# A Generalized Blakers-Massey Theorem

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# History/Acknowledgments

New proof of classical BMT in Homotopy Type Theory
Favonia-Finster-Licata-Lumsdaine
"A Mechanization of the Blakers-Massey Connectivity Theorem in Homotopy Type Theory"
arXiv:1605.03227

"Reverse engineered version"
Rezk
"Proof of the Blakers-Massey Theorem", homepage

Different approach
Chachólski-Scherer-Werndli
"Homotopy Excision and Cellularity"
arXiv:1408.3252

# Generalized Blakers-Massey Theorem

**Theorem (Anel-B-Finster-Joyal)**

*Let $(\mathcal{L}, \mathcal{R})$ be a modality in an $\infty$-topos. Consider a homotopy pushout*

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \longrightarrow & D. \end{array}$$

*If*

$$\Delta(f) \square \Delta(g) \in \mathcal{L}$$

*then the cartesian gap map*

$$(f, g)\colon A \to B \times^h_D C$$

*is in $\mathcal{L}$. In symbols: $(f, g) > \Delta(f) \square \Delta(g)$*

# $\infty$-topoi

**Definition**

An $\infty$-topos is a left exact localization of a simplicial presheaf category (on a small category).

**Example**

– simplicial sets $\mathcal{S}$ = "spaces"

– functors to simplicial sets (from a small category)

– $n$-excisive functors to simplicial sets

– spectra parametrized by spaces

# Gap and cogaps

Given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \longrightarrow & D. \end{array}$$

we call the canonical map

$$(f, g) \colon A \to B \times_D C$$

the *(cartesian) gap (map)*. The canonical map

$$B \sqcup_A C \to D$$

will be the *cocartesian gap map/cogap*.

# The Diagonal of a Map

**Definition**

Given a map

$$f\colon A \to B$$

the *diagonal* is the canonical map

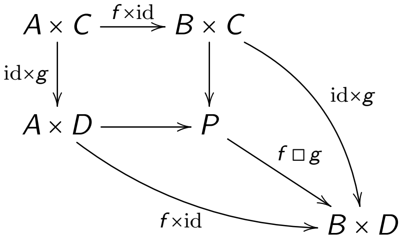
$$\Delta(f)\colon A \to A \times_B A.$$

**Example**

- $\Delta(* \to X) = * \to \Omega X$
- $\Delta(X \to *) = X \to X \times X$, the diagonal.

# The Pushout Product

Consider maps $f\colon A \to B$ and $g\colon C \to D$.




One has:

$$(f \,\square\, g) \,\square\, h = f \,\square\, (g \,\square\, h)$$

# Two examples

Given two objects $A$ and $B$.

**Example**

We have

$$(A \to *) \,\square\, (B \to *) = (A \star B \to *),$$

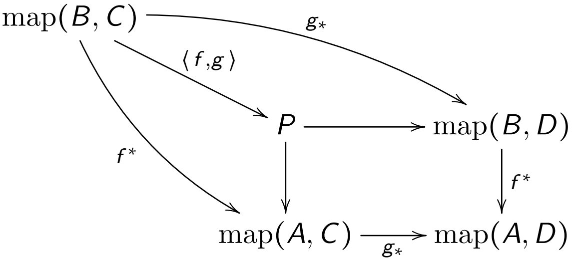
where $A \star B$ denotes the join of $A$ and $B$.

**Example**

We have

$$(* \to A) \,\square\, (* \to B) = (A \vee B \to A \times B).$$

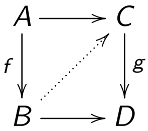
Adjointly:




One has:

$$\langle f, \langle g, h \rangle \rangle = \langle f \, \square \, g, h \rangle$$

# Lifting properties




## Definition

A map $f$ is *left orthogonal to* $g$, if $\langle f, g \rangle$ is a weak equivalence. We write:

$$f \perp g$$

Given a class $\mathcal{R}$ of maps, we write ${}^{\perp}\mathcal{R}$ for the class of maps that are left orthogonal to all maps in $\mathcal{R}$. Similarly, $\mathcal{L}^{\perp}$, $({}^{\perp}\mathcal{R})^{\perp}$, ...

# Factorization systems

Let $\mathcal{L}$ and $\mathcal{R}$ be two classes of maps. The pair $(\mathcal{L}, \mathcal{R})$ forms a *factorization system* if

1. each map $f$ can be functorially factored (uniquely up to homotopy) into $f = r\ell$ where $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$

and

2. $\mathcal{L}^{\perp} = \mathcal{R}$ and $\mathcal{L} = {}^{\perp}\mathcal{R}$.

**Example**

For $n \geq -1$ the pair ($n$-connected, $(n-1)$-truncated) form a factorization system on spaces.

# Modalities

**Definition**

A *modality* is a factorization system $(\mathcal{L}, \mathcal{R})$ such that the left class $\mathcal{L}$ is closed under base change.

**Example**

Given a left exact localization $F$ of an $\infty$-topos. Then

$$(F\text{-equivalences}, F\text{-local maps})$$

form a modality.

# Modality generated by $S$

**Proposition (ABFJ)**

*Given a set $S$ of maps in an $\infty$-topos, there exists a smallest modality such that $S \subset \mathcal{L}$.*

**Definition**

This smallest modality is called *the modality generated by $S$*.

We then write $\ell > S$ for any $\ell \in \mathcal{L}$.

In particular, if $S = \{f\}$, then $\ell > f$.

If $\ell \colon A \to *$ and $f \colon B \to *$, then this is equivalent to

$$A > B$$

in the sense of Dror Farjoun ("$A$ is killed by $B$", $P_B(A) = *$).

## Fiberwise Orthogonality

**Definition**

Fix a map $\ell$. We say that a map $f$ is *fiberwise right orthogonal* and write

$$\ell \perp\!\!\!\perp f$$

if $f$ is right orthogonal to any base change of $\ell$.

Observe: A modality $(\mathcal{L}, \mathcal{R})$ is just an ordinary factorization system where each map in $\mathcal{R}$ is fiberwise right orthogonal to any map in $\mathcal{L}$.

# Blakers-Massey Theorem

## Theorem (ABFJ)

*Let $(\mathcal{L}, \mathcal{R})$ be a modality in an $\infty$-topos. Consider a pushout*

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \longrightarrow & D. \end{array}$$

*If*

$$\Delta(f) \,\square\, \Delta(g) \in \mathcal{L}$$

*then the cartesian gap map*

$$(f, g) \colon A \to B \times_D C$$

*is in $\mathcal{L}$. In other words:*

$$(f, g) > \Delta(f) \,\square\, \Delta(g)$$

# Chacholski-Scherer-Werndli Theorem

Chacholski-Scherer-Werndli ("Homotopy Excision and Cellularity") prove for spaces:

$$\operatorname{fib}(f, g) > (\Omega \operatorname{fib} f) * (\Omega \operatorname{fib} g).$$

Note:

$$\operatorname{fib}\big(\Delta(f) \,\square\, \Delta(g)\big) = (\Omega \operatorname{fib} f) * (\Omega \operatorname{fib} g)$$

Observe that the following are equivalent:

– $h > (S^n \to *)$

– $h$ is $n$-connected

– $\Delta(h)$ is $(n-1)$-connected

Proof:

$$(f, g) > \Delta(f) \square \Delta(g) > (S^{m-1} \to *) \square (S^{n-1} \to *)$$
$$= (S^{m-1} \star S^{n-1} \to *) = (S^{m+n-1} \to *)$$

# “Dual” BMT

**Theorem (ABFJ)**

*In an ∞-topo consider a pullback:*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow f \\ C & \underset{g}{\longrightarrow} & D \end{array}$$

*Then for the cogap map*

$$B \cup_A C \to D > f \,\square\, g.$$

Let $\mathcal{F} = \mathrm{Fun}_{\mathcal{S}}(\mathcal{S}^{\mathrm{fin}}, \mathcal{S})$ or $\mathcal{F} = \mathrm{Fun}_{\mathcal{S}}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$; these are $\infty$-topoi.

A *homotopy functor* is a functor that preserves the class of weak equivalences.

Goodwillie: Treat homotopy functors as analogues of $C^\infty$-functions and study a Taylor expansion.

# The Goodwillie tower

**Theorem (Goodwillie)**

*For each homotopy functor $F$ exist a tower of functors*

$$F \to \cdots \to P_n F \to \cdots \to P_1 F \to P_0 F,$$

*such that $F \to P_n F$ is initial among all maps to n-excisive functors.*

# Higher excision

## Definition

A homotopy functor is *n-excisive* if it sends all strongly cocartesian $(n+1)$-cubes to cartesian ones.

## Examples

1. $F$ 0-excisive $\iff$ $F$ constant up to homotopy
2. $F$ 1-excisive $\iff$ $\pi_* F$ gen. homology theory

Here, $P_n F = \mathrm{hocolim}_k\, T_n^k F$ with

$$F(X) \to T_n F(X) = \mathrm{holim}_{U \neq \varnothing} F(X \star U),$$

where $U \subset \{1, \cdots, n+1\}$.

# Linearization

Let $F$ be reduced, ie. $F(*) = *$. Then

$$F(X) \to T_1 F(X) = \Omega F(\Sigma X),$$

and

$$P_1 F(X) = \mathrm{hocolim}_k\, \Omega^k F(\Sigma^k X) = \Omega^\infty F(\Sigma^\infty X)$$

For $F = \mathrm{id}$

$$P_1 \mathrm{id} = \Omega^\infty \Sigma^\infty.$$

Stable homotopy is the closest homology theory to the identity of pointed spaces.

# Modality associated to $P_n$

**Definition**

A map $f\colon F \to G$ is *$n$-excisive* (or $P_n$-local) if

$$\begin{array}{ccc} F & \longrightarrow & P_nF \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle P_nf} \\ G & \longrightarrow & P_nG \end{array}$$

is a homotopy pullback.

A map $f$ is a *$P_n$-equivalence* if the induced map $P_nf$ is an equivalence.

Recall: The pair ($P_n$-equiv., $n$-exc. maps) is a modality.
($P_n$ is left exact!)

# Goodwillie's Conjecture

**Theorem (Anel/B/Finster/Joyal)**

*In a homotopy pushout square in* $\mathcal{F} = \mathrm{Fun}(\mathcal{C}, \mathcal{S})$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

*let* $f$ *be a* $P_m$*-equivalence and* $g$ *a* $P_n$*-equivalence. Then*

$$(f, g) \colon A \to B \times_D C$$

*is a* $P_{m+n+1}$*-equivalence.*

*Holds also for functors to* $\mathcal{S}_*$*.*

**Corollary**

*Let $F$ be n-reduced. Then $P_{2n-1}F \simeq \Omega P_{2n-1}\Sigma F$.*

**Corollary (Arone-Dwyer-Lesh)**

*If a functor has derivatives only in the range between $n$ and $2n-1$ then it is infinitly deloopable.*

**Corollary (Goodwillie)**

*If a functor is n-homogeneous it is infinitly deloopable.*

In fact, $BF = P_n\Sigma F$ for $F$ $n$-homogeneous.

# Representable functors

We write for $K \in \mathcal{S}_*^{\mathrm{fin}}$

$$R^K = \mathrm{map}_{\mathcal{S}_*}(K, -).$$

We write

$$w_K \colon * \to R^K$$

for the canonical map picking out the constant map.

Idea: In the same way as $S^0 \to *$ generates the connected-truncated modalities, the maps $w_K$ generate the Goodwillie tower.

# Ganea Fibrations

Consider the map

$$w_{K_0} \square \cdots \square w_{K_n} \colon W_{n+1}(R^{K_0}, \cdots, R^{K_n}) \to R^{K_0} \times \cdots \times R^{K_n}.$$

For $K_0 = \cdots = K_n = K$ there is a pullback

$$\begin{array}{ccc} \Gamma_n(R^K) & \longrightarrow & W_{n+1}^{R^K} \\ {\scriptstyle \gamma_{n+1}^K}\downarrow & & \downarrow{\scriptstyle w_K^{\square n+1}} \\ R^K & \underset{\Delta}{\longrightarrow} & (R^K)^{n+1} \end{array}$$

and $\gamma_K$ is called the *n-th Ganea fibration* of $R^K$.

# Observations on this pullback

**Proposition (ABFJ)**

*Using Yoneda $\gamma_n^K$ induces the map*

$$t_n F \colon F \to T_n F$$

*used by Goodwillie to define $T_n F$ and then $P_n F$.*

**Proposition (ABFJ)**

*The maps $\gamma_n^K$ and $w_{n+1}^K$ possess the same fiberwise right orthogonal class of maps.*

**Corollary (ABFJ)**

$P_m$-equiv $\square$ $P_n$-equiv $\subset$ $P_{m+n+1}$-equiv

**Proposition**

*In an $\infty$-topos one has:*

1. *A map $f$ is mono iff $\pi_0 f$ is mono and the square*

$$\begin{array}{ccc} X & \longrightarrow & \pi_0 X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi_0 f} \\ Y & \longrightarrow & \pi_0 Y \end{array}$$

   *is a homotopy pullback.*

2. *A map $f$ is epi iff $\pi_0 f$ is epi.*

# Monos and Epis in $P_n\mathcal{F}$

**Proposition (ABFJ)**

*In the $\infty$-topos $P_n\mathcal{F}$ we have:*

1. *A map $f$ is mono iff $P_0 f$ is mono and the square*

$$\begin{CD} X @>>> P_0 X \\ @V{f}VV @VV{P_0 f}V \\ Y @>>> P_0 Y \end{CD}$$

   *is a homotopy pullback.*

2. *A map $f$ is epi iff $P_0 f$ is epi.*