Complete intersections on smooth affine varieties

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Plan of the talk

- A few questions
- Euler class groups
- The universal Segre class
- The unstable category
- Comparison results

A few questions

Let R be a (noetherian) ring. Given an ideal $I \subset R$, let $\mu(I)$ be the minimal number of generators of I.

Basic question: Can we compute $\mu(I)$ for any ideal I ?

The answer is rather depressing.

Macaulay showed a long time ago that given any integer $r \in \mathbb{N}$ there exists a prime ideal $\mathfrak{p} \subset \mathbb{C}[X_1, X_2, X_3]$ such that $\mu(\mathfrak{p}) \geq r$.

Murthy's conjecture

Despite the previous result, we would like to have a chance to compute $\mu(I)$ in some situations.

We can consider the R/I-module I/I^2 and an easy application of Nakayama's lemma yields

$$\mu(I/I^2) \leq \mu(I) \leq \mu(I/I^2) + 1$$

Murthy's conjecture: In case $R = k[X_1, \ldots, X_n]$ the equality $\mu(I) = \mu(I/I^2)$ holds for ideals of correct height.

A stronger question

Let R be a ring and I be an ideal. Given the following diagram

$$\begin{array}{ccc} R^n & \overset{\exists ?}{- - - \twoheadrightarrow} & I \\ \downarrow & & \downarrow \\ (R/I)^n & \underset{\omega_I}{\twoheadrightarrow} & I/I^2 \end{array}$$

does there exists a surjective lift?

The Euler class groups

The first "general" answer to the previous question was developed by the indian school (Bhatwadekar, Nori, Sridharan, etc…). Given $n \in \mathbb{N}$, consider the abelian group $E^n(R)$ freely generated by pairs (I, ω_I) where I is of height n, subject to the relations

- $(I, \omega_I) = (I, \omega_I \circ g)$ for any elementary operation g of $(R/I)^n$.
- $(I, \omega_I) = 0$ if there exists a lift as in the previous diagram.
- $(I, \omega_I) = (J, \omega_J) + (K, \omega_K)$ if J and K are comaximal of height n and $J \cap K = I$.

Results

In principle, this group allows to answer the previous question for any (noetherian) ring provided a few assumptions are satisfied.

Theorem. *Let R be a noetherian ring of Krull dimension d and let $n \in \mathbb{N}$ be such that $d \leq 2n-3$. Then the pair (I, ω_I) lifts if and only if $(I, \omega_I) = 0$ in $E^n(R)$.*

In particular, the above theorem is enough to answer Murthy's conjecture in case the ideal has correct height (basically due to Mohan Kumar), i.e. has height n and $d \leq 2n-3$.

Yet...

The Euler class group has some congenital defects.

- It deals only with ideals of a given height.
- It doesn't satisfy obvious functorial properties.

For instance, the Euler class groups of non affine varieties are not defined, and there is by consequence no long exact sequence of localization.
This certainly calls for a new approach.

The universal Segre class

Let then R be a k-algebra and let $\omega_I : (R/I)^n \to I/I^2$ be a surjective homomorphism. Choosing lifts of the images of the canonical basis of the left-hand term, we get elements $a_1, \ldots, a_n \in I$.

By Nakayama's lemma, there exists $s \in I$ such that

$$(1 - s)I \subset \langle a_1, \ldots, a_n \rangle$$

Consequently, we obtain $b_1, \ldots, b_n \in R$ such that

$$\sum_{i=1}^{n} a_i b_i = s(1 - s).$$

The (2n+1)-tuple

$$(a_1, \ldots, a_n, b_1, \ldots, b_n, s)$$

corresponds to a morphism

$$\mathrm{Spec}(R) \to Q_{2n}$$

where Q_{2n} is the affine variety whose ring of global sections is the following

$$A_{2n} = k[X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z]/\langle \sum X_i Y_i - Z(1-Z) \rangle$$

Naive homotopies

We've associated to $\omega_I : (R/I)^n \to I/I^2$ a morphism

$$f : \mathrm{Spec(R)} \to Q_{2n}$$

which depends on many choices...

Recall that two morphisms $f_0, f_1 : \mathrm{Spec}(R) \to Q_{2n}$ are naively homotopic if there exists

$$F : \mathrm{Spec}(R) \times \mathbb{A}^1_k \to Q_{2n}$$

such that $F(0) = f_0$ and $F(1) = f_1$.

Being naively homotopic generates an equivalence relation $\sim$ and we obtain a set

$$\pi_0(Q_{2n})(R) := \mathrm{Hom}(\mathrm{Spec}(R), Q_{2n})/\sim .$$

Lemma. *The assignment*

$$(I, \omega_I) \mapsto (a_1, \ldots, a_n, b_1, \ldots, b_n, s) \in \pi_0(Q_{2n})(R)$$

is well defined. We denote by $S(I, \omega_I)$ *the latter.*

The class $S(I, \omega_I)$ is called the universal Segre class of the pair (I, ω_I).

Why do we call this class "universal Segre class"? Imagine that you want obstruction for the lifting problem. The natural way to do it is to associate to any smooth affine scheme X a pointed set $F(X)$ and a class $s_F(I, \omega_I) \in F(X)$ satisfying the following properties:

- The above data are contravariantly functorial in X.
- We have $s_F(I, \omega_I) = * \in F(X)$ if there is a lift Ω.
- The map $F(X) \to F(X \times \mathbb{A}^1)$ induced by the projection is a bijection.

Consider Q_{2n} as pointed by $v_0 = (0, \ldots, 0)$. It is easy to check that the pointed functor

$$X \mapsto \pi_0(Q_{2n})(X)$$

together with the universal Segre class is initial among such functors. Consequently, it should detect all phenomena.

The theorem

In fact, one can prove the following theorem.

Theorem. *Let k be an infinite perfect field of characteristic different from 2 and $n \geq 2$. Then $S(I, \omega_I) = v_0$ if and only if (I, ω_I) lifts (in a strong sense).*

By construction, the set $\pi_0(Q_{2n})(X)$ doesn't see polynomial extensions. It follows that Murthy's conjecture is solved.

Sketch of the proof

In characteristic different from 2, the variety Q_{2n} can be seen as the homogeneous space of norm 1 elements of the split quadratic form

$$\sum_{i=1}^{n} x_i y_i + z^2 .$$

For a smooth k-algebra R, we find

$$Q_{2n}(R)/EO_{2n+1}(R) = \pi_0(Q_{2n})(R) .$$

It suffices then to check that the “strong” lifting property is preserved by the action of the generators of $EO_{2n+1}(R)$.

The strong lifting property

Let $v = (a_1, \ldots, a_n, b_1, \ldots, b_n, s) \in Q_{2n}(R)$ and set $I(v) = \langle a_1, \ldots, a_n, s \rangle$.

We say that the strong lifting property holds for if there exists $\lambda_1, \ldots, \lambda_n \in R$ such that

$$I(v) = \langle a_1 + \lambda_1 s^2, \ldots, a_n + \lambda_n s^2 \rangle .$$

Improving known results

Recall that the Euler class group $E^n(R)$ was doing a pretty good job in detecting liftable pairs (I, ω_I), with I of height n and $d \leq 2n-3$.

Why should the set $\pi_0(Q_{2n})(R)$ be better?

By construction, it is functorial and homotopy invariant. Yet, the same holds for Euler class groups (for smooth algebras)!

Can we endow $\pi_0(Q_{2n})(R)$ with an abelian group structure?

The unstable category

Let $\mathcal{H}(k)$ be the unstable $\mathbb{A}^1$-homotopy category of smooth schemes associated to an infinite perfect field k.
Basic features:

- There is a well-behaved notion of homotopy.
- Two kind of spheres: S^1, $\mathbb{G}_m$.
- Every (pointed) space has higher homotopy sheaves: $\pi_0^{\mathbb{A}^1}(X)$, $\pi_1^{\mathbb{A}^1}(X,x)$, etc...
- Classical theorems in algebraic topology have their analogues in this setting.

Results

In the category $\mathcal{H}(k)$, the following results hold:

Theorem (Asok,Wendt,Hoyois). *Let k be a field with $\mathrm{char}(k) \neq 2$. For any smooth k-algebra R, we have*

$$[\mathrm{Spec}(R), Q_{2n}]_{\mathbb{A}^1} = \pi_0(Q_{2n})(R).$$

Theorem (Asok, Doran, F.). *For any $n \in \mathbb{N}$, we have*

$$Q_{2n} \simeq S^n \wedge \mathbb{G}_m^{\wedge n}$$

in $\mathcal{H}(k)$.

Freudenthal's theorem

In topology, it is well-known that the set of maps

$$X \to S^n$$

up to homotopy is an abelian group, provided X is of "small size" relatively to n. The analogue result in $\mathcal{H}(k)$ is:

Theorem. *For $n \geq 2$, the set $\pi_0(Q_{2n})(R)$ is endowed with the structure of an abelian group provided R is of dimension $d \leq 2n-2$.*

In fact, $\pi_0(Q_{2n})(R) = [\mathrm{Spec}(R), Q_{2n}]_{\mathbb{A}^1}$ is S^1-stable.

The explicit group structure

Miraculously, the group structure provided by the above theorem can be explicitly described. Let $v, w \in Q_{2n}(R)$ with corresponding ideals $I(v) = \langle a_1, \ldots, a_n, s \rangle$ and $I(w) = \langle b_1, \ldots, b_n, z \rangle$.

Applying a moving lemma, we may suppose that

- The ideals $I(v)$ and $I(w)$ are of height n .
- They are comaximal.

Using the Chinese reminder lemma, we see that if

$$J = I(v) \cap I(w)$$

then $J/J^2 = I(v)/I(v)^2 \times I(w)/I(w)^2$ and we may find

$$c_1, \ldots, c_n \in R$$

such that the classes modulo the respective ideals generate $I(v)/I(v)^2$ and $I(w)/I(w)^2$.

Up to homotopy, we have $I(v) = \langle c_1, \ldots, c_n, u \rangle$ and $I(w) = \langle c_1, \ldots, c_n, u' \rangle$. One can check that

$$J = \langle c_1, \ldots, c_n, uu' \rangle$$

and we have $I(v) + I(w) = J$ in $[X, Q_{2n}]_{\mathbb{A}^1}$.

Consequences

As a consequence, we can obtain the following result:

Theorem (Asok, F.). *Let R be a smooth k-algebra of dimension d and $n \in \mathbb{N}$ be such that $d \leq 2n-3$. Then, the Segre class yields an isomorphism*

$$E^n(R) \simeq \pi_0(Q_{2n})(R).$$

So, we recover the Euler class group but not only. Indeed, the group on the right-hand side satisfies extended functorial properties, making computations easier.

Cohomological approach

In the category $\mathcal{H}(k)$, there is a useful construction called Postnikov towers. It allows to understand the set of morphisms $[X,Y]_{\mathbb{A}^1}$ for reasonable spaces X and Y.

The inputs needed are the homotopy sheaves $\pi_i^{\mathbb{A}^1}(Y,y)$ for $i \geq 2$ and the cohomology groups $H^j(X,\pi_i^{\mathbb{A}^1}(Y,y))$ for $j \leq d$, where d is the dimension of X.

Now, Q_{2n} is a sphere and there are lots of efforts to compute the homotopy sheaves of these objects.

Top dimension

As a test, let's try to compute the group

$$[X, Q_{2d}]_{\mathbb{A}^1} = \pi_0(Q_{2d})(X)$$

when $d \geq 3$. A formal argument shows that it is

$$H^d(X, \pi_d^{\mathbb{A}^1}(Q_{2d})) = \widetilde{CH}^d(X)$$

and we find a purely cohomological answer to our initial question. In particular, for a smooth affine algebra R,

$$E^d(R) \simeq \widetilde{CH}^d(\mathrm{Spec}(R)).$$

Beyond

The goal is of course to be able to go further than the dimension of the affine scheme X. Suppose that we want to understand the ideals of height $d-1$.

The required sheaf is $\pi^{\mathbb{A}^1}_{d-1}(Q_{2d-2})$ and we made a precise conjecture regarding its structure with A. Asok. An easy consequence of this conjecture is the following (conditional) theorem.

Theorem. *Let X be a smooth affine variety of dimension $d \geq 2$ over an algebraically closed field k. Then*

$$[X, Q_{2d-2}] = CH^{d-1}(X)$$

As a consequence, suppose that C is a curve in a smooth affine scheme X of dimension d.

Suppose moreover the the conormal bundle of C admits a trivialization. Then C is a complete intersection if and only if its associated class

$$[C] \in CH^{d-1}(X)$$

is trivial.

Thank your for your attention!