Bifibrations of model categories (and the Reedy construction)

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1. Liminaries

2. Bifibration of model categories

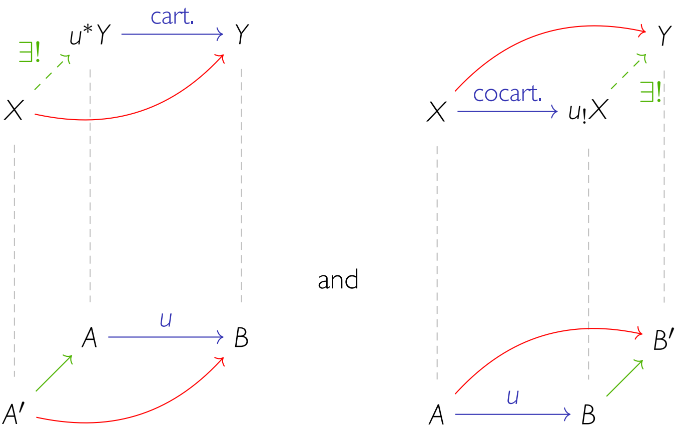
3. Comparison and applications

1

Liminaries

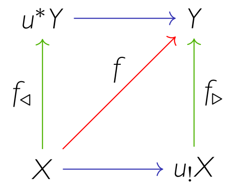
Grothendieck bifibrations

A cloven Grothendieck bifibration is a functor $p : \mathcal{E} \to \mathcal{B}$ together with

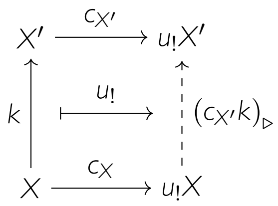


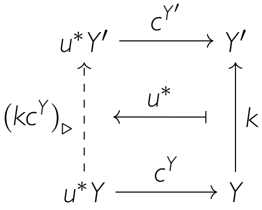
Push and pull

For any $f : X \to Y$ in $\mathcal{E}$ over $u : A \to B$,



Hence $u_!$ and u^* extend to functors:

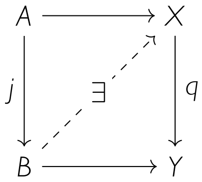




This produces an adjunction $u_! : \mathcal{E}_A \leftrightarrows \mathcal{E}_B : u^*$ between fibers.

Weak factorization system

In a category $\mathcal{M}$, denote $j \boxslash q$ when for any



Definition

A weak factorisation system on a category $\mathcal{M}$ is a couple $(\mathfrak{L}, \mathfrak{R})$ such that

- $\mathfrak{L} = \{j : \forall q \in \mathfrak{R}, j \boxslash q\}$ and $\{q : \forall j \in \mathfrak{L}, j \boxslash q\} = \mathfrak{R}$
- each morphism factors as $q \circ j$ for $j \in \mathfrak{L}$ and $q \in \mathfrak{R}$.

Model category

Definition

A model category is a complete and cocomplete category $\mathcal{M}$ together with $\mathbf{Cof}, \mathbf{W}, \mathbf{Fib}$ such that

- $\mathbf{W}$ has 2-out-of-3,
- $(\mathbf{Cof} \cap \mathbf{W}, \mathbf{Fib})$ and $(\mathbf{Cof}, \mathbf{W} \cap \mathbf{Fib})$ are weak factorisation systems.

An adjunction $L : \mathcal{M} \leftrightarrows \mathcal{N} : R$ is Quillen when:

$$L(\mathbf{Cof}_{\mathcal{M}}) \subseteq \mathbf{Cof}_{\mathcal{N}} \qquad R(\mathbf{Fib}_{\mathcal{N}}) \subseteq \mathbf{Fib}_{\mathcal{M}}$$

2

Bifibration of model categories

Lifting weak factorisation systems

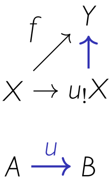
Define a wfs-adjunction as an adjunction $L : \mathcal{M} \leftrightarrows \mathcal{N} : R$ such that

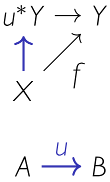
$$L(\mathfrak{L}_{\mathcal{M}}) \subseteq \mathfrak{L}_{\mathcal{N}} \qquad R(\mathfrak{R}_{\mathcal{N}}) \subseteq \mathfrak{R}_{\mathcal{M}}$$

Proposition (folklore)

Given a bifibration $p : \mathcal{E} \to \mathcal{B}$ with a wfs $(\mathfrak{L}, \mathfrak{R})$ on $\mathcal{B}$ and a wfs $(\mathfrak{L}_A, \mathfrak{R}_A)$ on each fiber $\mathcal{E}_A$, if each pair $(u_!, u^*)$ is a wfs-adjunction, the following classes yield a wfs on $\mathcal{E}$:

$$\mathfrak{L}_{\mathcal{E}} = \{f : p(f) \in \mathfrak{L}, f_{\triangleright} \in \mathfrak{L}_B\} \qquad \mathfrak{R}_{\mathcal{E}} = \{f : p(f) \in \mathfrak{R}, f_{\triangleleft} \in \mathfrak{R}_A\}$$





General principle

From now on, $p : \mathcal{E} \to \mathcal{B}$ is a bifibration with model structure $(\mathbf{Cof}, \mathbf{W}, \mathbf{Fib})$ on $\mathcal{B}$ and $(\mathbf{Cof}_A, \mathbf{W}_A, \mathbf{Fib}_A)$ on each fiber $\mathcal{E}_A$ such that every $(u_!, u^*)$ is Quillen.

Call f in $\mathcal{E}$

- a total acyclic cofibration if $p(f) \in \mathbf{Cof} \cap \mathbf{W}$ and $f_\triangleright \in \mathbf{Cof}_B \cap \mathbf{W}_B$,
- a total acyclic fibration if $p(f) \in \mathbf{Fib} \cap \mathbf{W}$ and $f_\triangleleft \in \mathbf{Fib}_A \cap \mathbf{W}_A$.

Denote $\mathbf{Cof}_\mathcal{E}$, $\mathbf{WCof}_\mathcal{E}$ and $\mathbf{Fib}_\mathcal{E}$, $\mathbf{WFib}_\mathcal{E}$ the corresponding classes.

Key observation

If we find a class $\mathbf{W}_\mathcal{E}$ of total weak equivalences such that

- $\mathbf{W}_\mathcal{E}$ has 2-out-of-3,
- $\mathbf{WCof}_\mathcal{E} = \mathbf{Cof}_\mathcal{E} \cap \mathbf{W}_\mathcal{E}$ and $\mathbf{WFib}_\mathcal{E} = \mathbf{Fib}_\mathcal{E} \cap \mathbf{W}_\mathcal{E}$.

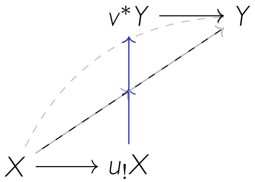
then $\mathbf{Cof}_\mathcal{E}, \mathbf{W}_\mathcal{E}, \mathbf{Fib}_\mathcal{E}$ is a model structure on $\mathcal{E}$.

Total weak equivalences

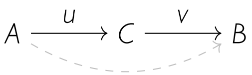
Main Idea

You don't have a choice for the total weak equivalences:

$$\mathbf{W}_{\mathcal{E}} \stackrel{\text{def}}{=} \mathbf{WFib}_{\mathcal{E}} \circ \mathbf{WCof}_{\mathcal{E}}$$



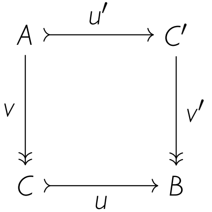
Call $f : X \to Y$ acyclic relatively to (u, v) if $p(f) = vu$ with $u \in \mathbf{Cof} \cap \mathbf{W}$ and $v \in \mathbf{Fib} \cap \mathbf{W}$, and $u_! X \to v^* Y \in \mathbf{W}_C$.



Define f to be a total weak equivalence if it is acyclic relatively to some pair.

Necessary conditions

- H$_1$ $u_!$ preserves and reflects weak equivalences if $u \in \mathbf{Cof} \cap \mathbf{W}$.
- H$_2$ v^* preserves and reflects weak equivalences if $v \in \mathbf{Fib} \cap \mathbf{W}$.
- hBC every square of the form



$u, u' \in \mathbf{Cof} \cap \mathbf{W}$
$v, v' \in \mathbf{Fib} \cap \mathbf{W}$

satisfies the homotopical Beck-Chevalley condition, i.e. the mate $u'_! v^* \to v'^* u_!$ is pointwise in $\mathbf{W}_{C'}$.

Main result

Theorem

The bifibration $p : \mathcal{E} \to \mathcal{B}$ satisfies H_1, H_2 and hBC if and only if $\mathbf{Cof}_{\mathcal{E}}, \mathbf{W}_{\mathcal{E}}, \mathbf{Fib}_{\mathcal{E}}$ give a model structure on $\mathcal{E}$ where

- $\mathbf{Cof}_{\mathcal{E}} = \{f : p(f) \in \mathbf{Cof},\ f_{\triangleright} \in \mathbf{Cof}_B\}$
- $\mathbf{Fib}_{\mathcal{E}} = \{f : p(f) \in \mathbf{Fib},\ f_{\triangleleft} \in \mathbf{Fib}_A\}$
- $\mathbf{W}_{\mathcal{E}} = \{f : p(f) \in \mathbf{W},\ f$ acyclic relatively to some $(u, v)\}$

What about completeness and cocompleteness of $\mathcal{E}$?

Limits and colimits in $\mathcal{E}$ can be constructed from those in the fibers and in the base.

3

Comparison and applications

Warm up example: the codomain bifibration

Given a category $\mathcal{C}$, the functor $\mathbf{cod} : \mathrm{Arr}(\mathcal{C}) \to \mathcal{C}$, $(A' \to A) \mapsto A$ is a bifibration where pulls and pushes are given as follow:

$$
\begin{array}{ccc}
A' & = & A' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle uf} \\
A & \xrightarrow{u} & B \\
\vdots & & \vdots \\
A & \xrightarrow{u} & B
\end{array}
\qquad\qquad
\begin{array}{ccc}
A \times_B B' & \longrightarrow & B' \\
\downarrow & & \downarrow{\scriptstyle f} \\
A & \xrightarrow{u} & B \\
\vdots & & \vdots \\
A & \xrightarrow{u} & B
\end{array}
$$

If $\mathcal{C}$ is a model category, each fiber $\mathbf{cod}_A \simeq \mathcal{C}_{/A}$ inherit a canonical model structure. The conditions H_1, H_2 and hBC are satisfied, hence we get a model structure on $\mathrm{Arr}(\mathcal{C})$... which is the injective model structure!

Related works

- *Model category structures in bifibred categories* (1994) Roig
  Bifibrations and Weak Factorisation Systems (2012) Stanculescu

  Defines $\mathbf{W}_{\mathcal{E}}$ as those f such that $p(f) \in \mathbf{W}$ and $f_{\triangleleft} \in \mathbf{W}_A$.

- *The Grothendieck construction for model categories* (2015) Harpaz, Prasma

  Defines $\mathbf{W}_{\mathcal{E}}$ as those $f : X \to Y$ such that $u = p(f) \in \mathbf{W}$
  and $X \to u^*Y \to u^*Y^{\mathbf{fib}} \in \mathbf{W}_A$.

In both cases, with <u>Y fibrant</u>, if f cartesian and $p(f) \in \mathbf{W}$, then $f \in \mathbf{W}_{\mathcal{E}}$.
Try to apply to the codomain bifibration:

$$\begin{array}{ccc} B' \times_B A & \longrightarrow & B' \\ \downarrow & & \downarrow\!\!\downarrow \\ A & \xrightarrow{u} & B \end{array}$$

(right properness)

Reedy categories

Definition (Reedy category)

A Reedy category is a category $\mathcal{R}$ together with subcategories $\mathcal{R}^+, \mathcal{R}^-$ such that

- there is a degree $d : \mathrm{Ob}\,\mathcal{R} \to \mu$ raised (lowered) by non-identity maps of $\mathcal{R}^+$ ($\mathcal{R}^-$),
- every morphism factors uniquely as f^+f^- with $f \in \mathcal{R}^+$ and $f^- \in \mathcal{R}^-$.

Define $\mathcal{R}_\lambda$ to be the full subcategory with objects of degree $< \lambda$.
An extension $X : \mathcal{R}_{\lambda+1} \to \mathcal{M}$ of $A : \mathcal{R}_\lambda \to \mathcal{M}$ is exaclty the choiche of factorizations

$$\mathrm{L}_r A \to X_r \to \mathrm{M}_r A \quad \forall r \text{ with degree } \lambda$$

Reedy model structure

Theorem

If $\mathcal{M}$ is a model category and $\mathcal{R}$ a Reedy category, then there is a model structure on $[\mathcal{R}, \mathcal{M}]$ where

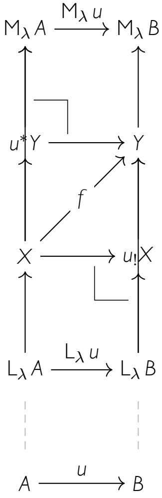
- the weak equivalences are pointwise,
- the cofibrations are those $f : A \to B$ such that the induced $\mathrm{L}_r B \sqcup_{\mathrm{L}_r A} A_r \to B_r$ is a cofibration in $\mathcal{M}$ for any $r \in \mathcal{R}$,
- the fibrations are those $f : A \to B$ such that the induced $A_r \to \mathrm{M}_r A \times_{\mathrm{M}_r B} B_r$ is a fibration in $\mathcal{M}$ for any $r \in \mathcal{R}$.

Why exactly do those maps naturally come into play?

The Reedy construction...

Because they secretly are fiber morphisms!

The restriction $p_\lambda : [\mathcal{R}_{\lambda+1}, \mathcal{M}] \to [\mathcal{R}_\lambda, \mathcal{M}]$ is a bifibration:



The fiber at A is equivalent to

$$\prod_{r\in\mathcal{R},\deg(r)=\lambda} {}^{L_r A\backslash}\mathcal{M}_{/M_r A}$$

hence is a model category.

If $[\mathcal{R}_\lambda, \mathcal{M}]$ has the Reedy model structure, then H_1, H_2 and hBC are satisfied, and... $[\mathcal{R}_{\lambda+1}, \mathcal{M}]$ gets the Reedy model structure by the theorem.

… and its generalizations

In *Reedy categories and their generalizations* (2015) by Mike Shulman:

Theorem 3.11

If $\mathcal{M}$ and $\mathcal{N}$ model categories, $F, G : \mathcal{M} \to \mathcal{N}$ and $\alpha : F \to G$ st

- F cocont. and maps $\mathbf{Cof} \cap \mathbf{W}$ to couniversal weak equivalences,
- G cont. and maps $\mathbf{Fib} \cap \mathbf{W}$ to universal weak equivalences,

then the biglueing $\mathrm{Gl}(\alpha)$ is a model category.

This is the key theorem to get a framework encompassing all known generalized Reedy situation: classical Reedy categories, Berger-Moerdijk, Cisinski, etc.

Glueing

The category $\mathrm{Gl}(\alpha)$ has

- as objects: the factorizations

$$FA \to X \to GA \quad \text{of } \alpha_A \qquad A \in \mathcal{M},\ X \in \mathcal{N}$$

- as morphisms: the commutative diagrams

$$\begin{array}{ccccc} FA & \longrightarrow & X & \longrightarrow & GA \\ {\scriptstyle Fu}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Gu} \\ FB & \longrightarrow & Y & \longrightarrow & GB \end{array} \qquad u \in \mathcal{M},\ f \in \mathcal{N}$$

The forgetful functor $\mathrm{Gl}(\alpha) \to \mathcal{M}$ is a Grothendieck bifibration. Hypothesis H_1, H_2 and hBC are met, hence a model structure on $\mathrm{Gl}(\alpha)$.

Thank you.

http://www.normalesup.org/~cagne/
https://pierrecagne.github.io