PIICQ seminar

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Resurgent large genus asymptotics of intersection numbers

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A case study: $m!$

Counting problem: $c_m = \#\left\{ \begin{array}{c} \text{arrangements of } m \text{ distinct objects} \\ \text{into } m \text{ distinct boxes} \end{array} \right\}$

Solution: $c_m = m! = \begin{cases} m \cdot c_{m-1} & m > 1 \\ 1 & m = 1 \end{cases}$

Pro: exact
Con: recursive

Asymptotics: $c_m = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \left(1 + O(m^{-1})\right)$

Con: asymptotically exact
Pro: closed-form

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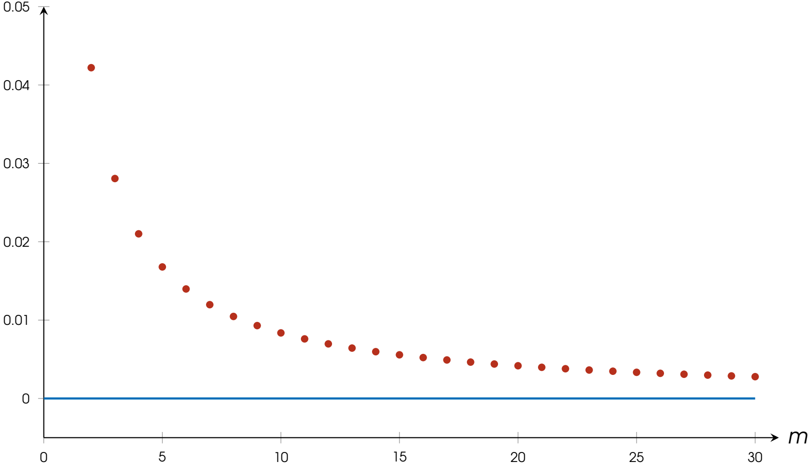
Visualising Stirling's formula

$$\frac{m!}{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m} - 1 = O(m^{-1})$$



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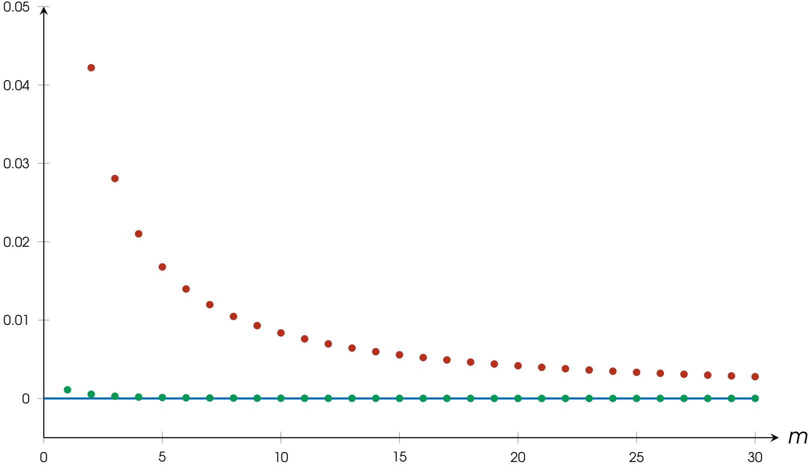

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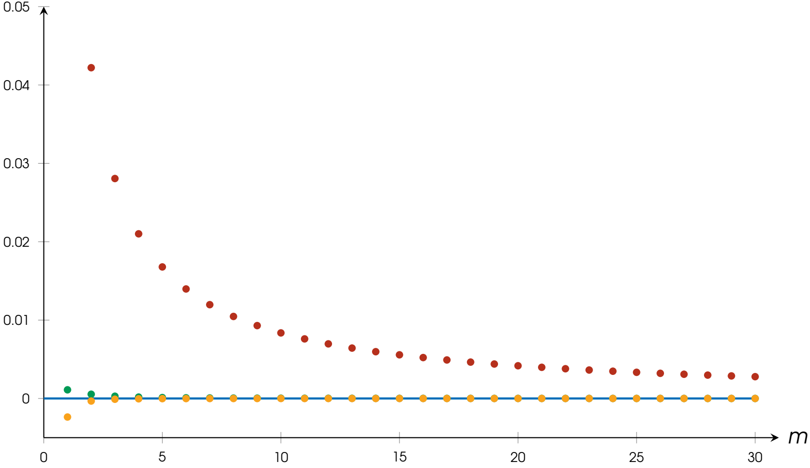

ψ-class intersection numbers

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \qquad d_i \geqslant 0, \qquad d_1 + \cdots + d_n = 3g - 3 + n$$

- Volumes of moduli spaces of metric ribbon graphs (maps)

$$V_{g,n}(L_1, \ldots, L_n) = \sum_{d_1 + \cdots + d_n = 3g-3+n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{i=1}^{n} \frac{L_i^{2d_i}}{2^{d_i} d_i!}$$

- Building block for all tautological intersection numbers:
  - Weil–Petersson volumes
  - Masur–Veech volumes
  - Hurwitz numbers
  - . . .
- Compute the perturbative expansion of topological 2d gravity

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Solution

Normalisation: $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle_g = \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{i=1}^{n} (2d_i + 1)!!$

Witten conjecture/Kontsevich theorem, early '90s:

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle_g = \sum_{m=2}^{n} (2d_m + 1) \langle\langle \tau_{d_1+d_m-1} \tau_{d_2} \cdots \widehat{\tau_{d_m}} \cdots \tau_{d_n} \rangle\rangle_g$$

$$+ \frac{1}{2} \sum_{a+b=d_1-2} \Big(\langle\langle \tau_a \tau_b \tau_{d_2} \cdots \tau_{d_n} \rangle\rangle_{g-1} + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{d_2,\dots,d_n\}}} \langle\langle \tau_a \tau_{I_1} \rangle\rangle_{g_1} \langle\langle \tau_b \tau_{I_2} \rangle\rangle_{g_2} \Big)$$

with initial data $\langle\langle \tau_0 \tau_0 \tau_0 \rangle\rangle_0 = 1$ and $\langle\langle \tau_1 \rangle\rangle_1 = \frac{1}{8}$.

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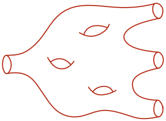

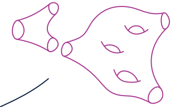

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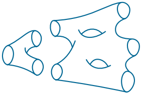

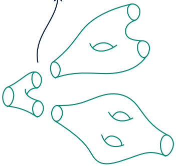

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Large genus asymptotics

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$:

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle_g = \frac{2^n}{4\pi} \frac{\Gamma(2g-2+n)}{\left(\frac{2}{3}\right)^{2g-2+n}} \left(1 + O(g^{-1})\right)$$

- Conjectured by Delecroix–Goujard–Zograf–Zorich, 2019
- Proved by Aggarwal, 2020
  (combinatorial/probabilistic analysis of Witten–Kontsevich topological recursion)
- Proved by Guo–Yang, 2021
  (combinatorial analysis of the determinantal formula)

Questions

- Universal strategy, adaptable to different problems?
- 'Geometric' meaning of the formula?
- Subleading corrections?

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Large genus asymptotics: new perspective

Answers

- Universal strategy: resurgent analysis of the determinantal formula
- Geometric meaning: Airy functions

$$y^2 - x = 0 \xrightarrow{\text{quantisation}} \left(\hbar^2 \frac{d^2}{dx^2} - x\right) \psi(x, \hbar) = 0$$

- Subleading corrections: algorithm + properties

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$S = 1$
Stokes constant

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$A = 2/3$

$\psi \sim \frac{1}{\sqrt{2}x^{1/4}} e^{\pm \frac{A}{\hbar} x^{-3/2}}$

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Computable; polynomial in n and multiplicities of d_i

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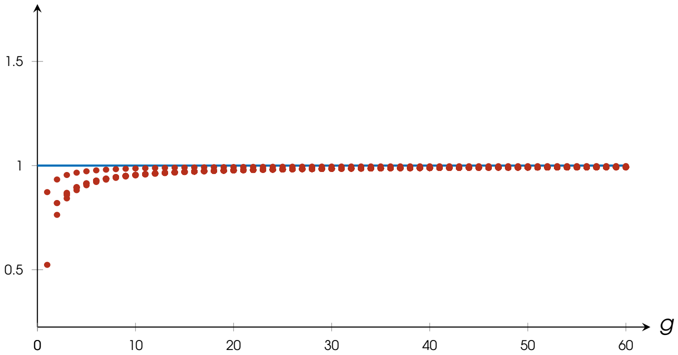
$\alpha_1 = -\frac{17-15n+3n^2}{12} - \frac{(3-n)(n-p_0)}{2} - \frac{(n-p_0)_2}{4}$

where $p_0 = \#\{d_i = 0\}$

Visualising the large genus asymptotics

$$\frac{\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle_g}{\frac{2^n}{4\pi} \frac{\Gamma(2g-2+n)}{(2/3)^{2g-2+n}}} = 1 + O(g^{-1})$$

For $n = 2$:



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$$\frac{2g-3+n}{2/3}\left(\frac{\langle\langle\tau_{d_1}\cdots\tau_{d_n}\rangle\rangle_g}{\frac{2^n}{4\pi}\frac{\Gamma(2g-2+n)}{(2/3)^{2g-2+n}}}-1\right)=\alpha_1(n,p_0)+O(g^{-1})$$

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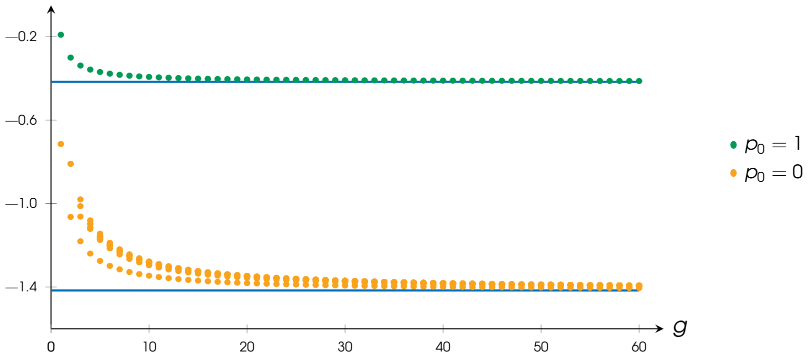

Borel meets Darboux

Borel's idea:

- Divergent power series:

$$\widetilde{\varphi}(\hbar) = \sum_{m \geqslant 0} a_m \hbar^m$$

with $|a_m| = O(R^{-m} m!)$.

- The Borel transform

$$\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$$

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Darboux's result: sketch of the proof

Take an abs. convergent power series: $\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$

Suppose its analytic continuation has a single log singularity at $s = A$:

$$\widehat{\varphi}(s) = (\text{holomorphic } @A) \log(s - A) + \text{holomorphic } @A$$

$$a_m = \frac{m!}{2\pi \mathrm{i}} \oint_{\mathcal{C}} \frac{\widehat{\varphi}(s)}{s^{m+1}} ds$$

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minor
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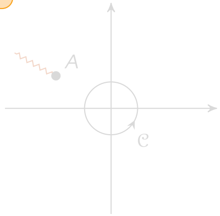

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$S \in \mathbb{C}$

minor
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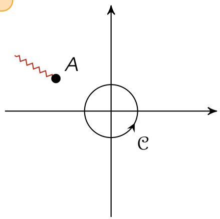

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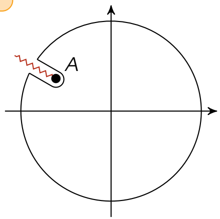

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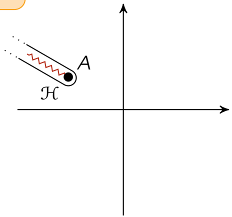

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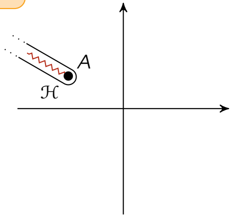

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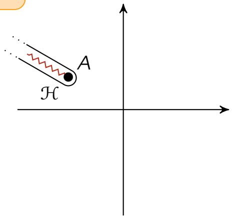

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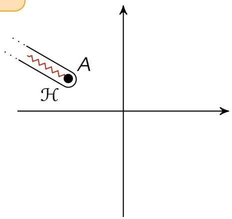

Borel meets Darboux: the algorithm

- Given: $\widetilde{\varphi}(\hbar) = \sum_m a_m \hbar^m$ divergent
- Borel transform: $\widehat{\varphi}(s) = \sum_m \frac{a_m}{m!} s^m$ abs. convergent
- Suppose you can compute:
  1. Log singularities: $A \in \mathrm{Sing}(\widehat{\varphi})$
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$$a_m = \frac{\Gamma(m)}{2\pi} \sum_{A \in \mathrm{Sing}(\widehat{\varphi})} \frac{S_A}{A^m} \left(b_{A,0} + \frac{A}{m-1} b_{A,1} + \frac{A^2}{(m-1)(m-2)} b_{A,2} + \cdots \right)$$

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Properties of the resurgence method

- **Algorithmic.**

$$\widetilde{\varphi} = \sum_m a_m \hbar^m \longrightarrow (S_A, \widehat{\psi}_A)_{A \in \mathrm{Sing}(\widehat{\varphi})} \longrightarrow \text{asymptotic of } a_m$$

- **Exponential integrals.** The singularity structure of exponential integrals is well-understood:

$$\widetilde{\varphi} = \mathrm{Asym}\left(\int e^{-\frac{1}{\hbar}S(t)}dt\right) \longrightarrow (S_A, \widehat{\psi}_A)_{A \in \mathrm{Sing}(\widehat{\varphi})}$$

- **Sums and products.** The singularity structure of sums and products of divergent series is well-understood:

$$\lambda_1 \widetilde{\varphi}_1 + \lambda_2 \widetilde{\varphi}_2 \longrightarrow (S_A^+, \widehat{\psi}_A^+)_{A \in \mathrm{Sing}(\widehat{\varphi}_1) \cup \mathrm{Sing}(\widehat{\varphi}_2)}$$

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Example: the Airy kernel

$$K(z,w;\hbar) = \frac{\mathrm{Ai}(z^2;\hbar)\mathrm{Bi}'(w^2;\hbar) - \mathrm{Ai}'(z^2;\hbar)\mathrm{Bi}(w^2;\hbar)}{z^2 - w^2} = \sum_{m\geqslant 0} a_m \hbar^m$$

$$= \frac{1}{2\sqrt{zw}(z-w)} - \frac{1}{(zw)^{3/2}}\Big(\frac{5}{96z^2} - \frac{7}{96zw} + \frac{5}{96w^2}\Big)\hbar$$
$$+ \frac{1}{(zw)^{3/2}}\Big(\frac{385}{9216z^5} - \frac{455}{9216z^4w} + \frac{385}{9216z^3w^2} - \frac{385}{9216z^2w^3} + \frac{455}{9216zw^4} - \frac{385}{9216w^5}\Big)\hbar^2 + \cdots$$

$\widehat{\mathrm{Ai}}(z^2;\hbar)$ and $\widehat{\mathrm{Ai}'}(z^2;\hbar)$ have

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$$\Longrightarrow \quad a_m = \frac{\Gamma(m)}{2\pi}\left(\frac{1}{(\frac{4}{3}z^3)^m}\frac{w-z}{2\sqrt{zw}(z^2-w^2)} + \frac{1}{(-\frac{4}{3}w^3)^m}\frac{z-w}{2\sqrt{zw}(z^2-w^2)} + \cdots\right)$$

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Example visualised

Write $a_m = \frac{(-1)^m}{(zw)^{3/2}} \sum_{k+\ell=3m-1} a_{k,\ell} \frac{1}{z^k(-w)^\ell}$.

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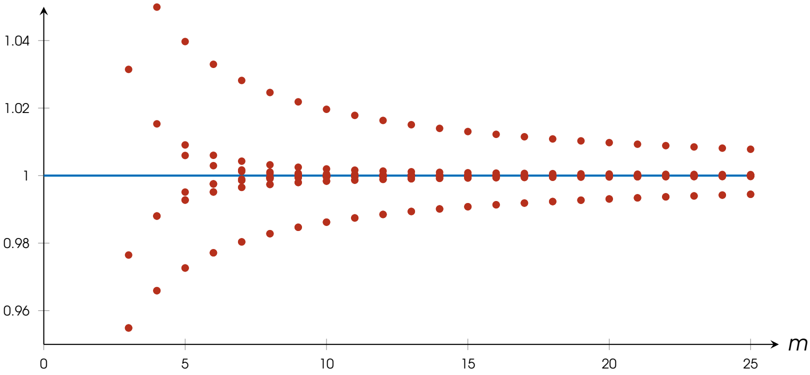

Strategy towards large genus asymptotics

Goal

Compute the large genus asymptotics of $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle_g$

1. Take the n-pnt fnct

$$W_n(z_1, \ldots, z_n; \hbar) = \sum_{g \geqslant 0} \hbar^{2g-2+n} W_{g,n}(z_1, \ldots, z_n)$$

2. W_n is a divergent series in $\hbar$. Take its Borel transform and study its singularity structure
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Determinantal formula

Define the disconnected n-pnt fnct and recall the Airy kernel

$$W_n^{\bullet}(z_1,\ldots,z_n;\hbar) = \sum_{P\in\mathrm{Part}(n)} W_{\ell(P)}(z_P;\hbar)\,,$$

$$K(z,w;\hbar) = \frac{\mathbf{Ai}(z^2;\hbar)\mathbf{Bi}'(w^2;\hbar) - \mathbf{Ai}'(z^2;\hbar)\mathbf{Bi}(w^2;\hbar)}{z^2-w^2}\,.$$

Determinantal formula (Eynard–Bergère, Bertola–Dubrovin–Yang):

$$W_n^{\bullet}(z_1,\ldots,z_n;\hbar) = \det_{1\leqslant i,j,\leqslant n} K(z_i,z_j;\hbar)$$

Example: $n=2$

$$W_2 = \frac{\mathrm{Ai}_1\mathrm{Bi}_1\mathrm{Ai}_2'\mathrm{Bi}_2' + \frac{1}{2}\mathrm{Ai}_1\mathrm{Bi}_1'\mathrm{Ai}_2\mathrm{Bi}_2' + \frac{1}{2}\mathrm{Ai}_1\mathrm{Bi}_1'\mathrm{Bi}_2\mathrm{Ai}_2'}{(z_1^2-z_2^2)^2} + (z_1 \leftrightarrow z_2)$$

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Singularity structure of W_n

Singularity strct of Ai, Bi $\Longrightarrow$ Singularity strct of W_n

- $2n$ log singularities of $\widehat{W}_n$, located at

$$+\tfrac{4}{3}z_i^3 \quad \text{and} \quad -\tfrac{4}{3}z_i^3, \qquad i = 1, \dots, n$$

- Stokes constants: $S = 1$
- Minors:
  - A at $+\frac{4}{3}z_i^3$: replace each $(\mathrm{Ai}_i, \mathrm{Ai}_i')$ with $(\mathrm{Bi}_i, \mathrm{Bi}_i')$
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Summary

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$:

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle_g = S \, \frac{2^n}{4\pi} \, \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n} \alpha_1 + \cdots + \frac{A^k}{(2g-3+n)_k} \alpha_k + O(g^{-k-1}) \right)$$

where:

- $S = 1$

  Stokes constants of the Airy ODE
- $A = 2/3$

  leading exp behaviour of Ai
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Bessel

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$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle_g^{\Theta} = \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^{n} \psi_i^{d_i} (2d_i+1)!!$$

$$= S \, \frac{2^n}{4\pi} \, \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \Bigg(1 + \frac{A}{2g-3+n} \alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}} \alpha_k + O(g^{-k-1}) \Bigg)$$

where:

- $S = 2$

  Stokes constants of the Bessel ODE
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r-Airy

Witten r-spin int. nmbrs (r-KdV τ-fnct (Faber–Shadrin–Zvonkine)):

$$\langle\langle \tau_{a_1,d_1}\cdots\tau_{a_n,d_n}\rangle\rangle_g^{r\text{-spin}} = \int_{\overline{\mathcal{M}}_{g,n}} c_W(a_1,\dots,a_n)\prod_{i=1}^{n}\psi_i^{d_i}(rd_i+a_i)!_{(r)}$$

$$= \frac{2^n}{2\pi}\frac{\Gamma(2g-2+n)}{r^{g-1-|d|}}\Bigg[\frac{S_{r,1}}{|A_{r,1}|^{2g-2+n}}\left(\alpha_0^{(r,1)}+\frac{|A_{r,1}|}{2g-3+n}\alpha_1^{(r,1)}+\cdots\right)$$

$$+\cdots$$

$$+\frac{S_{r,\lfloor\frac{r-1}{2}\rfloor}}{|A_{r,\lfloor\frac{r-1}{2}\rfloor}|^{2g-2+n}}\left(\alpha_0^{(r,\lfloor\frac{r-1}{2}\rfloor)}+\frac{|A_{r,\lfloor\frac{r-1}{2}\rfloor}|^K}{2g-3+n}\alpha_1^{(r,\lfloor\frac{r-1}{2}\rfloor)}+\cdots\right)$$

$$+\frac{\delta_r^{\text{even}}}{2}\frac{S_{r,\frac{r}{2}}}{|A_{r,\frac{r}{2}}|^{2g-2+n}}\left(\alpha_0^{(r,\frac{r}{2})}+\frac{|A_{r,\frac{r}{2}}|^K}{2g-3+n}\alpha_1^{(r,\frac{r}{2})}+\cdots\right)\Bigg]$$

where $S_{r,i}$, $A_{r,i}$, $\alpha_k^{(r,i)}$ are obtained the r-Airy ODE.

Thank you for the attention!