MINIMAL MODELS FOR OPERADIC ALGEBRAS OVER ARBITRARY RINGS

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Fernando Muro

Universidad de Sevilla

Differential graded algebras in topology

A DIFFERENTIAL GRADED ALGEBRA (DGA) A is a chain complex equipped with a binary associative product satisfying the Leibniz rule

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b).$$

Its HOMOLOGY $H_*(A)$ is a graded algebra.

Differential forms on a manifold $\Omega^*(M)$	$\rightsquigarrow$	$H^*_{DR}(M)$
Singular cochains on a space $C^*(X,k)$	$\rightsquigarrow$	$H^*(X,k)$
Singular chains on a top. group $C_*(G,k)$	$\rightsquigarrow$	$H_*(G,k)$
Sullivan's model of a space $A^*_{PL}(X)$	$\rightsquigarrow$	$H^*(X,\mathbb{Q})$
	$\dots$	

Reconstructing DGAs from their homology

k commutative ground ring.

A differential graded algebra.

$H_*(A)$ homology graded algebra.

Can we recover A from $H_*(A)$?

Theorem (Kadeishvili'80)

If $H_*(A)$ is projective, then it can be endowed with a minimal A_∞-algebra structure which allows to recover A up to quasi-isomorphism.

A_∞-algebras

An A_∞-ALGEBRA is a $\mathbb{Z}$-graded module X endowed with degree $n-2$ operations, $n \geq 1$,

$$m_n \colon X \otimes \overset{n}{\cdots} \otimes X \longrightarrow X$$

satisfying the following equations, $n \geq 1$,

$$\sum_{\substack{p+q=n+1 \\ 1 \leq i \leq p}} \pm m_p \circ_i m_q = 0,$$

- m_1 is a differential for X, $m_1^2 = 0$,
- m_1 satisfies the Leibniz rule w.r.t. $a \cdot b = m_2(a, b)$,
- m_2 is associative up to the chain homotopy m_3,
- $\ldots$

MINIMAL if $m_1 = 0$. DGAs are A_∞-algebras with $m_n = 0$, $n > 2$.

∞-morphisms

An ∞-MORPHISM of A_∞-algebras $f\colon X \dashrightarrow Y$ is a sequence of degree $n-1$ maps, $n \geq 1$,

$$f_n\colon X\otimes \overset{n}{\cdots} \otimes X \longrightarrow Y$$

satisfying the following equations, $n \geq 1$,

$$\sum_{\substack{p+q=n+1\\ 1\leq i\leq p}} \pm f_p \circ_i m_q^X = \sum_{i_1+\cdots+i_r=n} \pm m_r^Y(f_{i_1},\ldots,f_{i_r}),$$

- $f_1\colon X \to Y$ is a map of complexes,
- f_1 is multiplicative w.r.t. m_2 up to the chain homotopy f_2,
- ...

It is an ∞-QUASI-ISOMORPHISM if f_1 is a quasi-isomorphism, and a (STRICT) MORPHISM of A_∞-algebras when $f_n = 0$, $n > 1$.

Kadeishvili defined inductively an ∞-quasi-isomorphism

$$f \colon H_*(A) \overset{\sim}{\dashrightarrow} A.$$

There is a Quillen equivalence between model categories

$$A_\infty\text{-algebras} \rightleftarrows \text{DGAs} \qquad \text{[Hinich'97]}$$

whose weak equivalences are quasi-isomorphisms, and ∞-morphisms with projective source represent maps in

$$\mathrm{Ho}(A_\infty\text{-algebras}).$$

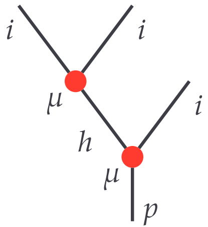
Kontsevich–Soibelman's formulas

We can obtain the A_∞-algebra structure on $H_*(A)$ and the ∞-quasi-isomorphism $f\colon H_*(A) \dashrightarrow A$ from an SDR

$$H_*(A) \underset{p}{\overset{i}{\rightleftarrows}} A \circlearrowleft_h$$

- i cycle selection map,
- $pi = 1$,
- h chain homotopy for $ip \simeq 1$,
- ...

$$m_n = \sum_{n \text{ leaves}} \pm$$



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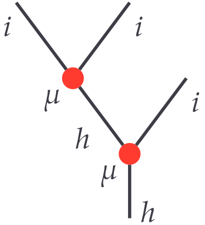
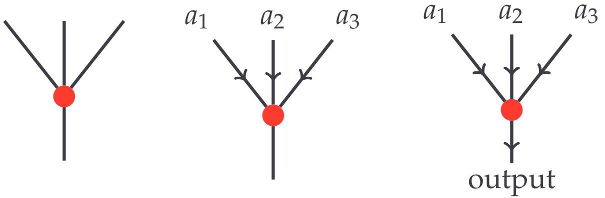


$$, \qquad f_1 = i.$$

Generalizations

An OPERAD $\mathcal{O} = \{\mathcal{O}_n\}_{n\geq 0}$ is an algebraic gadget defining a certain kind of algebras. It consists of:

- complexes of $k[\Sigma_n]$-modules $\mathcal{O}_n$ of ARITY n operations,
- COMPOSITION LAWS $\circ_i : \mathcal{O}_p \otimes \mathcal{O}_q \to \mathcal{O}_{p+q-1}$, $1 \leq i \leq p$,
- an IDENTITY operation $\mathrm{id} \in \mathcal{O}_1$,
- associativity, unit, and equivariance relations.



Generalizations

All previous results extend in the following way:

DGAs	$\leftrightsquigarrow$	algebras over a quadratic Koszul operad $\mathcal{O}$, e.g. $\mathcal{O} = \mathcal{A}s, \mathcal{C}om, \mathcal{L}ie, \mathcal{P}ois, \mathcal{G}erst, \dots$
A_∞-algebras	$\leftrightsquigarrow$	$\mathcal{O}_\infty$-algebras, $\mathcal{O}_\infty$ is the minimal resolution of $\mathcal{O}$, e.g. $\mathcal{A}s_\infty$ is the operad for A_∞-algebras.

We must require technical conditions so that the homotopy theories of operads and their algebras are well defined.

Theorem

Given an $\mathcal{O}$-algebra A with $H_*(A)$ PROJECTIVE, the homology can be endowed with a minimal $\mathcal{O}_\infty$-algebra structure with an ∞-quasi-isomorphism $H_*(A) \dashrightarrow A$.

There is a Quillen equivalence between model categories

$$\mathcal{O}_\infty\text{-algebras} \rightleftarrows \mathcal{O}\text{-algebras}$$

whose weak equivalences are quasi-isomorphisms, and ∞-morphisms with projective source represent maps in

$$\mathrm{Ho}(\mathcal{O}_\infty\text{-algebras}).$$

Removing the projectivity hypothesis

What if $H_*(A)$ is not projective?

Theorem (Sagave'10)

There is a projective resolution of $H_*(A)$ with a minimal derived A_∞-algebra structure which allows to recover A up to E^2-equivalence.

A derived A_∞-algebra is an $(\mathbb{N}, \mathbb{Z})$-bigraded module X such that the total graded module $\mathrm{Tot}(X)$

$$\mathrm{Tot}_n(X) = \bigoplus_{p+q=n} X_{p,q}$$

has an A_∞-structure compatible with the vertical filtration

$$F_m \,\mathrm{Tot}_n(X) = \bigoplus_{\substack{p+q=n \\ p \leq m}} X_{p,q}.$$

A **derived ∞-morphism** of derived A_∞-algebras $X \dashrightarrow Y$ is an ∞-morphism $\mathrm{Tot}(X) \dashrightarrow \mathrm{Tot}(Y)$ preserving the vertical filtration, and a **(strict) morphism** is a map preserving the bigrading and all the structure.

- Derived A_∞-algebras are also A_∞-algebras equipped with a split increasing filtration,
- derived ∞-morphisms are ∞-morphisms preserving the filtration,
- (strict) morphisms $X \to Y$ are filtered (strict) morphisms $\mathrm{Tot}(X) \to \mathrm{Tot}(Y)$ compatible with the splittings.

A DERIVED A_∞-ALGEBRA is the same as a bigraded module X equipped with bidegree $(-i, n-2+i)$ operations, $n \geq 1$, $i \geq 0$,

$$m_{i,n} \colon X^{\otimes n} \longrightarrow X$$

satisfying the following equations, $n \geq 1$, $i \geq 0$,

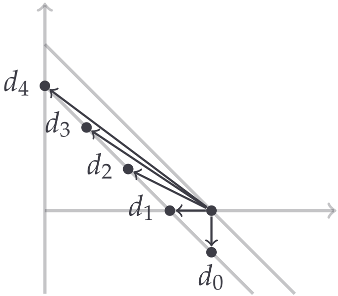
$$\sum_{\substack{p+q=n+1\\ 1\leq j\leq p\\ k+l=i}} \pm m_{k,p} \circ_j m_{l,q} = 0,$$

- $\{m_{0,n}\}_{n\geq 1}$ defines a usual A_∞-algebra,
- $\{m_{i,1}\}_{i\geq 0}$ forms a TWISTED COMPLEX,
- ...

We can similarly describe DERIVED ∞-MORPHISMS.

Twisted complexes

A TWISTED COMPLEX is a bigraded module X such that $\mathrm{Tot}(X)$ is equipped with a differential compatible with the vertical filtration,



- d_0 is a vertical differential, $d_0^2 = 0$, MINIMAL means $d_0 = 0$,
- d_1 is a map of vertical complexes (up to signs),
- d_1 squares to zero up to vertical chain homotopy d_2, $d_1^2 \simeq 0$,
- ...

Twisted complexes

A TWISTED MORPHISM of twisted complexes $X \dashrightarrow Y$ is a map of complexes $\mathrm{Tot}(X) \dashrightarrow \mathrm{Tot}(Y)$ preserving the vertical filtration, and a (STRICT) MORPHISM is a map preserving the bigrading and all the d_i.

- twisted complexes are also complexes equipped with a split filtration,
- twisted morphisms are maps preserving the filtration,
- (strict) morphisms are twisted morphisms compatible with the splittings.

Homotopy theory of derived A_∞-algebras

Sagave, like Kadeishvili, defined inductively a derived ∞-morphism inducing an isomorphism on the E^2-term of the associated spectral sequences,

$$f\colon \text{horizontal proj. resolution of } H_*(A) \overset{\sim}{\dashrightarrow} A.$$

Theorem

There is a model structure on the category of derived A_∞-algebras with total quasi-isomorphisms as weak equivalences, derived ∞-morphisms with projective source represent maps in the homotopy category, and there is a zig-zag of Quillen equivalences

$$\textit{derived } A_\infty\textit{-algebras} \rightleftarrows \bullet \leftrightarrows \textit{DGAs}.$$

Homotopy theory of different kinds of complexes

The category of (chain) complexes has a monoidal model structure with quasi-isomorphisms as weak equivalences and surjections as fibrations.

An $\mathbb{Z}$-GRADED COMPLEX is a $(\mathbb{Z}, \mathbb{Z})$-bigraded module equipped with a VERTICAL differential d_0

$$
\begin{array}{cccc}
 & \vdots & \vdots & \\
\cdots & X_{p-1,q} & X_{p,q} & \cdots \\
 & \downarrow{\scriptstyle d_0} & \downarrow{\scriptstyle d_0} & \\
\cdots & X_{p-1,q-1} & X_{p,q-1} & \cdots \\
 & \vdots & \vdots &
\end{array}
$$

They inherit a monoidal model structure from complexes.

Homotopy theory of different kinds of complexes

Modules in graded complexes over the ring of DUAL NUMBERS

$$\mathcal{D} = k[\epsilon]/(\epsilon^2) \cong k \cdot 1 \oplus k \cdot \epsilon, \qquad |\epsilon| = (-1, 0),$$

are the same as BICOMPLEXES with HORIZONTAL differential

$$d_1(x) = \epsilon \cdot x.$$

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ \cdots & & X_{p-1,q} & \xleftarrow{d_1} & X_{p,q} & & \cdots \\ & & \downarrow d_0 & & \downarrow d_0 & & \\ \cdots & & X_{p-1,q-1} & \xleftarrow{d_1} & X_{p,q-1} & & \cdots \\ & & \vdots & & \vdots & & \end{array}$$

They also inherit a VERTICAL model structure, which restricts to $(\mathbb{N}, \mathbb{Z})$-bicomplexes.

Proposition (total model structure)

The vertical model structure on $(\mathbb{N}, \mathbb{Z})$-bicomplexes has a left Bousfield localization with TOTAL *quasi-isomorphisms as weak equivalences. The inclusion on the* VERTICAL AXIS *defines a Quillen equivalence*

$$\textit{complexes} \rightleftarrows \textit{bicomplexes}.$$

Fibrations are surjections which are vertical quasi-isomorphisms in positive dimensions.

Homotopy theory of different kinds of complexes

$\mathcal{D}$ is a quadratic Koszul algebra and twisted complexes are the same as $\mathcal{D}_\infty$-modules.

Proposition

The category of twisted complexes has a model structure with total quasi-isomorphisms as weak equivalences and fibrations as in the previous slide. We also have Quillen equivalences

$$\textit{complexes} \rightleftarrows \textit{twisted complexes} \rightleftarrows \textit{bicomplexes}.$$

Twisted morphisms with projective source represent maps in

$$\mathrm{Ho}(\text{twisted complexes}).$$

Homotopy theory of biDGAs

A biDGA is a bicomplex with a compatible product. They yield examples of derived A_∞-algebras.

Theorem

The category of biDGAs has a model structure with the same weak equivalences and fibrations as in the total model structure for bicomplexes and there is a Quillen equivalence

$$\mathit{DGAs} \rightleftarrows \mathit{biDGAs}.$$

BiDGAs are algebras in graded complexes over an operad

$$d\mathcal{A}s = \mathcal{A}s \circ_\varphi \mathcal{D}.$$

Homotopy theory of biDGAs

Theorem (Livernet–Roitzheim–Whitehouse'13)

$d\mathcal{A}s$ is a quadratic Koszul operad of graded complexes and $d\mathcal{A}s_\infty$ is the operad for derived A_∞-algebras.

Theorem

The category of derived A_∞-algebras has a model structure with the same weak equivalences and fibrations as twisted complexes, derived ∞-morphisms with projective source represent maps in the homotopy category, and there is a Quillen equivalence

$$\text{derived } A_\infty\text{-algebras} \rightleftarrows \text{biDGAs}.$$

Proposition

Bicomplexes have yet another monoidal model structure:

- *weak equivalences are* E^2*-equivalences,*
- *fibrations are surjective horizontal quasi-isomorphisms which are also surjective on vertical cycles.*

A cofibrant replacement $\tilde{X}$ of a complex X concentrated in the vertical axis is a Cartan–Eilenberg resolution. Its vertical homology

$$H_*^v(\tilde{X})$$

is a projective resolution of $H_*(X)$.

The Cartan–Eilenbert model structure

There is a hierarchy of model structures on bicomplexes:

$$\text{Vertical} \to \text{Cartan–Eilenberg} \to \text{Total.}$$

Corollary

BiDGAs inherit a Cartan–Eilenberg model structure from bicomplexes.

A homotopical proof of Sagave's theorem

A DGA.

$\tilde{A}$ Cartan–Eilenberg cofibrant resolution (biDGA) $\tilde{A} \overset{\sim}{\twoheadrightarrow} A$.

We can therefore choose an SDR of graded complexes,

$$H_*^v(\tilde{A}) \underset{p}{\overset{i}{\rightleftarrows}} \tilde{A} \circlearrowleft_h$$

The transferred $d\mathcal{A}s_\infty$-algebra structure on the horizontal projective resolution $H_*^v(\tilde{A})$ of $H_*(A)$ given by Kontsevich–Soibelman's explicit formulas defines a minimal derived A_∞-algebra weakly equivalent to $\tilde{A}$, and hence to A,

$$H_*^v(\tilde{A}) \overset{\sim}{\dashrightarrow} \tilde{A} \overset{\sim}{\twoheadrightarrow} A.$$

We can replace $\mathcal{O} = \mathcal{A}s$ with any quadratic Koszul operad $\mathcal{O}$.

Bi-$\mathcal{O}$-algebras are $\mathcal{O}$-algebras in bicomplexes. They coincide with algebras in graded complexes over an operad

$$d\mathcal{O} = \mathcal{O} \circ_{\varphi} \mathcal{D}.$$

Derived $\mathcal{O}_\infty$-algebras are bigraded modules X such that $\mathrm{Tot}(X)$ is endowed with an $\mathcal{O}_\infty$-algebra structure compatible with the vertical filtration.

Theorem (Maes'16)

$d\mathcal{O}$ is a quadratic Koszul operad of graded complexes and $d\mathcal{O}_\infty$ is the operad for derived $\mathcal{O}_\infty$-algebras.

Generalizations

Theorem

There is a model structure on the category of derived $\mathcal{O}_\infty$-algebras with total quasi-isomorphisms as weak equivalences, derived ∞-morphisms with projective source represent maps in the homotopy category, and there is a zig-zag of Quillen equivalences

$$\text{derived } \mathcal{O}_\infty\text{-algebras} \rightleftarrows \text{bi-}\mathcal{O}\text{-algebras} \leftrightarrows \mathcal{O}\text{-algebras}.$$

Theorem

Given an $\mathcal{O}$-algebra A, there is a projective resolution of $H_*(A)$ with a minimal derived $\mathcal{O}_\infty$-algebra structure weakly equivalent to A.

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