Topological complexity of configuration spaces

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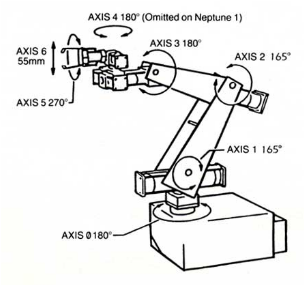
Configuration spaces in Robotics



Any mechanical system is parameterized by a topological space X, the so-called configuration space of the system.

Configurations of the system correspond to points in X.

Motions of the system correspond to paths in X.

Configuration spaces in Topology

For a manifold M and a natural number n, the manifold

$$F(M,n) := \{(x_1,\ldots,x_n) \in M^{\times n} \mid x_i \neq x_j \text{ for } i \neq j\}$$

is the n-th ordered configuration space of M.

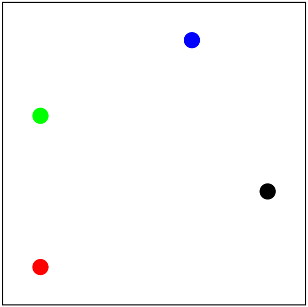
The manifold $F(M,n)$ has a free Σ_n-action by permutation of coordinates. The orbit manifold

$$B(M,n) := F(M,n)/\Sigma_n$$

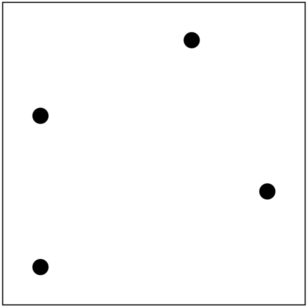
is called the n-th unordered configuration space of M.

Configurations of points in the plane

The case $M = \mathbb{C}$ is of particular importance. Here $F(\mathbb{C}, n)$ and $B(\mathbb{C}, n)$ model configurations of n robots moving in a planar region, avoiding collisions.

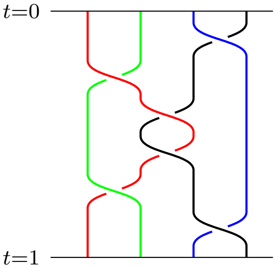


$\mathbf{x} \in F(\mathbb{C}, 4)$

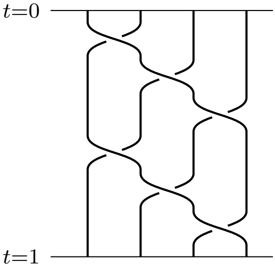


$[\mathbf{x}] \in B(\mathbb{C}, 4)$

Configurations of points in the plane



We have $\pi_1(F(\mathbb{C}, n)) \cong P_n$, the pure braid group.



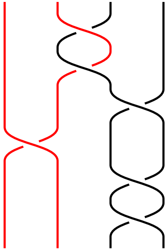
We have $\pi_1(B(\mathbb{C}, n)) \cong B_n$, the (full) braid group.

Configurations of points in the plane

For any subgroup $G \leq \Sigma_n$ we define

$$B_n^G := \pi_1\left(F(\mathbb{C}, n)/G\right).$$

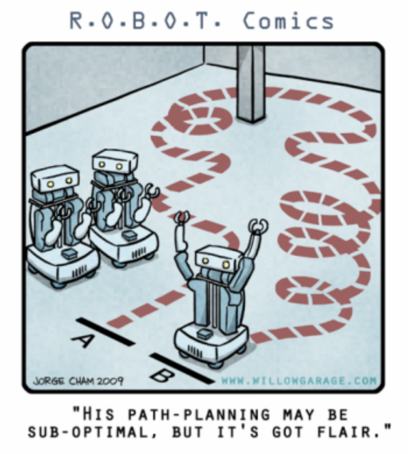
Then $B_n^G \leq B_n$ is the subgroup of G-braids.



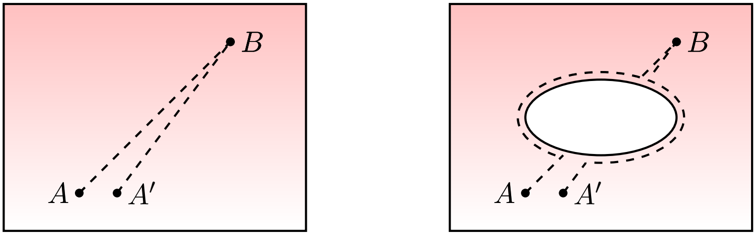
When $G = \Sigma_{n-k} \times \Sigma_k$ we obtain the mixed braid group $B_{n-k,k} \leq B_n$.

The motion planning problem

Find an algorithm which, given states A and B of the system, outputs a motion from A to B.



The topology of the configuration space plays an important role. It dictates whether motion planning algorithms exist which are continuous in the input configurations.



Premise

It is desirable to find motion planning algorithms with fewest domains of continuity, since these will be optimally stable.

In a system with configuration space X, the input of a motion planning algorithm is a point $(A, B) \in X \times X$, and the output is a path $\gamma\colon I = [0, 1] \to X$ with $\gamma(0) = A$ and $\gamma(1) = B$.

More formally, consider the endpoint fibration

$$\pi_X\colon X^I \to X \times X, \qquad \pi_X(\gamma) = \big(\gamma(0), \gamma(1)\big),$$

where X^I denotes the space of all paths in X.

A motion planning algorithm is a section of π_X. That is, it is a function $s\colon X \times X \to X^I$ such that $\pi_X \circ s = \mathrm{Id}_{X \times X}$.

Observation

There exists a continuous section $s\colon X \times X \to X^I$ of π_X if and only if X is contractible.

So motion planning algorithms in X often have essential discontinuities, due to the topology of X.

Topological complexity

Topological complexity is a numerical homotopy invariant which quantifies the complexity of the task of navigation in configuration spaces.

Definition (Farber 2002)

The topological complexity of a space X, denoted $\mathrm{TC}(X)$, is the least integer k such that $X \times X$ admits a cover by open sets $U_0, U_1, \dots, U_k$, on each of which π_X admits a local section (a continuous map $s_i \colon U_i \to X^I$ such that $\pi_X \circ s_i = \mathrm{incl} \colon U_i \subseteq X \times X$). If no such integer exists, we set $\mathrm{TC}(X) = \infty$.

Note that $\mathrm{TC}(X)$ is one less than the number of sets in the cover!

Topological complexity: basic properties

- If $X \simeq Y$ then $\mathrm{TC}(X) = \mathrm{TC}(Y)$ (homotopy invariance).
- $\mathrm{TC}(X) = 0$ if and only if X is contractible.
- $\mathrm{TC}(X \times Y) \leq \mathrm{TC}(X) + \mathrm{TC}(Y)$ (product formula).
- $\mathrm{cat}(X) \leq \mathrm{TC}(X) \leq 2\,\mathrm{cat}(X) \leq 2\dim(X)$,
  where $\mathrm{cat}(X)$ denotes the Lusternik–Schnirelmann category of X (the least integer k such that X admits a cover by open sets $U_0, U_1, \ldots, U_k$, with each inclusion $U_i \hookrightarrow X$ null-homotopic).

Cohomological lower bounds

Lower bounds are given by cohomology, in particular the zero-divisors cup-length.

Let $H^*(-) = H^*(-;\mathbb{k})$ with $\mathbb{k}$ a field. Recall that

$$\cup\colon\ H^*(X)\otimes H^*(X)\to H^*(X)$$

is a ring homomorphism. Its kernel $\ker(\cup)$ is the ideal of zero-divisors.

Theorem (Farber)

For any space X,

$$\mathrm{TC}(X)\geq \text{cup-length}\ \ker(\cup).$$

Example: Spheres

Example (Farber)

The topological complexity of the n-sphere ($n \geq 1$) is given by

$$\mathrm{TC}(S^n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Topological complexity of groups

Problem (Farber)

Given a discrete group π, describe

$$\mathrm{TC}(\pi) := \mathrm{TC}\left(K(\pi, 1)\right)$$

in terms of algebraic invariants of π.

Theorem (Eilenberg–Ganea, Stallings, Swan)

For a discrete group π we have

$$\mathrm{cat}(\pi) := \mathrm{cat}\left(K(\pi, 1)\right) = \mathrm{cd}(\pi),$$

where cd denotes the cohomological dimension.

Hence $\mathrm{cd}(\pi) \leq \mathrm{TC}(\pi) \leq 2\ \mathrm{cd}(\pi)$.

Topological complexity of configuration spaces

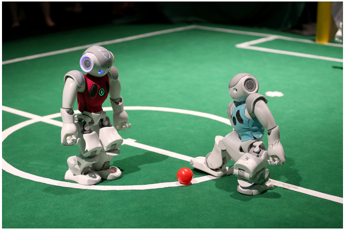
Problem

Compute the topological complexity

$$\mathrm{TC}\left(F(\mathbb{C}, n)/G\right)$$

for various subgroups $G \leq \Sigma_n$.

Relevant to motion planning of n robots partitioned into teams according to their function.



Note that $F(\mathbb{C}, n)$ is a $K(P_n, 1)$. This follows from the tower of Fadell–Neuwirth fibrations

$$\begin{array}{ccccccccc}
\mathbb{C}_{n-1} \longrightarrow & F(\mathbb{C}_{n-2}, 2) & \longrightarrow \cdots \longrightarrow & F(\mathbb{C}_2, n-2) & \longrightarrow & F(\mathbb{C}_1, n-1) & \longrightarrow & F(\mathbb{C}, n) \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& \mathbb{C}_{n-2} & \cdots & \mathbb{C}_2 & & \mathbb{C}_1 & & \mathbb{C}
\end{array}$$

where $\mathbb{C}_m := \mathbb{C} \setminus \{m \text{ points}\} \simeq \vee_m S^1$.

Likewise, $F(\mathbb{C}, n)/G$ is a $K(B_n^G, 1)$.

We can also deduce that $\mathrm{cd}(B_n^G) = n - 1$ for any $G \leq \Sigma_n$. Hence

$$n - 1 \leq \mathrm{TC}(F(\mathbb{C}, n)/G) = \mathrm{TC}(B_n^G) \leq 2n - 2.$$

Theorem (Farber–Yuzvinsky 2004)

The topological complexity of the ordered configuration spaces ($n \geq 2$) is

$$\mathrm{TC}\left(F(\mathbb{C}, n)\right) = 2n - 3.$$

The first two Fadell–Neuwirth fibrations are trivial, so

$$F(\mathbb{C}, n) \simeq \mathbb{C} \times \mathbb{C}_1 \times F(\mathbb{C}_2, n-2) \simeq S^1 \times X^{n-2}.$$

Then the product formula gives the upper bound

$$\mathrm{TC}(F(\mathbb{C}, n)) \leq \mathrm{TC}(S^1) + \mathrm{TC}(X^{n-2}) \leq 1 + 2(n-2) = 2n - 3.$$

The lower bound is obtained by exhibiting a nonzero product of $2n - 3$ zero-divisors in $H^*(F(\mathbb{C}, n); \mathbb{Q})$. □

The Cartan–Leray spectral sequence of the covering

$$F(\mathbb{C},n) \to F(\mathbb{C},n)/G$$

leads to a ring isomorphism

$$H^*(F(\mathbb{C},n)/G;\mathbb{Q}) \cong H^*(F(\mathbb{C},n);\mathbb{Q})^G.$$

This ring of invariants may be difficult to compute, or have few products. For example, when $G = \Sigma_n$ we have

$$H^k(B(\mathbb{C},n);\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & k = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

A lower bound for $\mathrm{TC}(\pi)$

Theorem (G.–Lupton–Oprea 2015)

Let A and B be subgroups of π such that $\gamma A \gamma^{-1} \cap B = \{1\}$ for every $\gamma \in \pi$. Then

$$\mathrm{cd}(A \times B) \leq \mathrm{TC}(\pi).$$

This lower bound does not require knowledge of the cohomology ring structure of π, and can improve on the zero-divisors cup-length lower bound.

For instance, one can use it to show that Higman's acyclic group

$$\mathcal{H} = \langle x, y, z, w \mid xyx^{-1}y^{-2}, yzy^{-1}z^{-2}, zwz^{-1}w^{-2}, wxw^{-1}x^{-2} \rangle$$

has $\mathrm{TC}(\mathcal{H}) = 4$.

Theorem (G.–Recio-Mitter 2016)

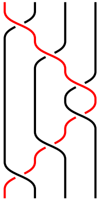
If $G \leq \Sigma_{n-2} \times \Sigma_2$, then

$$\mathrm{TC}(B_n^G) \geq 2n - 3.$$

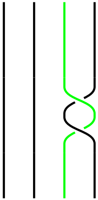
More generally, if $G \leq \Sigma_{n-k} \times \Sigma_k$, then

$$\mathrm{TC}(B_n^G) \geq 2n - k - 1.$$

Proof: We will prove the first statement, by exhibiting subgroups A and B of B_n^G such that $\gamma A \gamma^{-1} \cap B = \{1\}$ for every $\gamma \in B_n^G$, and $\mathrm{cd}(A \times B) = 2n - 3$.

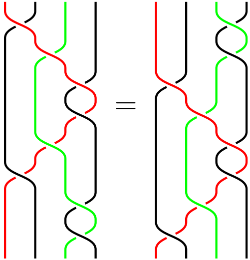


$\alpha_1 \in P_4$.



$\alpha_3 \in P_4$.

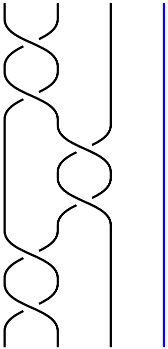
For $j = 1, \ldots, n-1$, let α_j be the pure braid which passes the j-th strand in front of the last $n-j$ strands, then behind the last $n-j$ strands to its original position.



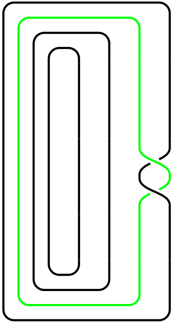
The relation $\alpha_1\alpha_3 = \alpha_3\alpha_1$.

The α_j commute pairwise, so they generate a free abelian subgroup $A \cong \mathbb{Z}^{n-1}$ of P_n (hence of B_n^G).

There is an obvious inclusion $\iota : P_{n-1} \hookrightarrow P_n$ given by introducing an n-th strand which does not interact with the others.

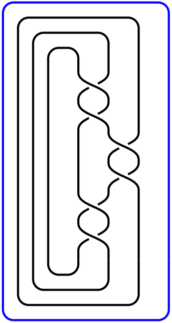


We let $B = \iota(P_{n-1}) \leq P_n \leq B_n^G$.

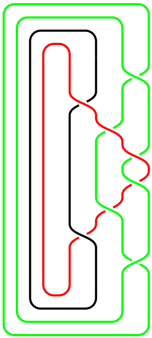


Closing off a non-trivial braid in A results in a link whose last 2 components cannot be split off.

Meanwhile, the closure of any braid in B is a link whose last component splits.



Conjugating by an element of B_n^G before closing results in an isotopic link, up to permutation of components by an element of $G \leq \Sigma_{n-2} \times \Sigma_2$.



It follows that $\gamma A \gamma^{-1} \cap B = \{1\}$ for every $\gamma \in B_n^G$, and so

$$\mathrm{TC}(B_n^G) \geq \mathrm{cd}(A \times B) = \mathrm{cd}(\mathbb{Z}^{n-1} \times P_{n-1}) = (n-1) + (n-2) = 2n-3.$$

An upper bound for $\mathrm{TC}(\pi)$

Theorem (G. 2012)

Let π be a torsion-free discrete group with centre $\mathcal{Z}(\pi)$. Identify $\mathcal{Z}(\pi)$ with its image under the diagonal embedding $d : \pi \to \pi \times \pi$. Then

$$\mathrm{TC}(\pi) \leq \mathrm{cd}\left(\frac{\pi \times \pi}{\mathcal{Z}(\pi)}\right).$$

Note that $\dfrac{\pi \times \pi}{\mathcal{Z}(\pi)}$ is torsion-free if and only if $\dfrac{\pi}{\mathcal{Z}(\pi)}$ is.

Theorem (G.–Recio-Mitter 2016)

Let $G \leq \Sigma_n$ be such that

- $G \leq \Sigma_{n-2} \times \{1\}^2$, or
- $G \leq \Sigma_{n-k} \times \Sigma_k$ where $(n,k) = (n-1,k) = (n-1,k-1) = 1$.

Then $\dfrac{B_n^G}{\mathcal{Z}(B_n^G)}$ is torsion-free, and $\mathrm{TC}(B_n^G) \leq 2n-3$.

Proof uses the characterisation of periodic braids (Eilenberg, Kerékjártó) and cycle structures in Σ_n.

Corollary (G.–Recio-Mitter 2016)

If $G \leq \Sigma_{n-2} \times \{1\}^2$ then $\mathrm{TC}(B_n^G) = 2n-3$.

Further problems

- Compute $\mathrm{TC}(B_n) = \mathrm{TC}(B(\mathbb{C}, n))$.
- Compute $\mathrm{TC}(F(\mathbb{R}^d, n)/G)$ for $d > 2$.

  (Note $\mathrm{cat}(B(\mathbb{R}^d, n)) = (d-1)(n-1)$ for $n = p^k$ by Blagojevic–Lück–Ziegler 2012).
- Compute $\mathrm{TC}(F(M, n)/G)$ for other manifolds M.

Merci pour votre attention !