

Existence and uniqueness of a class of semiclassical gravity solutions and the gravitational quantum stealth effect

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Outline

QFT in curved spacetime and semiclassical gravity

A new class of semiclassical gravity solutions

Remarks on the 'stealth' property and strong cosmic censorship

Outlook

QFT in curved spacetimes

- ▶ In recent years, QFT in curved spacetimes has gained increased attention – [Avenues QFT in CS 2025 @ Tours](#).
- ▶ The task at hand is to generalise well-known QFT in Minkowski spacetime to curved backgrounds.
- ▶ Examples: **Cosmology** (structure formation), **black holes** (Hawking radiation).
- ▶ Plenty of interesting questions, RQI, entropy, entanglement, etc.

QFT in curved spacetimes

- ▶ The language of **algebraic QFT** has proven to be extremely useful – almost necessary – to develop QFT in curved spacetimes.
- ▶ The approach here is to see
 1. **observables** as elements of an abstract (\star or $C\text{-}\star$ or $W\text{-}\star$) **algebra** (with some extra conditions) and
 2. **states** as **linear maps** from the algebra of observables to the complex numbers, which can be thought of as **expectation values** (with some extra conditions).
- ▶ The Hilbert space notion of QFT is then recovered by constructing a representation for observables in Hilbert space (via the GNS construction).
- ▶ Especially useful because in curved spacetime there are **no distinguished vacua** – hence no distinguished representations. (There is no Poincaré covariance!)
- ▶ Rather than give you a precise definition, let us look at a **simple example**.

QFT in curved spacetimes – KG field

- ▶ Let us give the simplest possible example.
- ▶ Klein-Gordon theory in curved spacetime:

Let (M, g_{ab}) be glob. hyp. The Klein-Gordon algebra, \mathcal{A}_{KG} , is the unital \star -algebra generated by “smeared fields” $\Phi(f)$ with $f \in C_0^\infty(M)$ subject to the relations

KG1 $f \mapsto \Phi(f)$ is linear (linearity),

KG2 $\Phi(f)^\star = \Phi(\bar{f})$ (hermiticity),

KG3 $\Phi(Pf) = 0$, where $P := \square - m^2 - \xi R$ (field equation) and

KG4 $[\Phi(f), \Phi(g)] = -iE_P(f, g)\mathbb{1}$, for $g \in C_0^\infty(M)$ (commutation relations),

where E_P^\pm are the (unique) advanced (-) and retarded (+) Green operators of P and $E_P := E_P^- - E_P^+$ is the causal propagator.

QFT in curved spacetimes: states

Where are the states?!

- ▶ States are linear maps from an algebra of observables to complex numbers.
- ▶ $\omega : \mathcal{A} \rightarrow \mathbb{C}$ is a state if (i) $\omega(\mathbb{1}) = 1$ and (ii) $\omega(A^*A) \geq 0$ for any $A \in \mathcal{A}$.

Where are the Hilbert spaces?!

- ▶ The GNS construction allows one to produce, out of an algebra with algebraic state a tuple $(\mathcal{D}_\omega \subset \mathcal{H}_\omega, \Omega_\omega, \pi_\omega)$ where observables are represented as operators in Hilbert space.
 - ▶ Here, $\pi_\omega(\Phi) = \hat{\Phi}$.
 - ▶ $\omega(\Phi(f_1) \dots \Phi(f_n)) = \langle \Omega_\omega | \hat{\Phi}(f_1) \dots \hat{\Phi}(f_n) \Omega_\omega \rangle$.
- ▶ Despite this huge generality, one can still calculate some stuff: E.g. distinguished vacua and representations, e.g. in spacetimes with symmetries or asymptotic symmetries.

What does QFT in curved spacetime tell us?

QFT in curved spacetime has made amazing predictions. For example:

1. That black holes radiate at infinity at a temperature proportional to their surface gravity, the **Hawking temperature**. E.g., for Schwarzschild $T_H = \kappa/(2\pi) = 1/(8\pi M)$ (Hawking, 1975).
2. That in cosmology (or more generally non-stationary situations) there is spontaneous particle creation (Parker, 1969).
3. That in non-inertial frames, observers can perceive the vacuum as filled with particles. Notably, uniformly linearly accelerated observers feel the vacuum as a thermal state at a temperature proportional to their acceleration, the **Unruh temperature**, $T_U = a/(2\pi)$ (Fulling, 1972), (Davies, 1975), (Unruh, 1976).

Semiclassical gravity: beyond QFT in curved spacetimes

- ▶ Question: How do quantum fields **affect the spacetime geometry**? QG...
- ▶ At least to a good approximation, in some regime, **semiclassical gravity** should provide us with some insights or questions.
- ▶ Consider a **black hole spacetime** with a **radiating quantum field**. **Radiation suggests evaporation**.
- ▶ Black hole information loss...

Semiclassical gravity

- ▶ Let us unfreeze the gravitational degrees of freedom!
- ▶ Consider the semiclassical gravity equations with a Klein-Gordon field,

$$G_{ab} + \Lambda^b g_{ab} = 8\pi G_N^b \omega(T_{ab}), \quad (1a)$$

$$(\square - m^2 - \xi R)\Phi = (g^{ab}\nabla_a\nabla_b - m^2 - \xi R)\Phi = 0. \quad (1b)$$

- ▶ g_{ab} : spacetime metric (with inverse g^{ab}).
- ▶ Φ : Klein-Gordon field with mass $m^2 \geq 0$ and curvature coupling $\xi \in \mathbb{R}$.
- ▶ $G_{ab} = R_{ab} - (1/2)g_{ab}R$: Einstein tensor, where $R^a{}_{bcd}$: Riemann tensor, $R_{bd} = R^c{}_{bcd}$: Ricci tensor and $R = g^{ab}R_{ab}$: Ricci scalar.
- ▶ $\omega(T_{ab})$: Expectation value of the stress-energy tensor of Φ in the (algebraic) state ω .
- ▶ Λ^b and G_N^b : “bare” cosmological and Newton’s constant, resp.

The stress-energy tensor & the need for renormalisation

- ▶ How can we define $\omega(T_{ab})$?
- ▶ Consider the classical analogue. We could define it for the KG field in a convoluted way as follows:

1. Define the “point-split” differential operator

$$\begin{aligned} \mathcal{T}_{ab} := & (1 - 2\xi)g_b{}^{b'}\nabla_a\nabla_{b'} + \left(2\xi - \frac{1}{2}\right)g_{ab}g^{cd'}\nabla_c\nabla_{d'} - \frac{1}{2}g_{ab}m^2 \\ & + 2\xi\left[-g_a{}^{a'}g_b{}^{b'}\nabla_{a'}\nabla_{b'} + g_{ab}g^{cd}\nabla_c\nabla_d + \frac{1}{2}G_{ab}\right]. \quad (2) \end{aligned}$$

2. Apply \mathcal{T}_{ab} to the product $\phi(x)\phi(x')$ at spacetime points x and x' .
 3. Define $T_{ab}^{\text{class}}(x) = \lim_{x' \rightarrow x} \mathcal{T}_{ab}\phi(x)\phi(x')$. This coincides with the usual definition.
- ▶ One could hope that replacing in the procedure above $\phi(x)\phi(x')$ by $\omega(\Phi(x), \Phi(x'))$ would yield a definition for $\omega(T_{ab})$.
 - ▶ However, typically the limit $x' \rightarrow x$ in $\omega(\Phi(x), \Phi(x'))$ diverges! (Convince yourself in Minkowski spacetime.)

Quantum field theory in curved spacetimes: states

- ▶ The **Wightman function** of the Klein-Gordon field:

$$\omega(\Phi(f_1)\Phi(f_2)) = \int_{M \times M} \text{dvol}(x)\text{dvol}(x') f_1(x)f_2(x') G^+(x, x').$$

$$(\square_x - m^2 - \xi R(x))G^+(x, x') = (\square_{x'} - m^2 - \xi R(x'))G^+(x, x') = 0.$$

- ▶ A state is **Hadamard** if in a convex normal neighbourhood

$$G^+(x, x') = \frac{1}{2(2\pi)^2} \left(\frac{\Delta^{1/2}(x, x')}{\sigma_\epsilon(x, x')} + v(x, x') \ln \left(\frac{\sigma_\epsilon(x, x')}{\ell^2} \right) \right) + w_\ell(x, x')$$

- ▶ Δ is the van Vleck-Morette determinant.
- ▶ $\sigma_\epsilon(x, x') := \sigma(x, x') + i\epsilon(t(x) - t(x')) + \frac{1}{2}\epsilon^2$: regularised half-squared geodesic distance.
- ▶ v and w_ℓ are smooth and symmetric coefficients – v is state independent and w_ℓ depends on the state.
- ▶ $\ell \in \mathbb{R}$ is an arbitrary length scale.

Quantum field theory: states

- ▶ Coefficient v fixed formally by **Hadamard recursion relations**:

$$v(x, x') = \sum_{n=0}^{\infty} v_n(x, x') \sigma^n(x, x'),$$

- ▶ Similar expansion for w , but w_0 is “free” – state-dependent.
- ▶ The first two terms in

$$G^+(x, x') = \frac{1}{2(2\pi)^2} \left(\frac{\Delta^{1/2}(x, x')}{\sigma_\epsilon(x, x')} + v(x, x') \ln \left(\frac{\sigma_\epsilon(x, x')}{\ell^2} \right) \right) + w_\ell(x, x')$$

depend only on the spacetime geometry, and the parameters m^2 and ξ of the field, and coincide with the **Hadamard bi-distribution of the KG operator** $\square - m^2 - \xi R$, H_ℓ , namely

$$H_\ell(x, x') = \frac{1}{2(2\pi)^2} \left(\frac{\Delta^{1/2}(x, x')}{\sigma_\epsilon(x, x')} + v(x, x') \ln \left(\frac{\sigma_\epsilon(x, x')}{\ell^2} \right) \right).$$

- ▶ **Distributional singularities in G^+ are state-independent.**

The renormalised stress-energy tensor

The expectation value of the **renormalised stress-energy tensor**:

$$\omega(T_{ab}(x)) := \lim_{x' \rightarrow x} \mathcal{T}_{ab}(G^+(x, x') - H_\ell(x, x')) + \frac{1}{4\pi^2} g_{ab} [v_1](x) + \Theta_{ab}(x).$$

$$\mathcal{T}_{ab} = (1 - 2\xi) g_b{}^{b'} \nabla_a \nabla_{b'} + \left(2\xi - \frac{1}{2}\right) g_{ab} g^{cd'} \nabla_c \nabla_{d'} - \frac{1}{2} g_{ab} m^2 \\ + 2\xi \left[-g_a{}^{a'} g_b{}^{b'} \nabla_{a'} \nabla_{b'} + g_{ab} g^{cd} \nabla_c \nabla_d + \frac{1}{2} G_{ab} \right],$$

$$[v_1] := \lim_{x' \rightarrow x} v_1(x, x') = \frac{1}{8} m^4 + \frac{1}{4} \left(\xi - \frac{1}{6} \right) m^2 R - \frac{1}{24} \left(\xi - \frac{1}{5} \right) \square R \\ + \frac{1}{8} \left(\xi - \frac{1}{6} \right)^2 R^2 - \frac{1}{720} R_{ab} R^{ab} + \frac{1}{720} R_{abcd} R^{abcd},$$

$$\Theta_{ab} = \alpha_1 g_{ab} + \alpha_2 G_{ab} + \alpha_3 I_{ab} + \alpha_4 J_{ab}, \quad \text{with}$$

$$I_{ab} = 2R_a{}^c{}_{;bc} - \square R_{ab} - \frac{1}{2} g_{ab} \square R + 2R_a{}^c R_{cb} - \frac{1}{2} g_{ab} R^{cd} R_{cd},$$

$$J_{ab} = 2R_{;ab} - 2g_{ab} \square R - \frac{1}{2} g_{ab} R^2 + 2R R_{ab}.$$

Semiclassical gravity

To summarise, semiclassical gravity is

$$G_{ab}(x) + \Lambda g_{ab}(x) = 8\pi G_N \lim_{x' \rightarrow x} \mathcal{T}_{ab}(G^+(x, x') - H_\ell(x, x')) \\ + \frac{2G_N}{\pi} g_{ab}[v_1](x) + \alpha I_{ab}(x) + \beta J_{ab}(x), \quad (3a)$$

$$(\square_x - m^2 - \xi R(x)) G^+(x, x') = (\square_{x'} - m^2 - \xi R(x')) G^+(x, x') = 0. \quad (3b)$$

S : initial value surface,

- ▶ **Metric initial data:** $g_{ab}|_S, \dot{g}_{ab}|_S, \ddot{g}_{ab}|_S, \ddot{\ddot{g}}_{ab}|_S$.
- ▶ **Matter initial data:** $\omega(\varphi\pi')$, $\omega(\varphi\pi')$, $\omega(\pi\varphi')$ and $\omega(\pi\pi')$ on S satisfying CCR. (Further data for one-point function if non-zero.)

Semiclassical gravity: Constraints

Semiclassical gravity contains four constraint equations.

- ▶ If the semiclassical gravity equations are $\mathcal{E}_{ab} = 0$, then
- ▶ **Hamilton**-like constraint: $n^a n^b \mathcal{E}_{ab} = 0$
- ▶ **Gauss**-like constraints: $n^a h^c_b \mathcal{E}_{ac} = 0$
- ▶ Constraints follow from the Gauss-Codazzi relations (as in GR).
- ▶ Constraints in the sense of semiclassical gravity means that they contain up to order 3 time derivatives.
- ▶ All good initial data should satisfy the constraints.

What can we say about semiclassical gravity?

- ▶ The mathematical structure of semiclassical gravity is not very well understood.
- ▶ Advances include:
 - ▶ Solutions in de Sitter space (BAJ-A, 2019), (Gottschalk, Rothe & Siemssen, 2022) and Anti-de Sitter (BAJ-A, Mamani-Leqqe, 2024).
 - ▶ Solutions in static situations (Sanders, 2020), (BAJ-A, 2021).
 - ▶ Cosmological solutions (FLRW symmetry) (Pinamonti & Siemssen, 2013), (Gottschalk & Siemssen, 2018), (Meda, Pinamonti & Siemssen, 2020).
 - ▶ Conformally static spacetimes, (BAJ-A & Modak, 2022).
 - ▶ Some advances in spherical symmetry – not there yet! (Meda, Pinamonti, Roncallo & Zanghì, 2022), (Verch & Janssen, 2023).
 - ▶ Some advances in perturbation theory – not there yet! (BAJ-A, Miramontes & Sudarsky, 2020).
 - ▶ Towards the IVP (BAJ-A, Kay, Miramontes & Sudarsky, 2023), (ibid, 2024).

A new class of semiclassical gravity solutions

- ▶ Let us take a different approach – instead of imposing constraints on spacetime, let us constraint the class of states. We consider the Klein-Gordon equation,

$$(\square_x - m^2 - (1/2)R(x))G^+(x, x') = 0, \quad (4)$$

and similarly in x' and look for some special Hadamard states.

- ▶ We can write the Hadamard property as

$$G^+(x, x') = \frac{1}{2(2\pi)^2} \left(\frac{\Delta^{1/2}(x, x')}{\sigma_\epsilon(x, x')} + v(x, x') \ln \left(\frac{\sigma_\epsilon(x, x')}{\ell^2} \right) \right) + w(x, x').$$

- ▶ We will expand the coefficients v and w as follows

A new class of semiclassical gravity solutions

$$v(x, x') = \sum_{n=0}^{\infty} V_n(x, x') \sigma^n(x, x'), \quad (5)$$

$$w(x, x') = \sum_{n=0}^{\infty} W_n(x, x') \sigma^n(x, x'), \quad (6)$$

and each bi-function V_n and W_n can be written using covariant Taylor series as

$$V_n(x, x') = v_n(x) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} v_{n(k)}(x, x'), \quad \text{with}$$
$$v_{n(k)}(x, x') = v_{na_1 \dots a_k}(x) \sigma^{i a_1}(x, x') \dots \sigma^{i a_k}(x, x'), \quad (7)$$

$$W_n(x, x') = w_n(x) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} w_{n(k)}(x, x'), \quad \text{with}$$
$$w_{n(k)}(x, x') = w_{na_1 \dots a_k}(x) \sigma^{i a_1}(x, x') \dots \sigma^{i a_k}(x, x'). \quad (8)$$

A new class of semiclassical gravity solutions

- ▶ The freedom in the coefficient W is **freedom of choosing W_0** , since the Klein-Gordon equation determines W_{n+1} in terms of W_n (and V) – these are the **Hadamard recursion relations**.
- ▶ Equivalently, **freedom providing covariant Taylor series even coefficients** of W_0 : $w_0, w_{0ab}, w_{0abcd}, \dots$
- ▶ It is possible to write the stress-energy tensor in terms of these coefficients. It takes the form

$$\begin{aligned}\omega(T_{ab}) = & \frac{1}{2(2\pi)^2} \left[-w_{0ab} + \frac{1}{4} w_{0c}{}^c g_{ab} \right. \\ & + \frac{1}{2} (1 - 2\xi) w_{0;ab} + \frac{1}{2} \left(2\xi - \frac{1}{2} \right) \square w_0 g_{ab} \\ & \left. + \left(-\frac{m^2}{4} g_{ab} + \xi \left(R_{ab} - \frac{1}{4} R g_{ab} \right) \right) w_0 \right] \\ & + \frac{1}{4(2\pi)^2} \nu_1 g_{ab} + \alpha_1 g_{ab} + \alpha_2 G_{ab} + \alpha_3 I_{ab} + \alpha_4 J_{ab}. \quad (9)\end{aligned}$$

A new class of semiclassical gravity solutions

- ▶ Can one obtain equations for these coefficients? The classical analogy is that

$$\nabla^a T_{ab}^{\text{class}} = \varphi_{;b}(\square - m^2 - \xi R)\varphi = 0. \quad (10)$$

- ▶ Imposing $\nabla^a \omega(T_{ab}) = 0$ gives a complicated equation – hard to solve, a priori third order.
- ▶ **Idea!** Constraint the class of states by constraining the covariant Taylor series coefficients of W_0 – **make this equations easy to solve.** We choose:

$$w_{0ab} = \frac{1}{2}R_{ab}w_0 - Jg_{ab}, \quad (11)$$

where J is an analytic function.

Conservation of stress-energy tensor imposes

$$(\square - m^2)w_0 = 12J - 2v_1 + 8(2\pi)^2\alpha_0, \quad (12)$$

where α_0 is a real constant.
A new class of semiclassical gravity solutions

A new class of semiclassical gravity solutions

- ▶ It turns out that these states have the property that

$$\omega(T_{ab}) = (\alpha_0 + \alpha_1)g_{ab} + \alpha_2 G_{ab} + \alpha_3 I_{ab} + \alpha_4 J_{ab}. \quad (13)$$

- ▶ The stress-energy tensor – up to ambiguities – gives only a cosmological constant contribution to the eq. of motion.
- ▶ This renders the semiclassical gravity equations solvable. Namely, one has the system

$$\begin{aligned} E_{ab} := & G_{ab} + (\Lambda - \kappa\alpha_0)g_{ab} \\ & - \kappa_1 \left(R_{;ab} - \frac{1}{2}g_{ab}\square R - \square R_{ab} + \frac{1}{2}g_{ab}R^{cd}R_{cd} - 2R^{cd}R_{cdab} \right) \\ & - \kappa_2 \left(2R_{;ab} - 2g_{ab}\square R + \frac{1}{2}g_{ab}R^2 - 2RR_{ab} \right) = 0, \\ (\square - m^2)w_0 = & 12J - 2v_1 + 8(2\pi)^2\alpha_0. \end{aligned}$$

A new class of semiclassical gravity solutions

$$\begin{aligned} E_{ab} := & G_{ab} + (\Lambda - \kappa\alpha_0)g_{ab} \\ & - \kappa_1 \left(R_{;ab} - \frac{1}{2}g_{ab}\square R - \square R_{ab} + \frac{1}{2}g_{ab}R^{cd}R_{cd} - 2R^{cd}R_{cdab} \right) \\ & - \kappa_2 \left(2R_{;ab} - 2g_{ab}\square R + \frac{1}{2}g_{ab}R^2 - 2RR_{ab} \right) = 0, \end{aligned} \quad (15a)$$

$$(\square - m^2)w_0 = 12J - 2v_1 + 8(2\pi)^2\alpha_0. \quad (15b)$$

- ▶ System (15) is indeed a well-posed system. This is higher derivative gravity (Noakes, 1983) + a linear wave equation.
- ▶ Out of w_0 , w_{0ab} (as prescribed) and other $w_{0ab\dots}$, we can in principle reconstruct a two-point function algebraically.
- ▶ Rather than state a precise theorem (Theorem 1 in BAJ-A, 2024) let me explain in words...

The 'stealth' property and strong cosmic censorship

- ▶ **Stealth property:** Clearly, if we set $\alpha_0 = 0$ (our cosmological constant contribution), the stress-energy tensor vanishes (up to renormalisation ambiguities).
- ▶ In classical gravity, non-trivial solutions in curved spacetimes that have a **trivial** T_{ab}^{class} are called 'stealth' configurations.
- ▶ Plenty of work by Zanelli, the Chilean gravitational community, etc. **I thank E Ayón-Beato for explaining me many things about stealths!**
- ▶ Classical stealths have complicated non-trivial potentials in general – in QFT this can be achieved for **linear quantum fields**.

The 'stealth' property and strong cosmic censorship

- ▶ Consider a spacetime like Reissner-Nordström-de Sitter. It possesses an inner horizon, which is a **Cauchy horizon**.
- ▶ We expect Cauchy horizons to be unstable. This is strong-cosmic censorship.
- ▶ There is a quantum version of strong cosmic censorship, see [\(BAJ-A, 2024\)](#).
- ▶ Our class of solutions includes spacetimes with Cauchy horizons and T_{ab} **regular (even analytic!) at the horizon!**
- ▶ Situation analogous to [\(Krasnikov, 1996\)](#), [\(Shushkov, 1995\)](#).
- ▶ KRW theorems [\(Kay, Radzikowski, Wald, 1997\)](#)– having to do with the details of w_0 near the horizon.

Final remarks

- ▶ We found an interesting (somewhat non-generic) class of solutions to semiclassical gravity **without spacetime symmetries**.
- ▶ The 'reduced' system (in terms of w_0) is possibly numerically implementable – opens up interesting explorations.
- ▶ We know there exist states with the properties that we demanded, but in general the positivity condition remain open.
- ▶ High regularity required for now.

Thanks!

More details here:

B. A. Juárez-Aubry, “A new class of semiclassical gravity solutions, gravitational quantum stealths and regular Cauchy horizons”, (2024) [[arXiv:2412.08402](https://arxiv.org/abs/2412.08402) [gr-qc]].

See you soon @ Avenues of Quantum Field Theory in Curved Spacetime 2025!