

Yang–Baxter equation, Young tableaux, group factorisations

Victoria LEBED, Trinity College Dublin

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2	6	6
1	3	4

$$(ab)c = a(bc)$$

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

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Yang-Baxter equation: basics

Data: monoidal category \mathcal{C} (mainly sets / vector spaces),
object V , $\sigma: V^{\otimes 2} \rightarrow V^{\otimes 2}$.

Yang-Baxter equation (YBE)

$$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2: V^{\otimes 3} \rightarrow V^{\otimes 3}$$

$$\sigma_1 = \sigma \otimes \text{Id}_V, \quad \sigma_2 = \text{Id}_V \otimes \sigma$$

Omnipresent:

- particle physics;
- statistical mechanics;
- quantum / conformal field theory;
- quantum groups;
- C^* algebras;

.....

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Omni-present:

→ low-dimensional topology

$$\sigma \leftrightarrow \text{crossing}$$



$$\text{YBE} \leftrightarrow \text{Reidemeister III move}$$

Reidemeister III
move

2 Basic examples

Data: monoidal category \mathcal{C} , object V , $\sigma: V^{\otimes 2} \rightarrow V^{\otimes 2}$.

YBE: $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2: V^{\otimes 3} \rightarrow V^{\otimes 3}$ $\sigma_1 = \sigma \otimes \text{Id}_V, \dots$

✓ $\sigma = \text{Id}_{V \otimes V}$;

✓ $\mathcal{C} = \mathbf{Set}$, $\sigma(x, y) = (y, x)$ $\xrightarrow{\text{deform}}$ $\mathcal{C} = \mathbf{Vect}_k$,

✓ $\sigma = R$ -matrix (quantum groups);

✓ Lie (Leibniz) algebra $(V, [])$, central element $1 \in V$,
 $\sigma(x \otimes y) = y \otimes x + \hbar 1 \otimes [x, y]$.

YBE for $\sigma \iff$ Jacobi identity for $[]$

✓ rack (S, \triangleleft) , $\sigma(x, y) = (y, x \triangleleft y)$

YBE for $\sigma \iff$ self-distributivity for \triangleleft

Self-distributivity: $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$

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Idempotent YBE solutions

✓ Monoid $(S, *, 1)$, $\sigma(x, y) = (1, x * y)$;

YBE for $\sigma \iff$ associativity for $*$

✓ Monoid factorisation $G = HK$,

$$S = H \cup K, \quad \sigma(x, y) = (h, k), \quad h \in H, k \in K, hk = xy.$$

✓ Ordered set S ,

$$\sigma(x, y) = (x, \max\{x, y\}).$$

$$\sigma(x, y) = (\min\{x, y\}, \max\{x, y\}).$$

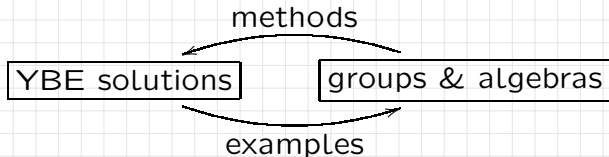
✓ Lattice (S, \wedge, \vee) , $\sigma(x, y) = (x \wedge y, x \vee y)$.

All these YBE solutions are idempotent: $\sigma^2 = \sigma$.

Un. env. monoid of a YBE solution (S, σ) in $\mathcal{C} = \mathbf{Set}$:

$$\text{Mon}(S, \sigma) = \langle S \mid xy = y'x' \text{ whenever } \sigma(x, y) = (y', x') \rangle$$

Un. env. (semi)groups and algebras are defined similarly.



Theorem: (S, σ) a “nice” finite YBE solution, $\sigma^2 = \text{Id} \implies$

- ✓ $\text{Mon}(S, \sigma)$ is of *I-type*, cancellative, Öre;
- ✓ $\text{Grp}(S, \sigma)$ is solvable, Garside;
- ✓ $\mathbb{k}\text{Mon}(S, \sigma)$ is Koszul, noetherian, Cohen–Macaulay, Artin–Schelter regular

(Manin, Gateva-Ivanova & Van den Bergh, Etingof–Schedler–Soloviev, Jespers–Okniński, Chouraqui 80’-...).

$$\text{Mon}(S, \sigma) = \langle S \mid xy = y'x' \text{ whenever } \sigma(x, y) = (y', x') \rangle$$

Examples:

✓ monoid $(S, *, 1)$, $\sigma(x, y) = (1, x * y)$,

$$S \simeq \text{Mon}(S, \sigma) /_{(\star)}$$

✓ Lie algebra $(V, [], 1)$, $\sigma(x \otimes y) = y \otimes x + 1 \otimes [x, y]$,

$$\text{UEA}(V, []) \simeq \text{Alg}(V, \sigma) /_{(\star)}.$$

(\star) : $1 = 1_{\text{Mon}}$

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Representations

$$\text{Mon}(S, \sigma) = \langle S \mid xy = y'x' \text{ whenever } \sigma(x, y) = (y', x') \rangle$$

Representations of (S, σ) := representations of $\mathbb{k}\text{Mon}(S, \sigma)$,
i.e. \mathbb{k} -vector spaces M with $M \times S \rightarrow M$ s.t.

$$(m \cdot x) \cdot y = (m \cdot y') \cdot x'$$

Examples:

- trivial representation: $M = \mathbb{k}$, $m \cdot x = m$;
- $M = \mathbb{k}\text{Mon}(S, \sigma)$, $m \cdot x = mx$;
- usual reps for monoids, Lie algebras, racks.

A cohomology theory for YBE solutions should:

1) Describe **deformations**: $\sigma_0 \rightsquigarrow \sigma_0 + \hbar\sigma_1 + \hbar^2\sigma_2 + \dots$.

Difficult! Pioneers: *Freyd–Yetter '89, Eisermann '05.*

First approximation: **diagonal deformations** (in some cases yield all deformations)

$$\sigma_q(x, y) = q^{\omega(x, y)} \sigma(x, y), \quad \omega: S \times S \rightarrow \mathbb{Z}.$$

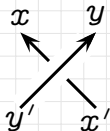
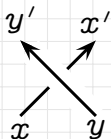
ω a 2-cocycle $\implies \sigma_q$ a YBE solution.

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A cohomology theory?

Theorem: For “nice” set-theoretic YBE solutions (S, σ) , one has efficient and easily computable **knot and knotted surface invariants**

$\#\{ (S, \sigma)\text{-colourings of diagrams} \}$.



$$\sigma(x, y) = (x', y')$$

Example: Solutions coming from racks are “nice”.

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A cohomology theory?

A cohomology theory for YBE solutions should:

2) Enhance the above invariants:

(S, σ) -coloured diagram (D, \mathcal{C}) & $\omega: S \times S \rightarrow \mathbb{Z}$
 \rightsquigarrow Boltzmann weight

$$\mathcal{B}_\omega(\mathcal{C}) = \sum_{\substack{y' \nearrow x' \\ x \searrow y}} \omega(x, y) - \sum_{\substack{x \nearrow y \\ y' \searrow x'}} \omega(x, y).$$

ω a 2-cocycle \implies a knot invariant is defined by
 $\{\mathcal{B}_\omega(\mathcal{C}) \mid \mathcal{C} \text{ is a } (S, \sigma)\text{-colouring of } D\}.$

A cohomology theory for YBE solutions should:

3) **Unify** cohomology theories for

- associative structures,
- Lie algebras,
- racks etc.

+ explain parallels between them,

+ suggest theories for new structures.

4) Be interpreted in terms of **classifying spaces**.

5) Compute the **cohomology of $\mathbb{k}\text{Mon}(S, \sigma)$** .

7 Braided cohomology

Data: $\mathcal{C} = \mathbf{Set}$, YBE solution (S, σ) , bimodule M over it.
 (any preadditive monoidal category will do)

Construction:

$$\mathcal{C}^n(S, \sigma; M) = \text{Maps}(S^{\times n}, M),$$

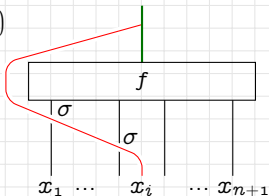
$$d^n = \sum_{i=1}^{n+1} (-1)^{i-1} (d_l^{n,i} - d_r^{n,i}): \mathcal{C}^n \rightarrow \mathcal{C}^{n+1},$$

$$x'_i \cdot f(x'_1 \dots x'_{i-1} x_{i+1} \dots x_{n+1})$$

$$d_l^{n,i} f:$$

$$x'_i x'_1 \dots x'_{i-1} x_{i+1} \dots x_{n+1}$$

$$\begin{array}{c} \uparrow \\ \sigma_1 \dots \sigma_{i-1} \uparrow \\ x_1 \dots x_{n+1} \end{array}$$



Theorems:

→ This defines a **precubical structure** $\implies d^{n+1} d^n = 0$.

$H^n(S, \sigma; M) = \text{Ker } d^n / \text{Im } d^{n-1}$ is the **nth cohomology group** of (S, σ) with coefficients in M .

Braided cohomology

Data: $\mathcal{C} = \mathbf{Set}$, YBE solution (S, σ) , bimodule M over it.

Construction:

$$C^n(S, \sigma; M) = \text{Maps}(S^{\times n}, M),$$

$$d^n = \sum_{i=1}^{n+1} (-1)^{i-1} (d_l^{n;i} - d_r^{n;i}): C^n \rightarrow C^{n+1}.$$

Theorems:

→ This defines a **precubical structure** $\implies d^{n+1}d^n = 0$.

→ For “nice” M (has a product compatible with the bimodule structure), there is a **cup product** on C^* inducing

$$\smile: H^n \otimes H^m \rightarrow H^{n+m},$$

which is **graded commutative** for “even nicer” M (trivial bimodule \mathbb{k}), with an explicit Steenrod-like homotopy.

→ Other good properties.

My favourite proofs:

✓ graphical calculus;

✓ algebraic properties of **quantum shuffles** (Rosso '95).

A good theory?

1) & 2) For $\omega \in C^2(S, \sigma; \mathbb{Z})$,

$d^2\omega = 0 \iff \omega$ yields Boltzmann weights
& diagonal deformations,

$\omega = d^1\theta \implies \omega$ yields trivial...

3) Unifies classical cohomology theories.

Example: monoid $(S, *, 1)$, $\sigma(x, y) = (1, x * y)$,

$$\begin{aligned}
 d_l^{n,i} f: & \quad \dots x_{i-2} \underline{x_{i-1} x_i} x_{i+1} \dots \xrightarrow{\sigma_{i-1}} \\
 & \quad \dots \underline{x_{i-2} 1} (x_{i-1} * x_i) x_{i+1} \dots \xrightarrow{\sigma_{i-2}} \\
 & \quad \dots \underline{1} x_{i-2} (x_{i-1} * x_i) x_{i+1} \dots \longrightarrow \dots \\
 & \quad 1 x_1 \dots x_{i-2} (x_{i-1} * x_i) x_{i+1} \dots \longrightarrow \\
 & \quad f(\dots x_{i-2} (x_{i-1} * x_i) x_{i+1} \dots).
 \end{aligned}$$

4) Can be interpreted in terms of classifying spaces.

A good theory?

5) Quantum symmetriser \mathcal{QS} :

a subcomplex of
braided complex for
(S, σ) with coefs in M

cup product

smaller complexes

$\xleftarrow{\mathcal{QS}}$

Hochschild complex for
 $\mathbb{k}\text{Mon}(S, \sigma)$ with coefs in M

cup product

tools

$$n=2: \mathcal{QS}(f)(x, y) = f(x, y) - f(y', x'), \quad \sigma(x, y) = (y', x').$$

Open problem (Yang, Farinati & García-Galofre '16):

How far is \mathcal{QS} from being a **quasi-iso**?

Theorem (FGG '16): \mathcal{QS} is a q-iso when $\sigma^2 = \text{Id}$, $\text{Char } \mathbb{k} = 0$,
the subcomplex is defined by

$$f(\dots, x, y, \dots) + f(\dots, y', x', \dots) = 0.$$

a subcomplex of
 braided complex for (S, σ) with coefs in M $\xrightarrow{\mathcal{QS}}$ Hochschild complex for $\mathbb{k}\text{Mon}(S, \sigma)$ with coefs in M

$$n=2: \mathcal{QS}(f)(x, y) = f(x, y) - f(y', x'), \quad \sigma(x, y) = (y', x').$$

Open problem: How far is \mathcal{QS} from being a quasi-iso?

Theorem (L '16): \mathcal{QS} is a quasi-iso when $\sigma^2 = \sigma$ & the subcomplex is defined by

$$f(\dots, x, y, \dots) = 0 \quad \text{whenever} \quad \sigma(x, y) = (x, y),$$

i.e., f is supported on critical n -tuples only:

$$\text{Crit}_n(S, \sigma) = \{(x_1, \dots, x_n) \in S^{\times n} \mid \forall i, \sigma(x_i, x_{i+1}) \neq (x_i, x_{i+1})\}.$$

Proof: Algebraic discrete Morse theory.

Examples:

✓ Ordered set S , $\sigma(x, y) = (\min\{x, y\}, \max\{x, y\})$,

$$\text{Mon}(S, \sigma) = \text{Sym}(S),$$

$$\text{Crit}_n(S, \sigma) = \{x_1 > x_2 \dots > x_n\}$$

↪ recovers the classical minimal resolution of $\mathbb{k}[S]$
& the resulting cohomology computations.

✓ Ordered set S , $\sigma(x, y) = (x, \max\{x, y\})$.

$$\text{Mon}(S, \sigma) = \langle S \mid \forall x > y, xy = xx \rangle,$$

$$\text{Crit}_n(S, \sigma) = \{x_1 > x_2 \dots > x_n\}$$

↪ improves Jöllenbeck–Welker '09.

9 Cohomology: idempotent case

Examples:

✓ Monoid factorisation $G = HK$,

$$S = H \cup K, \quad \sigma(x, y) = (h, k), \quad h \in H, \quad k \in K, \quad hk = xy.$$

$$\text{Mon}(S, \sigma) /_{(\star)} \simeq G, \quad (\star): \quad 1_G = 1_{\text{Mon}}$$

$$\text{Crit}_n(S, \sigma) = \bigsqcup_{p+q=n} \overline{K}^{\times p} \times \overline{H}^{\times q}, \quad \overline{K} = K \setminus \{1\}, \quad \overline{H} = H \setminus \{1\}$$

\rightsquigarrow a double complex

$$C^{p,q} = \text{Maps}(\overline{K}^{\times p} \times \overline{H}^{\times q}, M)$$

specialising to the **Künneth formula**
for the direct product $G = H \times K$.

Examples:

✓ Young tableaux on $\{1, \dots, n\}$.

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2	6	6
1	3	4

Robinson '38, Schensted' 61, Knuth '70:
 associative product $*$ on \mathbf{YT}_n .

Useful gadgets:

- representation theory of S_n , $GL_n(\mathbb{C})$ and $GL_n(F_q)$;
- intersections of Grassmannians;
- products of symmetric functions;
- lattice models;
- crystal bases for quantum groups.

Examples:

✓ Young tableaux on $\{1, \dots, n\}$.

Theorem (Cain et al., Bokut et al. '15): One-column tableaux \mathbf{Col}_n form a Gröbner–Shirshov basis for $(\mathbf{YT}_n, *)$.

Corollary (Lopatkin '16): First steps towards cohomology computations for $\mathbb{k}\mathbf{YT}_n$.

Work in progress: Idempotent YBE solutions σ_c on \mathbf{Col}_n and σ_r on \mathbf{Row}_n such that

$$(\mathbf{YT}_n, *) \simeq \text{Mon}(\mathbf{Col}_n, \sigma_c) /_{(*)} \simeq \text{Mon}(\mathbf{Row}_n, \sigma_r) /_{(*)}$$

Corollary: Manageable complexes computing the cohomology of $\mathbb{k}\mathbf{YT}_n$.