studied similar problems

- Modular Hamiltonians in excited states:

  Lashkari 2015, Sarosi-Ugajin 2016-2017, Casini-Teste-Torroba 2017, Lashkari et al. 2018, Arias et al. 2020, Lamprou-de Boer 2020 etc.

Earlier works

A good amount of literature already studied similar problems

- Modular Hamiltonians in excited states:

  Lashkari 2015, Sarosi-Ugajin 2016-2017, Casini-Teste-Torroba 2017, Lashkari et al. 2018, Arias et al. 2020, Lamprou-de Boer 2020 etc.

- Modular Hamiltonians under shape deformations:

  Allais-Mezei 2014, Faulkner et al. 2015-2016, Lewkowycz-Parrikar 2018, Balakrishnan-Parrikar 2020 etc.

Contributions to δH_A

Naively for spatial slice $\Sigma \in A \cup \bar{A}$, implies $G = G_A \otimes \mathbb{1}_{\bar{A}} + \mathbb{1}_A \otimes G_{\bar{A}}$.

Since G_A and $G_{\bar{A}}$ commute, the unitary transformation factors into

$$U = U_A \otimes U_{\bar{A}}$$

Contributions to δH_A

Naively for spatial slice $\Sigma \in A \cup \bar{A}$, implies $G = G_A \otimes \mathbb{1}_{\bar{A}} + \mathbb{1}_A \otimes G_{\bar{A}}$.

Since G_A and $G_{\bar{A}}$ commute, the unitary transformation factors into

$$U = U_A \otimes U_{\bar{A}}$$

The reduced density matrices, modular operators and modular Hamiltonian for the state $|\psi\rangle$ would then behave as

$$\rho_A = U_A \, \rho_A^{(0)} \, U_A^\dagger \,, \qquad \rho_{\bar{A}} = U_{\bar{A}} \, \rho_{\bar{A}}^{(0)} \, U_{\bar{A}}^\dagger$$
$$\Delta = \rho_A \otimes \rho_{\bar{A}}^{-1} = U \Delta^{(0)} \, U^\dagger \,, \qquad \overleftrightarrow{H} = U \overleftrightarrow{H}^{(0)} \, U^\dagger \,.$$

The first-order change in the modular Hamiltonian would then be given by

$$\delta \overleftrightarrow{H}^{\mathrm{commutator}} = -i\epsilon [G, \overleftrightarrow{H}^{(0)}]$$

Only true for factorized systems and doesn't strictly hold for FTs. An explicit 'endpoint' (surface separating A and $\bar{A}$) contribution arises to $\delta\overleftrightarrow{H}$ in various examples .

$$\delta\overleftrightarrow{H} = \delta\overleftrightarrow{H}^{commutator} + \delta\overleftrightarrow{H}^{endpoint}$$

Only true for factorized systems and doesn't strictly hold for FTs. An explicit 'endpoint' (surface separating A and $\bar{A}$) contribution arises to $\delta\overleftrightarrow{H}$ in various examples .

$$\delta\overleftrightarrow{H} = \delta\overleftrightarrow{H}^{commutator} + \delta\overleftrightarrow{H}^{endpoint}$$

- For perturbations via $J^{(n)}$, $\delta\overleftrightarrow{H}^{\mathrm{endpoint}}_{n=2,3,4\dots}$ is a sum of light-ray moments of $J^{(n)}$ and its descendants on the x^+ horizon (similar result for negative n).

$$-2\pi\epsilon\sum_{k=0}^{n-2}\frac{1}{k!}f^{(k)}(0)\sum_{l=0}^{\lfloor\frac{n-2-k}{2}\rfloor}\frac{(-\theta)^l}{l!}(n-k-2l-1)\int_{-\infty}^{\infty}dx^+(x^+)^{k+l}\partial_-^l J^{(n)}(x^+,0,0)$$

- The simplest example is for stress-tensor perturbations in CFT$_2$, for which

$$\delta\overleftrightarrow{H}^{endpoint}_{n=2} = -2\pi\epsilon\, f_0\int_{-\infty}^{\infty}dx^+\; T_{++}(x^+) = \text{ANEC operator.}$$

- The endpoint contribution is absent for modular weights $n = -1, 0, 1$.

A property of the endpoint term

To first order in ϵ, modular Hamiltonian annihilating the state implies

$$\left(\overleftrightarrow{H}^{(0)} + \delta\overleftrightarrow{H}\right)\left(\mathbb{1} - i\epsilon G\right)|0\rangle = 0$$

For $\delta\overleftrightarrow{H} = -i\epsilon[G, \overleftrightarrow{H}^{(0)}] + \delta\overleftrightarrow{H}^{\mathrm{endpoint}}$, we should have $\delta\overleftrightarrow{H}^{\mathrm{endpoint}}|0\rangle = 0$.

This is indeed how the lightray operators and the lightray moments behave.

Kravchuk- Simmons-Duffin 2018, Kologlu et al. 2019

The light-ray moments of the form

$$\mathcal{L}^k[J^{(n)}] = \int_{-\infty}^{\infty} dx^+ (x^+)^k J^{(n)}(x^+, x^-, \mathbf{x}_\perp) \qquad \text{for } k = 0, 1, 2, \ldots$$

annihilates the conformal vacuum both to the left and the right provided $k < \Delta + n - 1$.

$$\mathcal{L}^k[J^{(n)}]|0\rangle = 0 = \langle 0|\mathcal{L}^k[J^{(n)}] \qquad \text{for } k < \Delta + n - 1\,.$$

Same goes for the descendants.

Outline

- CFT vacuum perturbed by stress tensor (4 ways)

- Conformal transformation,
- General operator methods (two ways),
- Path integral method.

These methods can be generalized to obtain higher order $\mathcal{O}(\epsilon^m)$ contributions as well.

- CFT vacuum perturbed by higher modular weight operators (2 ways)

CFT$_2$ vacuum perturbed by stress tensor

Start with

$$\overleftrightarrow{H}^{(0)}_{(u,v)} = 2\pi \int_{-\infty}^{\infty} dz\, \frac{(v-z)(z-u)}{v-u}\, T_{++}(z) + (\text{right-movers})$$

. . . Casini-Huerta-Myers 2011

CFT$_2$ vacuum perturbed by stress tensor

Start with

$$\overleftrightarrow{H}^{(0)}_{(u,v)} = 2\pi \int_{-\infty}^{\infty} dz \, \frac{(v-z)(z-u)}{v-u} \, T_{++}(z) + (\text{right-movers})$$

... Casini-Huerta-Myers 2011

Under the action of a conformal transformation $z \to g(z)$

$$\overleftrightarrow{H}_{(u,v)} = 2\pi \int_{-\infty}^{\infty} dz \, \frac{(g(v)-g(z))(g(z)-g(u))}{g'(z)(g(v)-g(u))} \, T_{++}(z)$$

Das-Ezhuthachan 2018

CFT_2 vacuum perturbed by stress tensor

Start with

$$\overset{\leftrightarrow}{H}^{(0)}_{(u,v)} = 2\pi \int_{-\infty}^{\infty} dz \, \frac{(v-z)(z-u)}{v-u} \, T_{++}(z) + \text{(right-movers)}$$

. . . Casini-Huerta-Myers 2011

Under the action of a conformal transformation $z \to g(z)$

$$\overset{\leftrightarrow}{H}_{(u,v)} = 2\pi \int_{-\infty}^{\infty} dz \, \frac{(g(v)-g(z))(g(z)-g(u))}{g'(z)(g(v)-g(u))} \, T_{++}(z)$$

Das-Ezhuthachan 2018

We will be interested in the linearized perturbation $g(z) = z + \epsilon f(z)$ which is equivalent to perturbation by a stress tensor with the generator

$$G = \int dz \, f(z) T_{++}(z).$$

Resulting change in the modular Hamiltonian

$$
\begin{aligned}
\delta\overleftrightarrow{H}_{(u,v)} &= \overleftrightarrow{H}_{(u,v)} - \overleftrightarrow{H}^{(0)}_{(u,v)} \\
&= 2\pi\epsilon \int_{-\infty}^{\infty} dz\, T_{++}(z) \Bigg[-f'(z)\frac{(v-z)(z-u)}{v-u} + f(z)\frac{u+v-2z}{v-u} \\
&\qquad + f(v)\left(\frac{z-u}{v-u}\right)^2 - f(u)\left(\frac{v-z}{v-u}\right)^2 \Bigg]
\end{aligned}
$$

Resulting change in the modular Hamiltonian

$$\begin{aligned}
\delta \overleftrightarrow{H}_{(u,v)} &= \overleftrightarrow{H}_{(u,v)} - \overleftrightarrow{H}^{(0)}_{(u,v)} \\
&= 2\pi\epsilon \int_{-\infty}^{\infty} dz\, T_{++}(z) \Bigg[-f'(z) \frac{(v-z)(z-u)}{v-u} + f(z) \frac{u+v-2z}{v-u} \\
&\qquad + f(v) \left(\frac{z-u}{v-u}\right)^2 - f(u) \left(\frac{v-z}{v-u}\right)^2 \Bigg]
\end{aligned}$$

- The first line is commutator contribution $-i\epsilon \left[G, \overleftrightarrow{H}^{(0)} \right]$.

Resulting change in the modular Hamiltonian

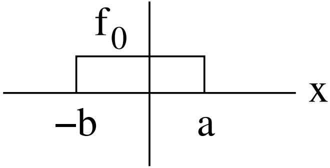
$$\begin{aligned}
\delta \overleftrightarrow{H}_{(u,v)} &= \overleftrightarrow{H}_{(u,v)} - \overleftrightarrow{H}^{(0)}_{(u,v)} \\
&= 2\pi\epsilon \int_{-\infty}^{\infty} dz\, T_{++}(z) \bigg[-f'(z) \frac{(v-z)(z-u)}{v-u} + f(z) \frac{u+v-2z}{v-u} \\
&\qquad + f(v) \left(\frac{z-u}{v-u} \right)^2 - f(u) \left(\frac{v-z}{v-u} \right)^2 \bigg]
\end{aligned}$$

- The first line is commutator contribution $-i\epsilon \left[G, \overleftrightarrow{H}^{(0)} \right]$.
- The second line is an endpoint contribution. Vanishes for $f(u) = f(v) = 0$.

Endpoint contribution for Rindler half-space

We can zoom in on endpoints by choosing $(a, b > 0)$. For example,

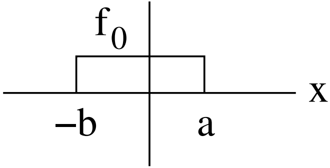
$$f(x) = f_0\theta(x + b) - f_0\theta(x - a)$$



Endpoint contribution for Rindler half-space

We can zoom in on endpoints by choosing $(a, b > 0)$. For example,

$$f(x) = f_0\theta(x+b) - f_0\theta(x-a)$$



The resulting change in Rindler modular Hamiltonian $(u = 0, v \to \infty)$

$$\delta \overset{\leftrightarrow}{H}_{(0,\infty)} = 2\pi\epsilon f_0 \int_{-\infty}^{\infty} dz\, T_{++}(z)\big[b\delta(z+b) + a\delta(z-a) + \theta(z+b) - \theta(z-a) - 1\big]$$

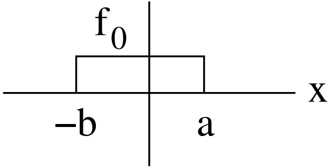
$a, b \to 0^+$ gives

$$\delta \overset{\leftrightarrow}{H}^{endpoint}_{(0,\infty)} \to -2\pi\epsilon f_0 \int_{-\infty}^{\infty} dz\, T_{++}(z).$$

Endpoint contribution for Rindler half-space

We can zoom in on endpoints by choosing $(a, b > 0)$. For example,

$$f(x) = f_0\theta(x+b) - f_0\theta(x-a)$$



The resulting change in Rindler modular Hamiltonian $(u = 0, v \to \infty)$

$$\delta\overleftrightarrow{H}_{(0,\infty)} = 2\pi\epsilon f_0 \int_{-\infty}^{\infty} dz\, T_{++}(z)\big[b\delta(z+b) + a\delta(z-a) + \theta(z+b) - \theta(z-a) - 1\big]$$

$a, b \to 0^+$ gives

$$\delta\overleftrightarrow{H}^{endpoint}_{(0,\infty)} \to -2\pi\epsilon f_0 \int_{-\infty}^{\infty} dz\, T_{++}(z).$$

Note that naively taking $a, b \to 0^+$ to begin with, gives vanishing $f(x)$ and G.

Endpoint effects at higher orders

$$\delta^{(n)}\overleftrightarrow{H} = \delta^{(n)}\overleftrightarrow{H}^{\mathrm{com}} + \delta^{(n)}\overleftrightarrow{H}^{\mathrm{ep}} \quad \text{with} \quad \delta^{(n)}\overleftrightarrow{H}^{\mathrm{com}} = \frac{(i\epsilon)^n}{n!}\left[\ldots\left[\left[\left[\overleftrightarrow{H}^{(0)}, G\right], G\right], G\right]\ldots\right]$$

Endpoint effects at higher orders

$$\delta^{(n)}\overleftrightarrow{H} = \delta^{(n)}\overleftrightarrow{H}^{\mathrm{com}} + \delta^{(n)}\overleftrightarrow{H}^{\mathrm{ep}} \quad \text{with} \quad \delta^{(n)}\overleftrightarrow{H}^{\mathrm{com}} = \frac{(i\epsilon)^n}{n!}\left[\ldots\left[\left[\left[\overleftrightarrow{H}^{(0)}, G\right], G\right], G\right]\ldots\right]$$

The endpoint contribution depends on the endpoint effect of the conformal transformation at that order.

$$g(z) = \sum_{n=0}^{\infty} \epsilon^n h_n(z) = \sum_{n=0}^{\infty} g^{(n)}(z)\,.$$ For example, $h_0(z) = z\,,\; h_1(z) = f(z)$ etc.

Endpoint effects at higher orders

$$\delta^{(n)}\overleftrightarrow{H} = \delta^{(n)}\overleftrightarrow{H}^{\rm com} + \delta^{(n)}\overleftrightarrow{H}^{\rm ep} \quad \text{with} \quad \delta^{(n)}\overleftrightarrow{H}^{\rm com} = \frac{(i\epsilon)^n}{n!}\left[\ldots\left[\left[\left[\overleftrightarrow{H}^{(0)}, G\right], G\right], G\right]\ldots\right]$$

The endpoint contribution depends on the endpoint effect of the conformal transformation at that order.

$$g(z) = \sum_{n=0}^{\infty} \epsilon^n h_n(z) = \sum_{n=0}^{\infty} g^{(n)}(z)\,.\ \text{For example, } h_0(z) = z\,,\ \ h_1(z) = f(z) \text{ etc.}$$

$$\begin{aligned}
\delta^{(1)}\overleftrightarrow{H}^{\rm endpoint}_{(0,\infty)} &= [\overleftrightarrow{H}^{(0)}, E^{(1)}] \\
\delta^{(2)}\overleftrightarrow{H}^{\rm endpoint}_{(0,\infty)} &= (i\epsilon)[[\overleftrightarrow{H}^{(0)}, E^{(1)}], G] + [\overleftrightarrow{H}^{(0)}, E^{(2)}] \\
\delta^{(3)}\overleftrightarrow{H}^{\rm endpoint}_{(0,\infty)} &= \frac{(i\epsilon)^2}{2!}[[[\overleftrightarrow{H}^{(0)}, E^{(1)}], G], G] + (i\epsilon)[[\overleftrightarrow{H}^{(0)}, E^{(2)}], G] + [\overleftrightarrow{H}^{(0)}, E^{(3)}] \\
&\ldots
\end{aligned}$$

Endpoint effects at higher orders

$$E^{(1)} = -i\epsilon f(0) \int_{-\infty}^{+\infty} dx^+ \, T_{++}(x^+) \,, \quad E^{(2)} = -\frac{i\epsilon^2}{2!} f(0) f'(0) \int_{-\infty}^{+\infty} dx^+ \, T_{++}(x^+) \,,$$

$$E^{(3)} = -i\frac{\epsilon^3}{3!} \left(f(0)(f'(0))^2 + (f(0))^2 f''(0) \right) \int_{-\infty}^{+\infty} dx^+ \, T_{++}(x^+) \ \ldots$$

$$E^{(n)} = -ig^{(n)}(0) \int_{-\infty}^{+\infty} dx^+ \, T_{++}(x^+) = -ig^{(n)}(0) \, (ANEC) \,,$$

Endpoint effects at higher orders

$$E^{(1)} = -i\epsilon f(0) \int_{-\infty}^{+\infty} dx^+ \, T_{++}(x^+) \,, \quad E^{(2)} = -\frac{i\epsilon^2}{2!} f(0) f'(0) \int_{-\infty}^{+\infty} dx^+ \, T_{++}(x^+) \,,$$

$$E^{(3)} = -i\frac{\epsilon^3}{3!} \left(f(0)(f'(0))^2 + (f(0))^2 f''(0) \right) \int_{-\infty}^{+\infty} dx^+ \, T_{++}(x^+) \quad \ldots$$

$$E^{(n)} = -ig^{(n)}(0) \int_{-\infty}^{+\infty} dx^+ \, T_{++}(x^+) = -ig^{(n)}(0) \, (ANEC) \,,$$

In total, the endpoint effect is also a unitary transformation

$$\sum_n \delta^{(n)} \overleftrightarrow{H}^{\mathrm{endpoint}} = e^{-i\epsilon G} \left(\left[\overleftrightarrow{H}^{(0)}, -i \sum_{n=0}^{\infty} g^{(n)}(0) \, (ANEC) \right] \right) e^{i\epsilon G}$$
$$= -2\pi \, g(0) \, U \, (ANEC) \, U^\dagger \,.$$

Endpoint effects at higher orders

$$E^{(1)} = -i\epsilon f(0) \int_{-\infty}^{+\infty} dx^+ \, T_{++}(x^+)\,, \quad E^{(2)} = -\frac{i\epsilon^2}{2!} f(0) f'(0) \int_{-\infty}^{+\infty} dx^+ \, T_{++}(x^+)\,,$$

$$E^{(3)} = -i\frac{\epsilon^3}{3!} \left(f(0)(f'(0))^2 + (f(0))^2 f''(0) \right) \int_{-\infty}^{+\infty} dx^+ \, T_{++}(x^+) \; \ldots$$

$$E^{(n)} = -ig^{(n)}(0) \int_{-\infty}^{+\infty} dx^+ \, T_{++}(x^+) = -ig^{(n)}(0)\,(ANEC)\,,$$

In total, the endpoint effect is also a unitary transformation

$$\sum_n \delta^{(n)} \overleftrightarrow{H}^{\text{endpoint}} = e^{-i\epsilon G} \left(\left[\overleftrightarrow{H}^{(0)}, -i \sum_{n=0}^{\infty} g^{(n)}(0)\,(ANEC) \right] \right) e^{i\epsilon G}$$
$$= -2\pi\, g(0)\, U\,(ANEC)\, U^\dagger\,.$$

$$\overleftrightarrow{H}^{excited} = U \left\{ \overleftrightarrow{H}^{(0)} - 2\pi\, g(0)\,(ANEC) \right\} U^\dagger\,.$$

δH_A for more general perturbations

δH_A for more general perturbations

Start with

$$|0\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\beta E_i/2} |i\rangle_A \otimes |i\rangle_{\bar{A}}$$

$$\rho_A^{(0)} = \mathrm{Tr}_{\bar{A}} |0\rangle\langle 0| = \frac{1}{Z} \sum_i e^{-\beta E_i} |i\rangle_{AA}\langle i|$$

$$\rho_{\bar{A}}^{(0)} = \mathrm{Tr}_{A} |0\rangle\langle 0| = \frac{1}{Z} \sum_i e^{-\beta E_i} |i\rangle_{\bar{A}\bar{A}}\langle i|$$

Similarly for ρ_A etc. for state $|\psi\rangle$.

δH_A for more general perturbations

Start with

$$|0\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\beta E_i/2} \, |i\rangle_A \otimes |i\rangle_{\bar{A}}$$

$$\rho_A^{(0)} = \mathrm{Tr}_{\bar{A}} \, |0\rangle\langle 0| = \frac{1}{Z} \sum_i e^{-\beta E_i} |i\rangle_{A\,A}\langle i|$$

$$\rho_{\bar{A}}^{(0)} = \mathrm{Tr}_{A} \, |0\rangle\langle 0| = \frac{1}{Z} \sum_i e^{-\beta E_i} |i\rangle_{\bar{A}\,\bar{A}}\langle i|$$

Similarly for ρ_A etc. for state $|\psi\rangle$.

We assume $G = G_A \otimes G_{\bar{A}}$ and expand ρ_A to first order in ϵ

$$\begin{aligned} \langle l|\delta\rho_A|m\rangle &= \frac{1}{Z} \sum_i e^{-\beta(E_i+E_m)/2} (-i\epsilon) \langle l|G_A|i\rangle\langle m|G_{\bar{A}}|i\rangle \\ &\quad + \frac{1}{Z} \sum_i e^{-\beta(E_i+E_l)/2} (i\epsilon) \langle i|G_A|m\rangle\langle i|G_{\bar{A}}|l\rangle \end{aligned}$$

δH_A for more general perturbations

Start with

$$|0\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\beta E_i/2} \, |i\rangle_A \otimes |i\rangle_{\bar{A}}$$

$$\rho_A^{(0)} = \mathrm{Tr}_{\bar{A}} \, |0\rangle\langle 0| = \frac{1}{Z} \sum_i e^{-\beta E_i} |i\rangle_{AA}\langle i|$$

$$\rho_{\bar{A}}^{(0)} = \mathrm{Tr}_{A} \, |0\rangle\langle 0| = \frac{1}{Z} \sum_i e^{-\beta E_i} |i\rangle_{\bar{A}\bar{A}}\langle i|$$

Similarly for ρ_A etc. for state $|\psi\rangle$.

We assume $G = G_A \otimes G_{\bar{A}}$ and expand ρ_A to first order in ϵ

$$\begin{aligned} \langle l|\delta\rho_A|m\rangle \quad &= \quad \frac{1}{Z} \sum_i e^{-\beta(E_i+E_m)/2} (-i\epsilon) \langle l|G_A|i\rangle\langle m|G_{\bar{A}}|i\rangle \\ &\quad + \frac{1}{Z} \sum_i e^{-\beta(E_i+E_l)/2} (i\epsilon) \langle i|G_A|m\rangle\langle i|G_{\bar{A}}|l\rangle \end{aligned}$$

Readily generalizable (for first order perturbation) to $G = \sum_i c_i G_{A,i} \otimes G_{\bar{A},i}$

δH_A for more general perturbations (cont'd)

The expression can be brought to a simpler form by defining modular conjugated operators $\tilde{G}_{\bar{A}}$ by their matrix elements

$$ {}_A\langle i|\tilde{G}_{\bar{A}}|j\rangle_A = {}_{\bar{A}}\langle j|G_{\bar{A}}|i\rangle_{\bar{A}} \quad \Rightarrow \quad \tilde{G}_{\bar{A}} = JG_{\bar{A}}J $$

...Witten 2018

giving

$$ \delta\rho_A = -i\epsilon G_A(\rho_A^{(0)})^{1/2}\tilde{G}_{\bar{A}}(\rho_A^{(0)})^{1/2} + i\epsilon(\rho_A^{(0)})^{1/2}\tilde{G}_{\bar{A}}(\rho_A^{(0)})^{1/2}G_A $$

δH_A for more general perturbations (cont'd)

The expression can be brought to a simpler form by defining modular conjugated operators $\widetilde{G}_{\bar{A}}$ by their matrix elements

$$_A\langle i|\widetilde{G}_{\bar{A}}|j\rangle_A = {}_{\bar{A}}\langle j|G_{\bar{A}}|i\rangle_{\bar{A}} \quad \Rightarrow \quad \widetilde{G}_{\bar{A}} = JG_{\bar{A}}J$$

...Witten 2018

giving

$$\delta\rho_A = -i\epsilon G_A(\rho_A^{(0)})^{1/2}\widetilde{G}_{\bar{A}}(\rho_A^{(0)})^{1/2} + i\epsilon(\rho_A^{(0)})^{1/2}\widetilde{G}_{\bar{A}}(\rho_A^{(0)})^{1/2}G_A$$

Generalizable to

$$\delta^{(n)}\rho_A = \frac{(i\epsilon)^n}{n!}\Big[\Big[\cdots\Big[\Big[(\rho_A^{(0)})^{\frac{1}{2}}\tilde{G}_{\bar{A}}^n(\rho_A^{(0)})^{\frac{1}{2}}, G_A\Big], G_A\Big]\cdots\Big].$$

δH_A for more general perturbations (cont'd)

The expression can be brought to a simpler form by defining modular conjugated operators $\widetilde{G}_{\bar{A}}$ by their matrix elements

$$ {}_A\langle i|\widetilde{G}_{\bar{A}}|j\rangle_A = {}_{\bar{A}}\langle j|G_{\bar{A}}|i\rangle_{\bar{A}} \quad \Rightarrow \quad \widetilde{G}_{\bar{A}} = J G_{\bar{A}} J $$

. . . Witten 2018

giving

$$ \delta\rho_A = -i\epsilon G_A (\rho_A^{(0)})^{1/2} \widetilde{G}_{\bar{A}} (\rho_A^{(0)})^{1/2} + i\epsilon (\rho_A^{(0)})^{1/2} \widetilde{G}_{\bar{A}} (\rho_A^{(0)})^{1/2} G_A $$

Generalizable to

$$ \delta^{(n)}\rho_A = \frac{(i\epsilon)^n}{n!} \Big[\Big[\cdots \Big[\Big[(\rho_A^{(0)})^{\frac{1}{2}} \tilde{G}_{\bar{A}}^{n} (\rho_A^{(0)})^{\frac{1}{2}}, G_A \Big], G_A \Big] \cdots \Big]. $$

A nice formula exists converting between $\delta\rho_A$ and δH_A to first order

$$ \begin{aligned} H_A^{(0)} + \delta H_A &= -\log\left(\rho_A^{(0)} + \delta\rho_A\right) \\ &= -\log \rho_A^{(0)} - \frac{1}{2}\int_{-\infty}^{\infty} \frac{ds}{1+\cosh s} (\rho_A^{(0)})^{-\frac{1}{2}-\frac{is}{2\pi}} \, \delta\rho_A \, (\rho_A^{(0)})^{-\frac{1}{2}+\frac{is}{2\pi}}. \end{aligned} $$

Sarosi-Ugajin 2017

δH_A for more general perturbations (cont'd)

In our case,

$$\delta H_A = \frac{i\epsilon}{2}\int_{-\infty}^{\infty}\frac{ds}{1+\cosh s}$$
$$\left((\rho_A^{(0)})^{-\frac{1}{2}-\frac{is}{2\pi}}G_A(\rho_A^{(0)})^{1/2}\widetilde{G}_{\bar{A}}(\rho_A^{(0)})^{\frac{is}{2\pi}} - (\rho_A^{(0)})^{-\frac{is}{2\pi}}\widetilde{G}_{\bar{A}}(\rho_A^{(0)})^{1/2}G_A(\rho_A^{(0)})^{-\frac{1}{2}+\frac{is}{2\pi}}\right)$$

In terms of modular-flowed operators (the second equality holds for operators that just act on $\mathcal{H}_A$)

$$\mathcal{O}\big|_s = \Delta^{-\frac{is}{2\pi}}\mathcal{O}\Delta^{\frac{is}{2\pi}} = (\rho_A^{(0)})^{-\frac{is}{2\pi}}\mathcal{O}(\rho_A^{(0)})^{\frac{is}{2\pi}}$$

we have

$$\boxed{\delta H_A = \frac{i\epsilon}{2}\int_{-\infty}^{\infty}\frac{ds}{1+\cosh s}\left(G_A\big|_{s-i\pi}\widetilde{G}_{\bar{A}}\big|_s - \widetilde{G}_{\bar{A}}\big|_s G_A\big|_{s+i\pi}\right)}$$

Two special cases

- If $G_A = \mathbb{1}_A$, then

$$\delta H_A = \frac{i\epsilon}{2} \int_{-\infty}^{\infty} \frac{ds}{1 + \cosh s} \left(G_A\big|_{s-i\pi} \widetilde{G}_{\bar{A}}\big|_s - \widetilde{G}_{\bar{A}}\big|_s G_A\big|_{s+i\pi} \right)$$

gives $\delta H_A = 0$.

Two special cases

- If $G_A = \mathbb{1}_A$, then

$$\delta H_A = \frac{i\epsilon}{2} \int_{-\infty}^{\infty} \frac{ds}{1+\cosh s} \left(G_A\big|_{s-i\pi} \widetilde{G}_{\bar{A}}\big|_s - \widetilde{G}_{\bar{A}}\big|_s G_A\big|_{s+i\pi} \right)$$

  gives $\delta H_A = 0$.

- If $G_{\bar{A}} = \mathbb{1}_{\bar{A}}$, then

$$\delta H_A = \frac{i\epsilon}{2} \int_a^{\infty} dx\, f(x) \left[\int_{-\infty}^{\infty} + \int_{\infty+2\pi i}^{-\infty+2\pi i} \right] \frac{ds}{1+\cosh s} \Delta^{-\frac{1}{2}-\frac{is}{2\pi}} J^{(n)}(x) \Delta^{\frac{1}{2}+\frac{is}{2\pi}}$$

  Using

$$\oint \frac{ds}{1+\cosh s} g(s) = -4\pi i\, g'(s=i\pi)\,,$$

  one can show

$$\delta H_A = -i\epsilon \int_a^{\infty} dx\, f(x)\, [J^{(n)}, \overleftrightarrow{H}^{(0)}]$$

Spacelike perturbation by $J^{(n)}$

The best way to approach is to instead write the perturbation as

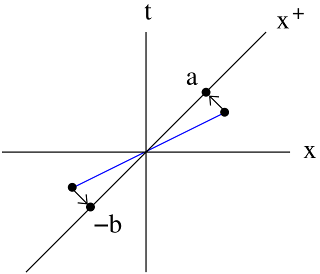
$$G = G_A \otimes \mathbb{1}_{\bar{A}} + \mathbb{1}_A \otimes G_{\bar{A}}$$

with

$$G_A = \int_0^a dx\, f(x)\, J^{(n)}(x^+ = x, x^- = -\theta x, 0)$$

and

$$G_{\bar{A}} = \int_{-b}^0 dx\, f(x)\, J^{(n)}(x^+ = x, x^- = -\theta x, 0)\,.$$



Spacelike perturbation by $J^{(n)}$ (cont'd)

In this case, a careful analysis at early and late modular times ($s \to \pm\infty$) gives the required endpoint effects.

The resulting Sarosi-Ugajin formula looks like ($w = e^{-s}$)

$$\delta H_A = \frac{i\epsilon}{2} \int_{-\infty}^{\infty} \frac{ds}{1 + \cosh s} \left[\left(G_A\big|_{s-i\pi} + \widetilde{G}_{\bar{A}}\big|_s \right) - \left(G_A\big|_{s+i\pi} + \widetilde{G}_{\bar{A}}\big|_s \right) \right]$$
$$= i\epsilon \int_{0+}^{\infty-} \frac{dw}{(w+1)^2} \left[\left(G_A\big|_{s-ir} + \widetilde{G}_{\bar{A}}\big|_s \right) - \left(G_A\big|_{s+ir} + \widetilde{G}_{\bar{A}}\big|_s \right) \right]$$

Spacelike perturbation by $J^{(n)}$ (cont'd)

In this case, a careful analysis at early and late modular times ($s \to \pm\infty$) gives the required endpoint effects.

The resulting Sarosi-Ugajin formula looks like ($w = e^{-s}$)

$$\delta H_A = \frac{i\epsilon}{2}\int_{-\infty}^{\infty} \frac{ds}{1+\cosh s}\left[\left(G_A\big|_{s-i\pi} + \widetilde{G}_{\bar{A}}\big|_s\right) - \left(G_A\big|_{s+i\pi} + \widetilde{G}_{\bar{A}}\big|_s\right)\right]$$

$$= i\epsilon\int_{0+}^{\infty-} \frac{dw}{(w+1)^2}\left[\left(G_A\big|_{s-ir} + \widetilde{G}_{\bar{A}}\big|_s\right) - \left(G_A\big|_{s+ir} + \widetilde{G}_{\bar{A}}\big|_s\right)\right]$$

For concreteness, study δH_A inside a correlation function with $J^{(n)}(y)$. Only in conjunction with $\widetilde{G}_{\bar{A}}$, well-behaved correlator independent of n at $s \to \pm\infty$

$$\left\langle \left(G_A\big|_{s\pm i\pi} + \widetilde{G}_{\bar{A}}\big|_s\right) J^{(n)}(y)\right\rangle \sim \begin{cases} 1/w^{\Delta} & \text{for } w \to \infty \\ w^{\Delta} & \text{for } w \to 0 \end{cases}$$

Spacelike perturbation by $J^{(n)}$ (cont'd)

$$\langle G_A\big|_{s\pm ir} J^{(n)}(y)\rangle = \left(\frac{e^{\pm ir}}{w}\right)^n \int_0^a dx^+ f(x^+)\langle J^{(n)}\big(\frac{e^{\pm ir}x^+}{w}, -\frac{\theta w x^+}{e^{\pm ir}}, 0\big) J^{(n)}(y)\rangle$$

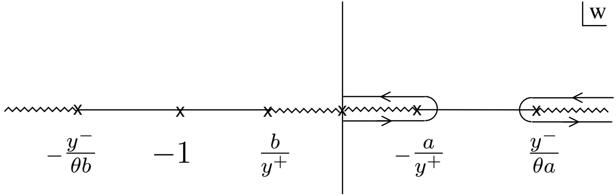
$$\langle \widetilde{G}_{\bar{A}}\big|_s J^{(n)}(y)\rangle = \left(-\frac{1}{w}\right)^n \int_{-b}^0 dx^+ f(x^+)\langle J^{(n)}\big(-\frac{x^+}{w}, \theta w x^+, 0\big) J^{(n)}(y)\rangle$$

Spacelike perturbation by $J^{(n)}$ (cont'd)

$$\langle G_A\big|_{s\pm ir} J^{(n)}(y)\rangle = \left(\frac{e^{\pm ir}}{w}\right)^n \int_0^a dx^+ f(x^+)\langle J^{(n)}\big(\frac{e^{\pm ir}x^+}{w}, -\frac{\theta w x^+}{e^{\pm ir}}, 0\big) J^{(n)}(y)\rangle$$

$$\langle \widetilde{G}_{\bar{A}}\big|_s J^{(n)}(y)\rangle = \left(-\frac{1}{w}\right)^n \int_{-b}^0 dx^+ f(x^+)\langle J^{(n)}\big(-\frac{x^+}{w}, \theta w x^+, 0\big) J^{(n)}(y)\rangle$$

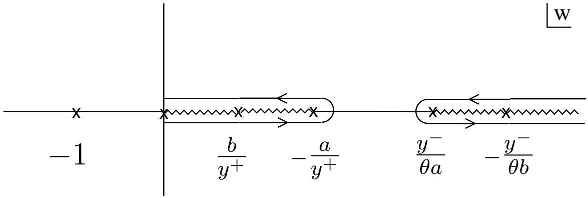
There are also a few branch-cuts which rotate as $r \to \pi$



$$\langle \delta H_A J^{(n)}(y)\rangle = i\epsilon \int_{\supset\subset} \frac{dw}{(w+1)^2}\left(-\frac{1}{w}\right)^n \int_{-b}^a dx^+ f(x^+)\langle J^{(n)}\big(-\frac{x^+}{w}, \theta w x^+, 0\big) J^{(n)}(y)\rangle$$

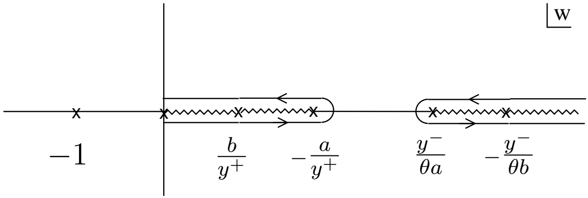
Perturbation either on A or $\bar{A}$

Perturbation acting on A alone: $a > -b > 0 \qquad \Rightarrow \qquad \delta H_A = i\epsilon\big[\overleftrightarrow{H}^{(0)}, G\big]$

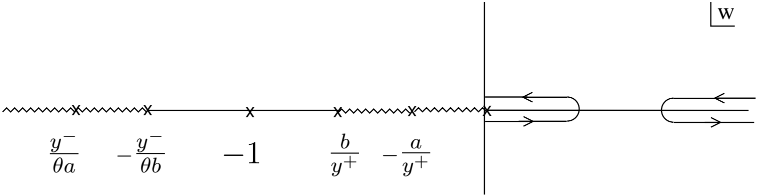


Perturbation either on A or $\bar{A}$

Perturbation acting on A alone: $a > -b > 0 \quad \Rightarrow \quad \delta H_A = i\epsilon[\overleftrightarrow{H}^{(0)}, G]$



Perturbation acting on $\bar{A}$ alone: $-b < a < 0 \quad \Rightarrow \quad \delta H_A = 0$



Perturbation straddles the endpoint $(a, b > 0)$

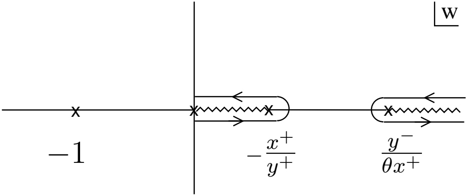
Switching x^+ and w integrals we have $\delta H_A = i\epsilon \int_{-b}^{a} dx^+ f(x^+) I(x^+)$ with

$$I(x^+) = \int_{\supset\subset} \frac{dw}{(w+1)^2} \left(-\frac{1}{w}\right)^n J^{(n)}\left(-\frac{x^+}{w}, \theta w x^+, 0\right).$$

Study $\langle \delta H_A J^{(n)}(y) \rangle$ leading to

$$\langle I(x^+) J^{(n)}(y) \rangle = \int_{\supset\subset} \frac{dw}{(w+1)^2} \left(-\frac{1}{w}\right)^n \frac{1}{\left(-\frac{x^+}{w} - y^+\right)^{\Delta+n} (-\theta w x^+ + y^-)^{\Delta-n}}$$

For $x^+ > 0$ e.g.



Perturbation straddles the endpoint (cont'd)

Large w, $n \geqslant 0$:

$$\langle I(x^+) J^{(n)}(y) \rangle \sim \begin{cases} 1/w^n & \text{if } x^+ = 0 \\ 1/w^\Delta & \text{if } x^+ \neq 0 \end{cases}$$

No problem closing the $\subset$ -shaped contour.

Perturbation straddles the endpoint (cont'd)

Large w, $n \geqslant 0$:

$$\langle I(x^+) J^{(n)}(y) \rangle \sim \begin{cases} 1/w^n & \text{if } x^+ = 0 \\ 1/w^\Delta & \text{if } x^+ \neq 0 \end{cases}$$

No problem closing the $\subset$ -shaped contour.

$w \to 0$, $n \geqslant 0$:

$$\langle I(x^+) J^{(n)}(y) \rangle \sim \begin{cases} 1/w^n & \text{if } x^+ = 0 \\ w^\Delta & \text{if } x^+ \neq 0 \end{cases}$$

As long as x^+ is non-zero, no problem closing the $\supset$ -shaped contour.

Perturbation straddles the endpoint (cont'd)

Large w, $n \geqslant 0$:

$$\langle I(x^+)J^{(n)}(y)\rangle \sim \begin{cases} 1/w^n & \text{if } x^+ = 0 \\ 1/w^\Delta & \text{if } x^+ \neq 0 \end{cases}$$

No problem closing the $\subset$ -shaped contour.

$w \to 0$, $n \geqslant 0$:

$$\langle I(x^+)J^{(n)}(y)\rangle \sim \begin{cases} 1/w^n & \text{if } x^+ = 0 \\ w^\Delta & \text{if } x^+ \neq 0 \end{cases}$$

As long as x^+ is non-zero, no problem closing the $\supset$ -shaped contour.

<u>x^+ large and negative</u>: Contours don't encircle cuts. $\langle I(x^+)J^{(n)}(y)\rangle = 0$

Perturbation straddles the endpoint (cont'd)

Large w, $n \geqslant 0$:

$$\langle I(x^+) J^{(n)}(y) \rangle \sim \begin{cases} 1/w^n & \text{if } x^+ = 0 \\ 1/w^\Delta & \text{if } x^+ \neq 0 \end{cases}$$

No problem closing the $\subset$ -shaped contour.

$w \to 0$, $n \geqslant 0$:

$$\langle I(x^+) J^{(n)}(y) \rangle \sim \begin{cases} 1/w^n & \text{if } x^+ = 0 \\ w^\Delta & \text{if } x^+ \neq 0 \end{cases}$$

As long as x^+ is non-zero, no problem closing the $\supset$ -shaped contour.

<u>x^+ large and negative</u>: Contours don't encircle cuts. $\langle I(x^+) J^{(n)}(y) \rangle = 0$

<u>x^+ large and positive</u>: Contours give commutator contribution.

$$I(x^+) = \theta(x^+) \left[\overleftrightarrow{H}^{(0)}, J^{(n)}(x^+, -\theta x^+, 0) \right]$$

Perturbation straddles the endpoint (cont'd)

Near $x^+ = 0$, the singularity involves $(-1)^k \delta^{(k)}(x^+)$ for $k = 0, \ldots, n-2$. The coefficients are found by computing

$$\int_{-\beta}^{\alpha} dx^+ (x^+)^k \langle I(x^+) J^{(n)}(y) \rangle$$

Perturbation straddles the endpoint (cont'd)

Near $x^+ = 0$, the singularity involves $(-1)^k \delta^{(k)}(x^+)$ for $k = 0, \ldots, n-2$. The coefficients are found by computing

$$\int_{-\beta}^{\alpha} dx^+ (x^+)^k \langle I(x^+) J^{(n)}(y) \rangle$$

$\underline{x^+ \to 0}$: Consider (possible for our relevant moments)

$$\int_{-\beta}^{\alpha} dx^+ (x^+)^k \langle I(x^+) J^{(n)}(y) \rangle \to \left(-\int_{\alpha}^{\infty} - \int_{-\infty}^{-\beta} \right) dx^+ \left((x^+)^k \langle I(x^+) J^{(n)}(y) \rangle \right)$$

take $x^+ \to -wx^+$ and expanding in $\theta w x^+$

$$\int_{-\beta}^{\alpha} dx^+ (x^+)^k \langle I(x^+) J^{(n)}(y) \rangle$$

$$= -\sum_{l=0}^{\infty} \frac{(-\theta)^l}{l!} \int_{\supset} \frac{dw}{(w+1)^2} \left(-\frac{1}{w} \right)^{n-k-2l-1}$$

$$\times \left(\int_{-\alpha/w}^{\infty} + \int_{-\infty}^{\beta/w} \right) dx^+ (x^+)^{k+l} \langle \partial_-^l J^{(n)}(x^+, 0, 0) J^{(n)}(y^+, y^-, 0) \rangle$$

Perturbation straddles the endpoint (cont'd)

We can strip off $J^{(n)}(y)$ and obtain (for $k+2l > n-1$)

$$I(x^+) = \sum_k \frac{(-1)^k}{k!} \delta^{(k)}(x^+)\,\mathcal{E}_k \qquad \text{with } z = 1/w$$

$$\mathcal{E}_k = -\sum_{l=0}^{\infty} \frac{(-\theta)^l}{l!} \oint_{z=0,-1} \frac{dz}{(z+1)^2}(-z)^{n-k-2l-1} \int_0^\infty dx^+ (x^+)^{k+l} \partial_-^l J^{(n)}(x^+,0,0)$$

Perturbation straddles the endpoint (cont'd)

We can strip off $J^{(n)}(y)$ and obtain (for $k + 2l > n - 1$)

$$I(x^+) = \sum_k \frac{(-1)^k}{k!} \delta^{(k)}(x^+)\, \mathcal{E}_k \qquad \text{with } z = 1/w$$

$$\mathcal{E}_k = -\sum_{l=0}^{\infty} \frac{(-\theta)^l}{l!} \oint_{z=0,-1} \frac{dz}{(z+1)^2} (-z)^{n-k-2l-1} \int_0^{\infty} dx^+ (x^+)^{k+l} \partial_-^l J^{(n)}(x^+, 0, 0)$$

Putting everything together, for $n \geqslant 2$ we have

$$I(x^+) = \theta(x^+)\, \mathcal{C} + \sum_{k=0}^{n-2} \frac{(-1)^k}{k!} \delta^{(k)}(x^+)\, \mathcal{E}_k \qquad \text{with}$$

$$\mathcal{C} = \left[\overleftrightarrow{H}^{(0)}, J^{(n)}(x^+, -\theta x^+, 0)\right] \qquad \text{and}$$

$$\mathcal{E}_k = 2\pi i \sum_{l=0}^{\lfloor \frac{n-2-k}{2} \rfloor} \frac{(-\theta)^l}{l!} (n-k-2l-1) \int_0^{\infty} dx^+ (x^+)^{k+l} \partial_-^l J^{(n)}(x^+, 0, 0)$$

Takeaways

For a general perturbative excitation

$$G = \int_{-b}^{a} dx^+ \int d^{d-2}x_\perp\, f(x^+, \mathbf{x}_\perp) J^{(n)}(x^+, -\theta x^+, \mathbf{x}_\perp)$$

algebraic non-bipartition manifested in $\delta \overleftrightarrow{H} = \delta \overleftrightarrow{H}^{commutator} + \delta \overleftrightarrow{H}^{endpoint}$

$$\delta \overleftrightarrow{H}^{\text{endpoint}} = -2\pi\epsilon \int d^{d-2}x_\perp \sum_{k=0}^{n-2} \frac{1}{k!} f^{(k)}(0, \mathbf{x}_\perp) \sum_{l=0}^{\left\lfloor \frac{n-2-k}{2} \right\rfloor} \frac{(-\theta)^l}{l!} (n-k-2l-1) \int_{-\infty}^{\infty} dx^+ (x^+)^{k+l} \partial_-^l J^{(n)}(x^+, 0, \mathbf{x}_\perp) \qquad (\text{for} \quad n \geqslant 2)$$

For $n \leqslant -2$

$$\delta \overleftrightarrow{H}^{\text{endpoint}} = +2\pi\epsilon \int d^{d-2}x_\perp \sum_{k=0}^{|n|-2} \frac{1}{k!} f^{(k)}(0, \mathbf{x}_\perp) \sum_{l=0}^{\left\lfloor \frac{|n|-2-k}{2} \right\rfloor} \frac{(-1)^{k+l}(|n|-k-2l-1)}{l!\, \theta^{k+l+1}} \int_{-\infty}^{\infty} dx^- (x^-)^{k+l} \partial_+^l J^{(n)}(0, x^-, \mathbf{x}_\perp)\,.$$

Final comments

- Discrete systems
- KMS conditions
- Perturbations over a causal diamond?
- Similar contribution in related quantities?
- Algebraic derivation
- Applications in holography

and many more ...

Thank you for your attention

Backup slides

Why $n \geqslant 2$: some intuitive remarks

Endpoint contributions arise when the generator G can move operators from A to $\bar{A}$ or vice versa along the x^+ Rindler horizon. Can be explicitly seen for CFT_2 case.

Why $n \geqslant 2$: some intuitive remarks

Endpoint contributions arise when the generator G can move operators from A to $\bar{A}$ or vice versa along the x^+ Rindler horizon. Can be explicitly seen for CFT_2 case.

Insert $\mathcal{O}(x^+, x^-, \mathbf{x}_\perp)$ at the endpoint $x^+ = 0$ and look for $\partial_+\mathcal{O}$ in

$$J^{(n)}(0)\mathcal{O}(x) \sim \cdots + B^{(n)\,\lambda}\partial_\lambda O(x) + \cdots$$

Why $n \geqslant 2$: some intuitive remarks

Endpoint contributions arise when the generator G can move operators from A to $\bar{A}$ or vice versa along the x^+ Rindler horizon. Can be explicitly seen for CFT_2 case.

Insert $\mathcal{O}(x^+, x^-, \mathbf{x}_\perp)$ at the endpoint $x^+ = 0$ and look for $\partial_+ \mathcal{O}$ in

$$J^{(n)}(0)\mathcal{O}(x) \sim \cdots + B^{(n)\,\lambda} \partial_\lambda O(x) + \cdots$$

On general grounds, for $x^+ = 0$, the modular weight of $B^{(n)\,\lambda}$ is $\geqslant 0$ as it is made up of either metric or x^μ. So, $\partial_+ \mathcal{O}$ can only appear in the OPE with an operator of weight $n \geqslant 1$.

Why $n \geqslant 2$: some intuitive remarks

Endpoint contributions arise when the generator G can move operators from A to $\bar{A}$ or vice versa along the x^+ Rindler horizon. Can be explicitly seen for CFT_2 case.

Insert $\mathcal{O}(x^+, x^-, \mathbf{x}_\perp)$ at the endpoint $x^+ = 0$ and look for $\partial_+\mathcal{O}$ in

$$J^{(n)}(0)\mathcal{O}(x) \sim \cdots + B^{(n)\,\lambda}\partial_\lambda O(x) + \cdots$$

On general grounds, for $x^+ = 0$, the modular weight of $B^{(n)\,\lambda}$ is $\geqslant 0$ as it is made up of either metric or x^μ. So, $\partial_+\mathcal{O}$ can only appear in the OPE with an operator of weight $n \geqslant 1$.

However, in the equal-time commutator $[J^{(n)}(0), \mathcal{O}(x)]$, it can only appear for $n \geqslant 2$.

Effects on entanglement entropy

$$H^{exc}_{\text{mod}}(\eta) = H^{exc}_{\text{mod}}(z) + \frac{A}{4G} - \frac{c\epsilon}{12}\int_u^v dz\, \frac{(v-z)(z-u)}{v-u} f'''(z)$$
$$-\frac{c\epsilon}{24}\int_u^v dz \left(\frac{u-v}{(v-z)(z-u)} f'(z) + \frac{(v-u)(v+u-2z)}{(v-z)^2(z-u)^2} f'(z) - \frac{f(u)}{(z-u)^2 + \frac{f(v)}{(v-z)^2}} \right)$$

Now take f as constant perturbation as $f(z) = f_0\theta(z+b) - f_0\theta(z-a)$;

$$H^{exc}_{\text{mod}}(\eta) = H^{exc}_{\text{mod}}(z) + \frac{A}{4G} - \frac{c\epsilon f_0}{6a^3} - \frac{c\epsilon f_0}{24a} + \frac{c\epsilon f_0}{24}\left(\frac{1}{z}\right)\Big|_0^\infty$$

Finally, for a localized perturbation inside the (u, v) region:

$$H^{exc}_{\text{mod}}(\eta) = H^{exc}_{\text{mod}}(z) + \frac{A}{4G} + \frac{c\,\epsilon\, f_0\,(v-u)}{24}\left(\frac{1}{b(v+u+b)} - \frac{1}{(a-2u)(v+u-a)} \right.$$
$$\left. - \frac{1}{(v-b)(b-u)} + \frac{1}{(v-a)(a-u)} \right)$$