Phase Transitions in Loewner Evolution:

A Mathematical Proof of Concept

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We introduce the Weierstrass Drift as a deterministic toy model for the Schramm–Loewner evolution process (SLE) driven by Brownian motion. More precisely, given $T > 0$, we are interested in the solution $g : [0,T] \times \mathbb{H} \to \mathbb{C}$ of the chordal backward Loewner (partial) differential equation (in short, Loewner equation),

$$\forall\,(t,z) \in [0,T]\times\mathbb{H} : \quad \frac{\partial g}{\partial t} = -\frac{2}{g(t,z) - c\,\mathcal{W}(t)} \quad , \qquad (\mathcal{L}_{\mathcal{W}})$$

where $\mathbb{H} = \{z = x + i\,y\,,\, \mathcal{I}m(z) = y > 0\}$ denotes the upper half-plane, $\mathcal{W}$ the Weierstrass function $x \mapsto \mathcal{W}(x) = \sum\limits_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\,N_b^n\,x\right)$, with $\lambda \in]0,1[$, $N_b \geqslant 2$, $c > 0[$, with the initial condition $g(t_0, z) = z$.

Given $t \in [0,T]$, we define *the hull* K_t as the two-dimensional domain such that

$$K_t = \left\{z \in \overline{\mathbb{H}}\,,\, g(s,z) = \mathcal{W}(s) \quad \text{for some } s \leqslant t\right\}. \qquad (\mathcal{R}\,1)$$

The now classical condition obtained by Joan Lind and Jessica Robbins in [Lin05] ensures that, if the Lip $\left(\frac{1}{2}\right)$ norm of the drift is below 4, the resulting SLE trace is a simple curve.

We hereafter revisit this condition, by relying on our previous works [DL25d], [DL24c], [DL24a], [DL25a], [DL24b], [DL25b], [DL25c], and show that not only we can obtain more general results, namely, for $\delta \in \left]0, \frac{1}{2}\right]$, bounds for the Lip (δ) norm of the Weierstrass function, but, also, for $\delta = \frac{1}{2}$, significantly improve the bounds obtained by J. Lind and J. Robbins in [LR17]. These results allow us to determine precise thresholds for phase transitions between simple and non-simple SLE traces. We also obtain the results associated with the less restrictive general case of a Lip (δ) drift, with $0 < \delta \leqslant \frac{1}{2}$, a configuration all the more interesting, insofar as the Brownian motion is Hölder continuous with associated Hölder exponent $\delta \in \left]0, \frac{1}{2}\right[$. Our general Weierstrass toy model could, therefore, provide a better understanding of the singularities of multifractal SLE traces.

We go much further than what is done in [LR17], since we conduct a comparative study, in the case of Weierstrass-prefractal driven hulls – when the drift is a (piecewise linear) Weierstrass-prefractal function, corresponding to a realistic approximation of the Weierstrass function, which enables us to take into account its roughness – and in the case of Weierstrass-truncated driven hulls – when the drift is a sharp truncation of the Weierstrass function, by nature a smooth function, as is done in [LR17]. By computing the associated Chhabra–Jensen multifractal spectra, we are finally able to show that the polygonal depth of the drift controls the appearance of new scales both in the drift and in the resulting hull, which highlights a fundamental connection between the regularity of the drift and the geometric complexity of the trace: small oscillations of the drift lead

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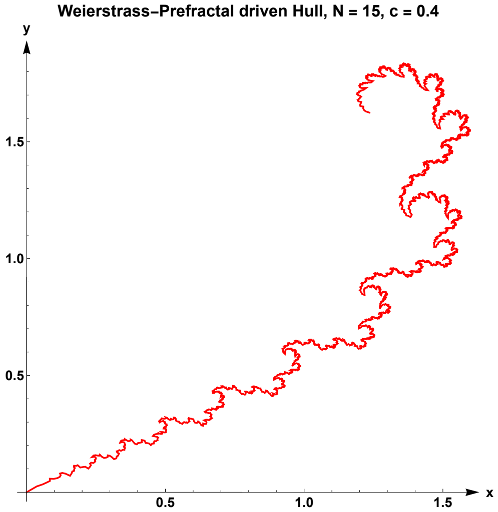
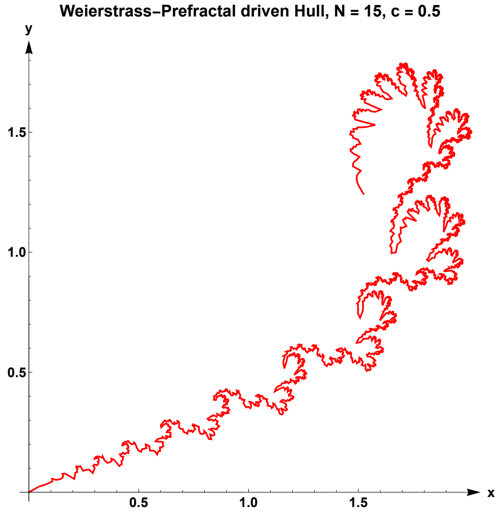
to oscillations of the hull.

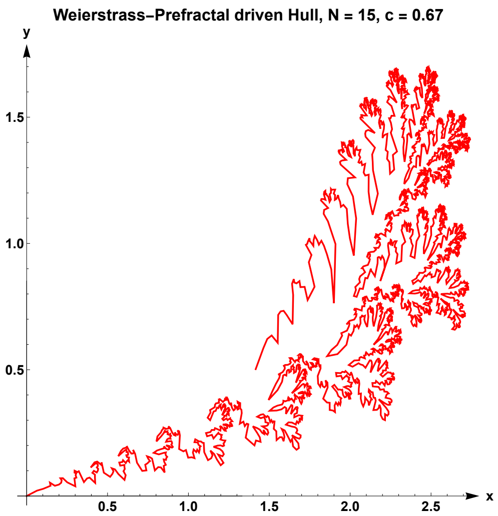
We therefore provide numerical evidence of the fact that the Weierstrass drift mimics key features of Brownian driving terms – such as the roughness, the box-counting dimensions of the resulting hulls, and *quasi self-similarity* (or self-shape similarity; see [DL25d]). This suggests that it can serve as a tractable toy model for the multifractality and phase transition study of SLE traces.



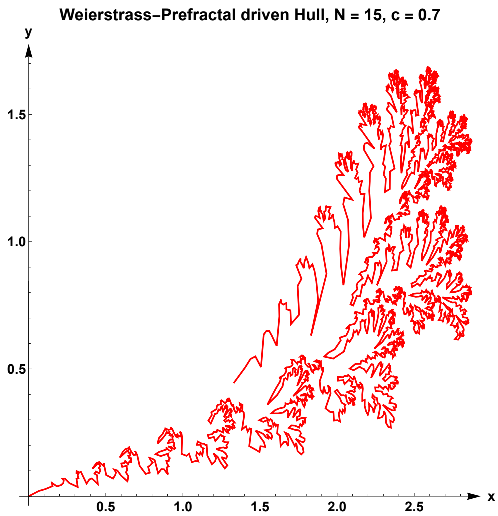
(a) **The Weierstrass prefractal driven hull, for $\lambda = \frac{1}{\sqrt{2}}$, $N_b = 2$, $N = 15$, $c = 0.4$.**



(b) **The Weierstrass prefractal driven hull, for $\lambda = \frac{1}{\sqrt{2}}$, $N_b = 2$, $N = 15$, $c = 0.5$.**



(c) **The Weierstrass prefractal driven hull, for $\lambda = \frac{1}{\sqrt{2}}$, $N_b = 2$, $N = 15$, $c = 0.67$.**



(d) **The Weierstrass prefractal driven hull, for $\lambda = \frac{1}{\sqrt{2}}$, $N_b = 2$, $N = 15$, $c = 0.7$.**

Figure 1: **The phase transition, in the case of Weierstrass-prefractal driven hulls, when $\lambda = \dfrac{1}{\sqrt{2}}$, $N_b = 2$, $N = 15$.**

References

[DL24a] Claire David and Michel L. Lapidus. Fractal complex dimensions and cohomology of the Weierstrass curve. In Patricia Alonso Ruiz, Michael Hinz, Kasso A. Okoudjou, Luke G. Rogers, and Alexander Teplyaev, editors, *From Classical Analysis to Analysis on Fractals: A Tribute to Robert Strichartz*, volume 2 of *Applied and Numerical Harmonic Analysis*. Birkhäuser, Boston, in press, 2024. URL: `https://hal.science/hal-03797595v2`.

[DL24b] Claire David and Michel L. Lapidus. From Weierstrass to Riemann: The Frobenius pass – When fractal flows come into play with zeta functions. In John Friedlander, Carl Pomerance, and Michael Th. Rassias, editors, *Essays in Analytic Number Theory – in Honor of Helmut Maier's 70th Birthday*. Springer Nature, Cham, in press, 2024. URL: `https://hal.sorbonne-universite.fr/hal-04614665`.

[DL24c] Claire David and Michel L. Lapidus. Weierstrass fractal drums - II - Towards a fractal cohomology. *Mathematische Zeitschrift*, 308(2), 2024. URL: `https://hal.science/hal-03758820`.

[DL25a] Claire David and Michel L. Lapidus. Iterated fractal drums ~ Some new perspectives: Polyhedral measures, atomic decompositions and Morse theory. In Hafedh Herichi, Maria Rosaria Lancia, Therese-Basa Landry, Anna Rozanova-Pierrat, and Steffen Winter, editors, *Fractal Geometry in Pure and Applied Mathematics*, volume 828 of *Contemporary Mathematics*, pages 61–130. Amer. Math. Soc., [Providence], RI, 2025. URL: `https://hal.sorbonne-universite.fr/hal-03946104v3`.

[DL25b] Claire David and Michel L. Lapidus. Polyhedral neighborhoods vs tubular neighborhoods: New insights for fractal zeta functions. *The Ramanujan Journal*, 67(73), 2025. URL: `https://rdcu.be/en4MM`.

[DL25c] Claire David and Michel L. Lapidus. Revisiting microlocal analysis: A bridge between fractal cohomology and classical function spaces. *Asymptotic Analysis, in press*, 2025.

[DL25d] Claire David and Michel L. Lapidus. Weierstrass fractal drums - I - A glimpse of complex dimensions. *Advances in Mathematics*, 481:110545, 2025. URL: `https://hal.sorbonne-universite.fr/hal-03642326`.

[Lin05] Joan R. Lind. A sharp condition for the Loewner equation to generate slits. *Annales Academiæ Scientiarum Fennicæ. Mathematica*, 30(1):143–158, 2005.

[LR17] Joan Lind and Jessica Robins. Loewner deformations driven by the Weierstrass function. *Involve. A Journal of Mathematics*, 10(1):151–164, 2017. URL: `https://msp.org/involve/2017/10-1/involve-v10-n1-p10-s.pdf`.