Elliptic Orthogonal Polynomials and their integrability

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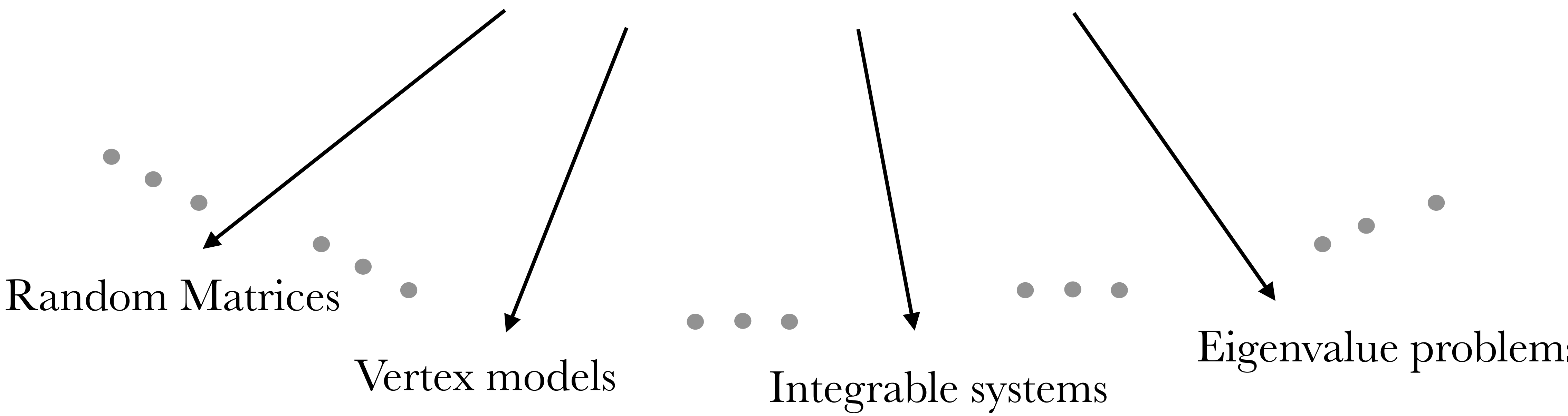
PIICQ Seminar

25th Nov, 2024

Joint work with T. L. Latimer, P. Roffelsen (Constructive approximation 2024, arXiv:2305.04404)

Motivation

Orthogonal polynomials are important to describe several phenomena in mathematics and physics.



Orthogonal polynomials

Definition: Orthogonal polynomials are a sequence of polynomials in z that are defined by the following inner product on an interval $[a, b]$ w.r.t the weight $\mathsf{w}(z)$:

$$\int_a^b P_m(z)P_n(z)\mathsf{w}(z)dz = h_n\delta_{mn}\,.$$

Colloquially, $P_n(z)$ has a zero of order n at $z = 0$:

$$P_n(z) \sim z^n(1 + \mathcal{O}(z))$$

Necessary ingredients

- Basis
- Interval [a,b]
- Weight

Examples:

1. Legendre polynomials: [-1,1] with $\mathsf{w}(z) = 1$
2. Laguerre polynomials: $[0,\infty)$ with $\mathsf{w}(z) = e^{-z}$
3. Hermite polynomials: $(-\infty, \infty)$ with $\mathsf{w}(z) = e^{-z^2}$

Properties of Orthogonal Polynomials

1. Moment representation
2. Three term recurrence relation
3. Riemann-Hilbert problem

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3. Riemann-Hilbert problem

1. Moments are defined as $m_k := \int_a^b z^k \mathrm{w}(z) dz$. The polynomials can then be written as

$$P_n = d_n \det \begin{pmatrix} m_0 & m_1 & \dots & m_n \\ m_1 & m_2 & \dots & m_{n+1} \\ \dots & \dots & \dots & \\ m_{n-1} & m_n & \dots & m_{2n-1} \\ 1 & z & \dots & z^n \end{pmatrix}$$

This uncovers the Hankel determinant representation.

2. All OPs satisfy the three term recurrence relation

$$zP_n(z) = a_n P_{n+1}(z) + b_n P_n(z) + c_n P_{n-1}(z)\,.$$

3. Riemann-Hilbert problem

Statement: The polynomials P_n appear as 11 entries of the solution of the RHP given by

$$Y_n(x) = \begin{pmatrix} P_n(x) & \mathscr{C}(P_n)(x) \\ \frac{2\pi i}{h_{n-1}} P_{n-1}(x) & \frac{2\pi i}{h_{n-1}} \mathscr{C}(P_{n-1})(x) \end{pmatrix}.$$

Was introduced in the seminal work of Deift, Kricherbauer, McLaughlin, Venakides, Zhou with contributions from many many others…

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Upshot:

- Let us obtain asymptotic behaviour of the polynomials for $n \to \infty$,
- Reveals integrable structures
- Several applications to Random matrices: key to obtain universality results among others

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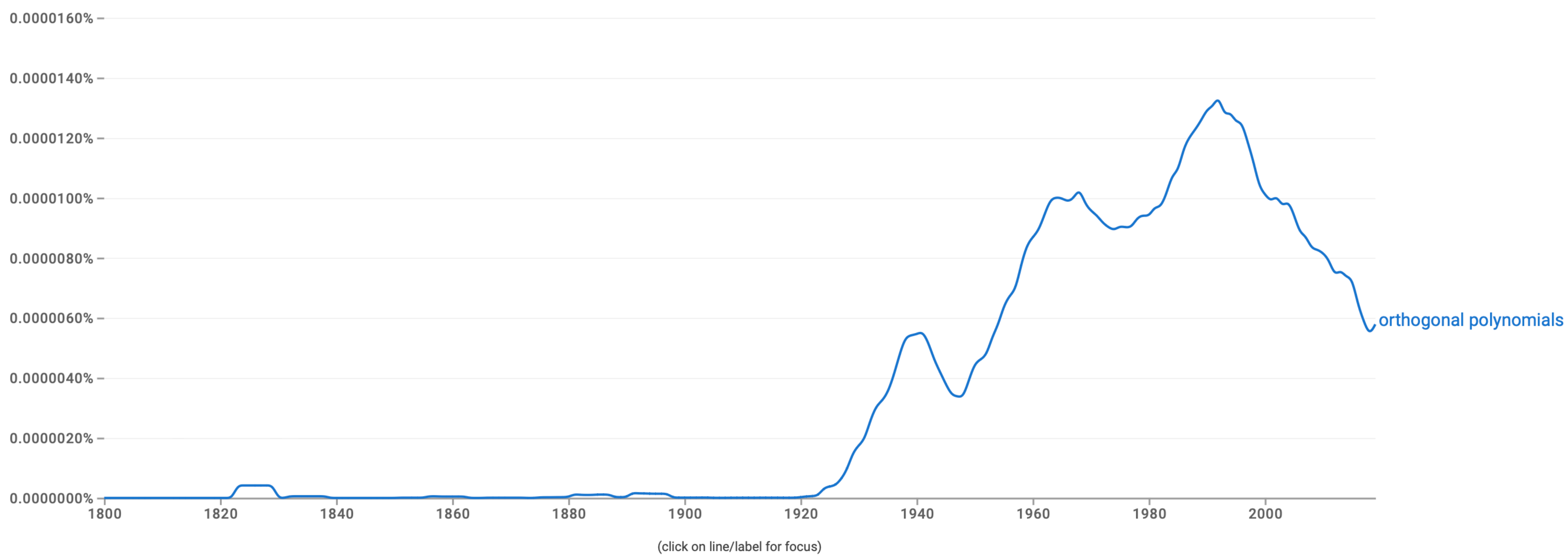

*only counts articles with ‘orthogonal polynomials’ in their title

To the elliptic world…

Systems on higher genus surfaces:

- Reveals highly non-trivial, topological properties of the system. For example, some properties of tau-function become apparent when we go to genus 1.

  F. Del Monte, H.D, P. Gavrylenko; Isomonodromic tau functions on a torus as Fredholm determinants, and charged partitions. CMP '22; arXiv:2011.06292

- Systems on higher genus can define a 'master' class of equations. The Landau-Lifshitz equation is one such example.

  H.D, A. Its, A. Prokhorov; Nonlinear steepest descent on a torus: a case study of the Landau-Lifshitz equation, arXiv: 2405.17662

- We are discovering several structures that want us to study elliptic generalisations: see for example Bergenn-Borodin, Kuijlaars-Piorkowski, …

History of Elliptic Orthogonal Polynomials

1945-70 — Rees, Carlitz, Heine, …

Generalized Jacobi polynomials are related to elliptic functions

2000s — Ismail, Valent, Yoon; Its, Chen; Vinet, Zhedanov; Nijhoff…

Generalization of Chebyshev polynomials, Akhiezer polynomials, …

2005-15 — Basor, Chen, Dai, Erhardt…

Generalized Jacobi polynomials and Painlevé equations

2019-21 — Bertola, Groot, Kuijlaars

Bi-orthogonality, Nonlinear steepest descent for polynomials on elliptic curves

2021-23 — Fasondini, Olver, Xu

Bi-orthogonal polynomials on algebraic curves and numerical methods

Summary

Step 1: Construction of EOPs

1. Basis
2. Support
3. Weight

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Step 2: Properties

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Step 2: Properties

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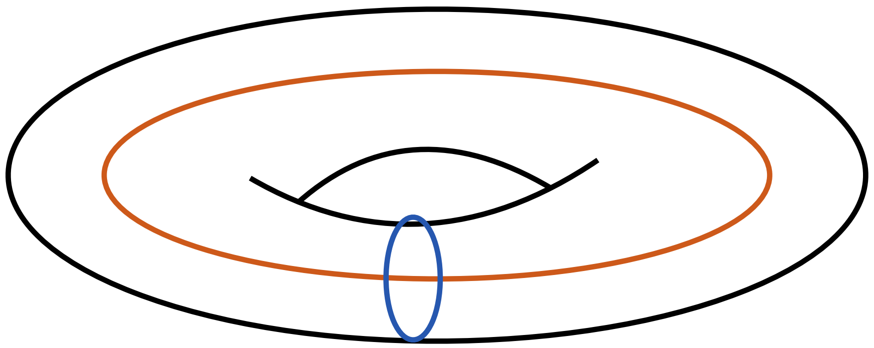
Step 3: Consequence

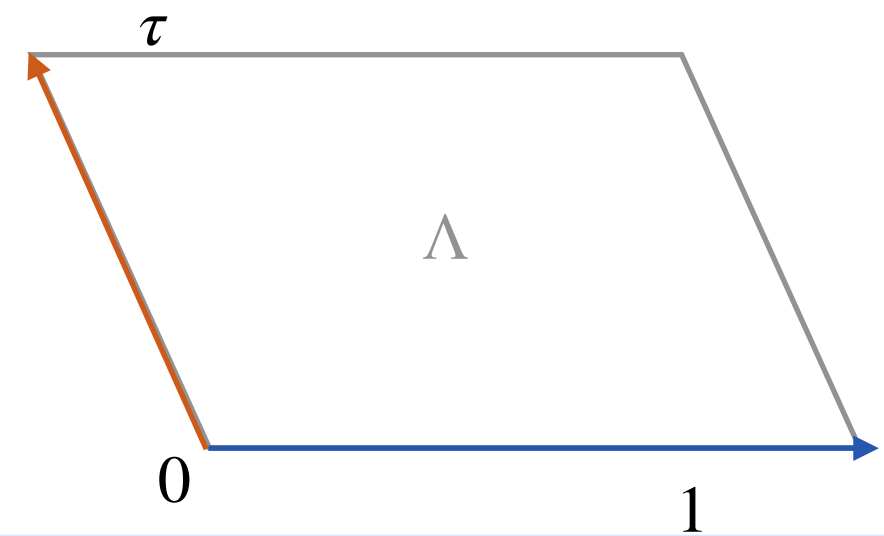
1. Recurrence relation
2. Integrability

Sanity check: Even polynomials give back the cases we already know!

Step 1: Construction of EOPs

There are no holomorphic functions on the torus. So what do we mean by polynomials?





Statement: Given an elliptic curve

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau),$$

a basis of functions on the surface is constructed in terms of x, y.

Basis: $\{1, x, y, x^2, xy, y^2, \ldots\}$

So, it is appropriate to consider polynomials in x, y.

For doubly periodic functions, $x = \wp(z), \quad y = \wp'(z)$.

Step 1: Construction of EOPs

Weirestrass $\wp$ function is defined on the lattice Λ as

$$\wp(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

Properties:

1. It is doubly periodic

$$\wp(z) = \wp(z+1) = \wp(z+\tau).$$

2. The only singularity is a double pole at $z = 0$ and consequently, it has two zeros.

3. It is an even function

$$\wp(-z) = \wp(z).$$

Note: Elliptic functions are unique once we specify the poles, zeros, and periodicity.

Step 1: Construction of EOPs

Weirestrass $\wp'$ function is defined on the lattice Λ as

$$\wp'(z) := -\frac{2}{z^3} + \sum_{\lambda \in \Lambda \setminus \{0\}} -\left(\frac{2}{(z-\lambda)^3} - \frac{1}{\lambda^2}\right).$$

Properties:

1. It is doubly periodic

$$\wp'(z) = \wp'(z+1) = \wp'(z+\tau).$$

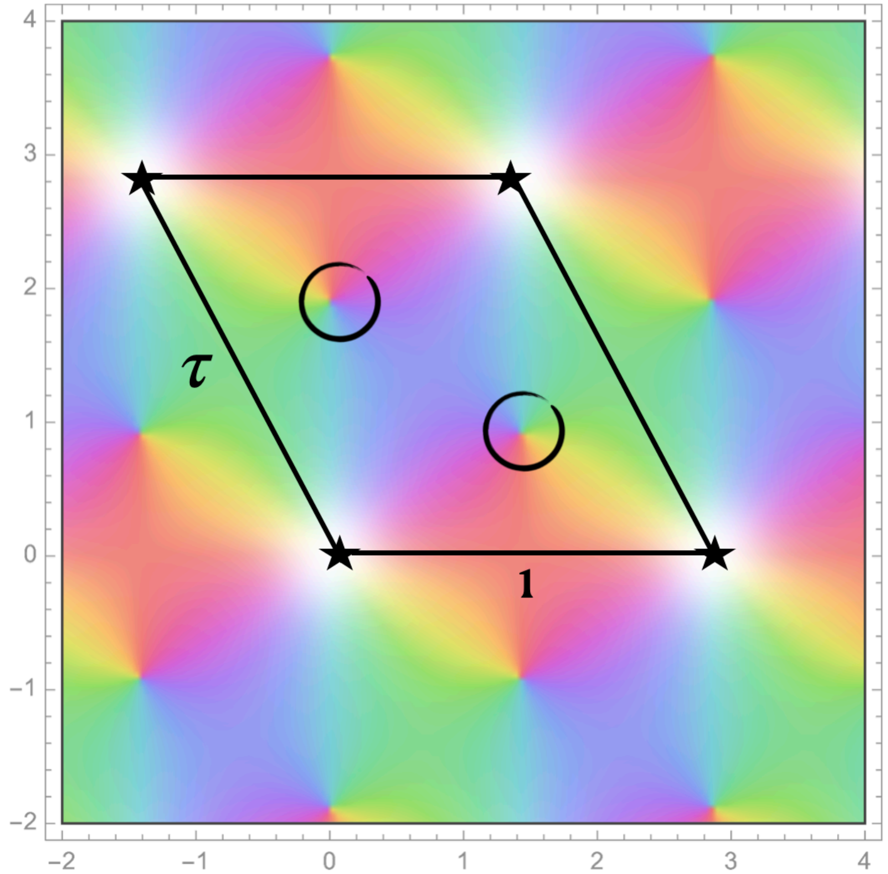
2. The only singularity is a triple pole at $z = 0$ and consequently, it has three zeros.

3. It is an odd function

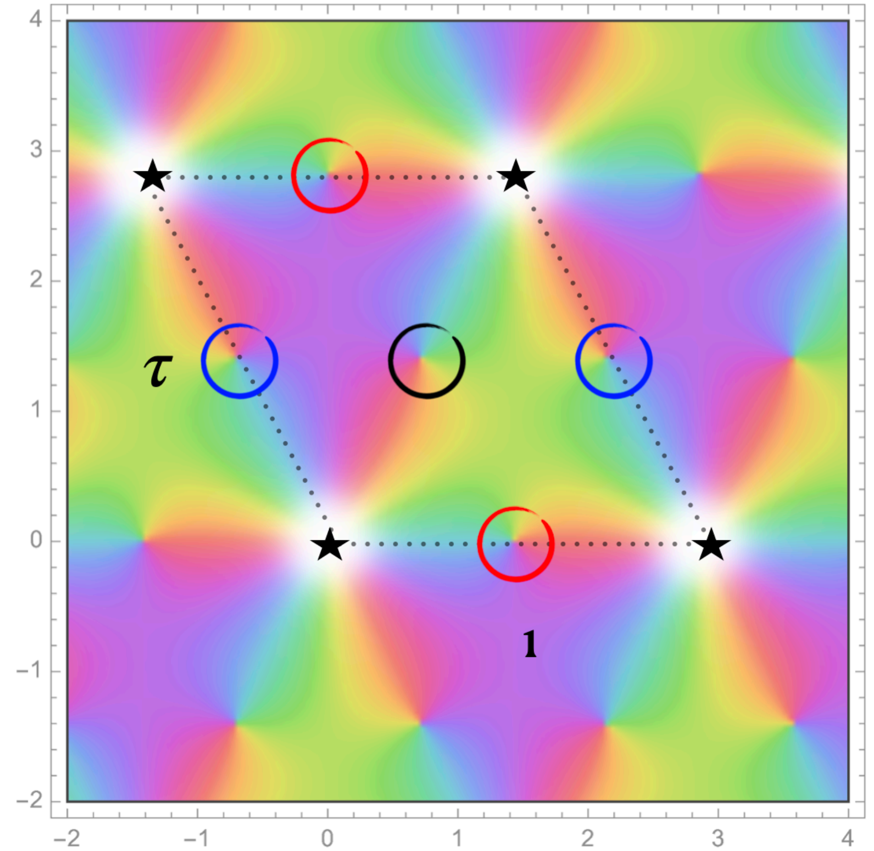
$$\wp'(-z) = -\wp'(z).$$

Step 1: Construction of EOPs

Pictures!



$\wp(z)$



$\wp'(z)$

Step 1: Construction of EOPs

We can now make our basis precise:

$$\mathscr{B} = \{\mathscr{E}_n\}_{n\geq 0, n\neq 1}, \qquad \mathscr{E}_{2k} = \wp(z)^k, \qquad \mathscr{E}_{2k+3} = -\frac{1}{2}\wp'(z)\wp(z)^k, \qquad k \geq 0,$$

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Properties:

1. The only poles are at $z = 0$ and have the degree

$$\mathscr{E}_{2k} \sim z^{-2k}, \qquad \mathscr{E}_{2k+3} \sim z^{-2k-3}.$$

2. The basis is doubly periodic, the even and odd modes are even and odd respectively.

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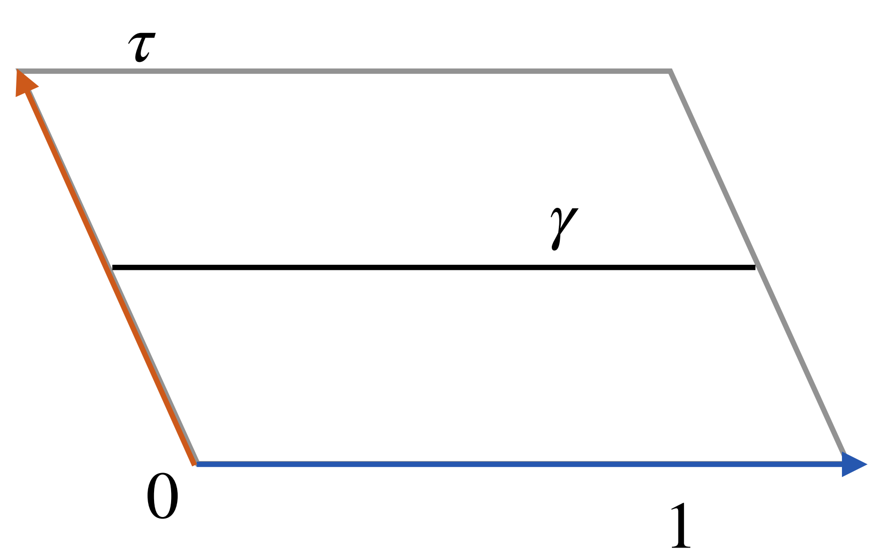
Support:



Weight:

Generic weight is too difficult at the moment.

Manageable case: n independent, even weight.

Step 1: Construction of EOPs

Definition: Elliptic Orthogonal polynomials are a sequence of polynomials in x, y that are defined by the following inner product on the interval γ w.r.t the weight $\mathsf{w}(z)$:

$$\int_{\gamma} \pi_m(z)\pi_n(z)\mathsf{w}(z)dz = h_n\delta_{mn}\,.$$

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$$\int_\gamma \pi_m(z)\pi_n(z)\mathsf{w}(z)dz = h_n\delta_{mn}\,.$$

In other words, $\pi_n(z)$ has a pole of order n at $z = 0$:

$$\pi_n(z) \sim z^{-n}(1 + \mathcal{O}(z))\,.$$

Note: $\pi_1(z)$ does not exist

Task for the future: Systematic classification of EOPs

Assumptions: weight is strictly positive, and time is pure imaginary so that $\wp(z)$ is real on γ.

Step 2: Properties

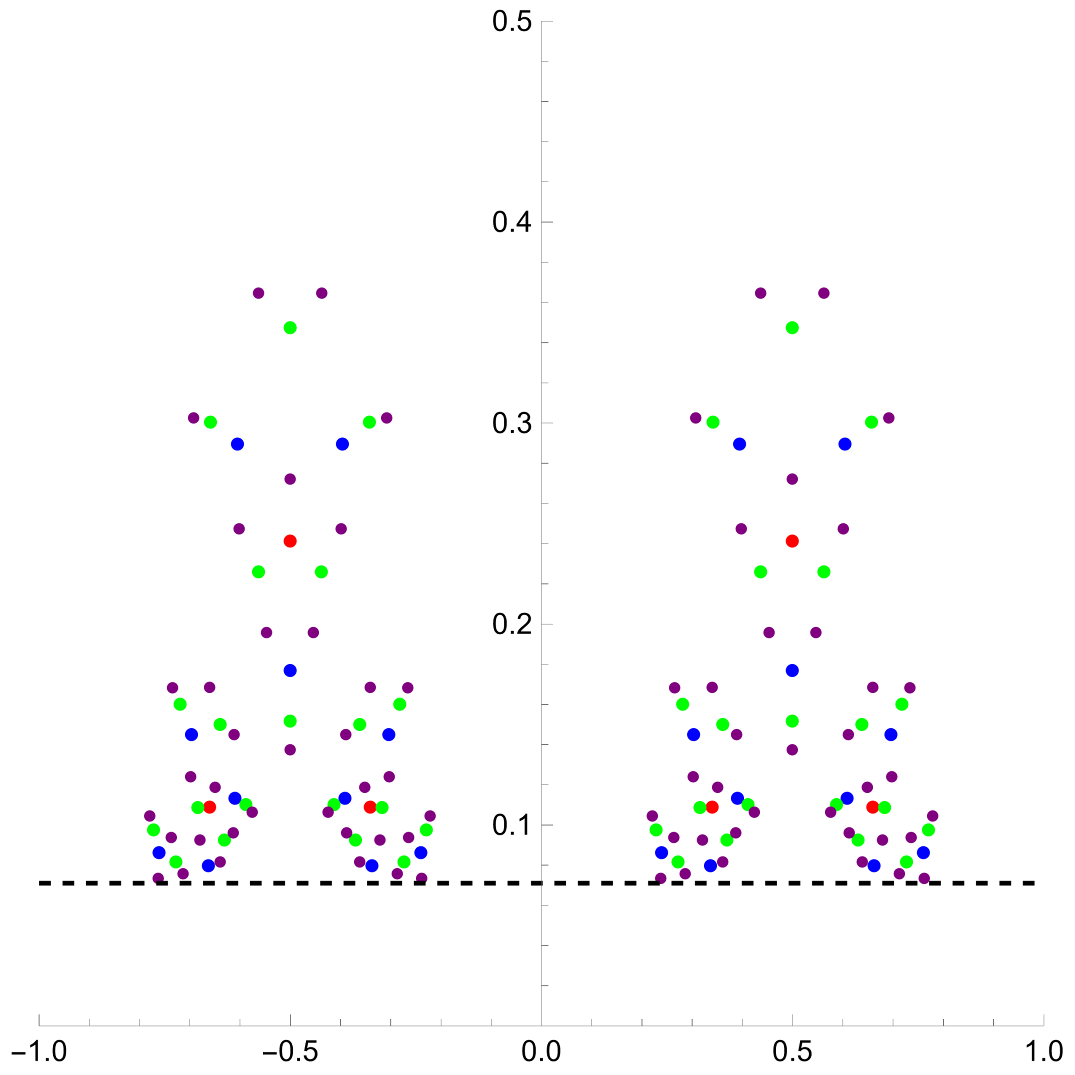
Much like OPs, EOPs have a moment representation.

Definition: $\mu_{i,j} := \int_\gamma \mathscr{E}_i(z)\mathscr{E}_j(z)\mathrm{w}(z)dz \qquad (i,j \in \mathbb{N}_{\neq 1})$

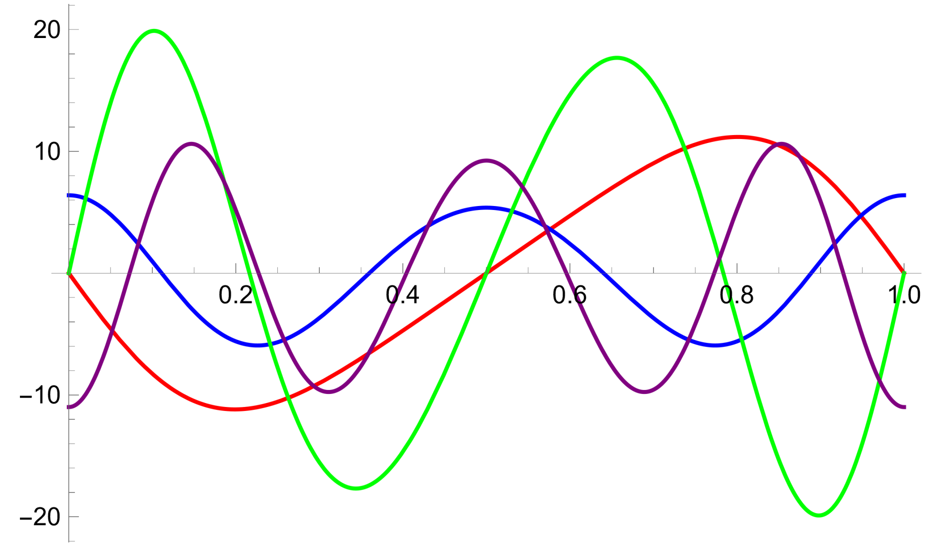
$\mu_{0,0}$	$\mu_{0,2}$	0	$\mu_{0,4}$	0	$\mu_{0,6}$	0	$\mu_{0,8}$
$\mu_{2,0}$	$\mu_{2,2}$	0	$\mu_{2,4}$	0	$\mu_{2,6}$	0	$\mu_{2,8}$
0	0	$\mu_{3,3}$	0	$\mu_{3,5}$	0	$\mu_{3,7}$	0
$\mu_{4,0}$	$\mu_{4,2}$	0	$\mu_{4,4}$	0	$\mu_{4,6}$	0	$\mu_{4,8}$
0	0	$\mu_{5,3}$	0	$\mu_{5,5}$	0	$\mu_{5,7}$	0
$\mu_{6,0}$	$\mu_{6,2}$	0	$\mu_{6,4}$	0	$\mu_{6,6}$	0	$\mu_{6,8}$
0	0	$\mu_{7,3}$	0	$\mu_{7,5}$	0	$\mu_{7,7}$	0
$\mu_{8,0}$	$\mu_{8,2}$	0	$\mu_{8,4}$	0	$\mu_{8,6}$	0	$\mu_{8,8}$

The odd and even moments form a checkerboard pattern.

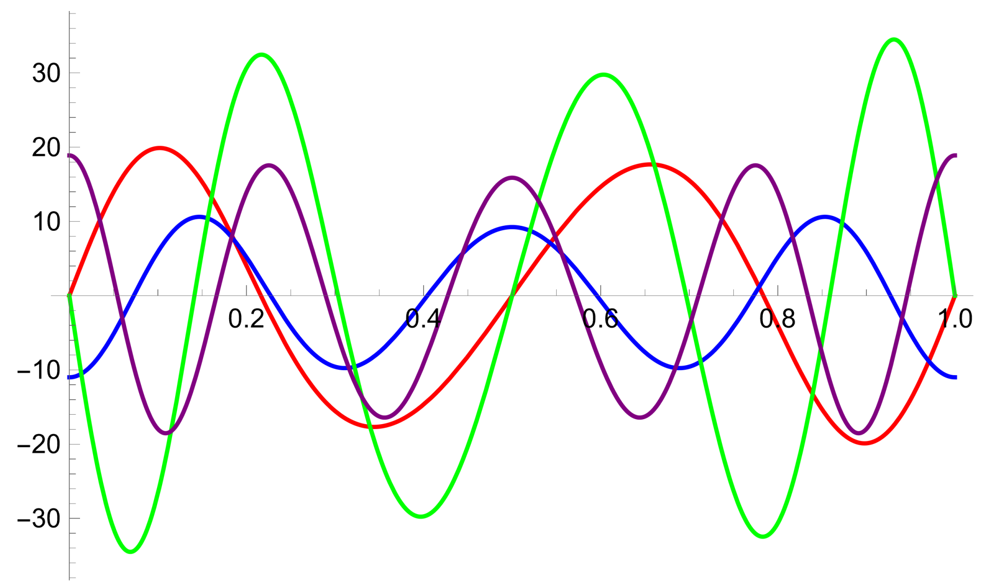
Step 2: Properties



Zeros of moment matrices for $n = 2,3,4,5$



Plots of $\pi_n(z)$ for $n = 2,3,4,5$



Plots of $\pi_n(z)$ for $n = 6,7,8,9$

Step 2: Properties

Riemann-Hilbert problem:

$$Y_n(z,\tau) = \begin{pmatrix} \pi_n(z) & \mathscr{C}(\pi_n)(z) \\ \frac{2\pi i}{h_{n-1}}\pi_{n-1}(z) & \frac{2\pi i}{h_{n-1}}\mathscr{C}(\pi_{n-1})(z) \end{pmatrix}.$$

- $Y_n(z,\tau)$ is analytic in $z \in \mathbb{T}\backslash(\gamma \cup \{0\})$.
- For $z \in \gamma$ the following jump condition holds

$$Y_{n,+}(z,\tau) = Y_{n,+}(z,\tau)\begin{pmatrix} 1 & \mathsf{w}(z) \\ 0 & 1 \end{pmatrix}.$$

- In the asymptotic limit $z \to 0$:

$$Y_n(z,\tau) = (1+\mathcal{O}(z))\begin{pmatrix} z^{-n} & 0 \\ 0 & z^{n-2} \end{pmatrix}.$$

- It is doubly periodic by definition.

Step 2: Properties

Riemann-Hilbert problem:

$$Y_n(z,\tau) = \begin{pmatrix} \pi_n(z) & \mathscr{C}(\pi_n)(z) \\ \frac{2\pi i}{h_{n-1}}\pi_{n-1}(z) & \frac{2\pi i}{h_{n-1}}\mathscr{C}(\pi_{n-1})(z) \end{pmatrix}.$$

Differences with the genus 0 case:

1. The Cauchy kernel needs to be generalised to genus 1:

$$C(w,z) = \zeta(w-z) - \zeta(w)\,.$$

2. The determinant is now z dependent. For instance, if the weight is doubly periodic,

$$\det Y_n(z,\tau) = \wp(z,\tau) + \alpha_n(\tau) =: f_n, \qquad \alpha_n := c_{2,n} + \widetilde{c}_{2,n-1} - \frac{h_n}{h_{n-1}}\,.$$

Theorem (D. - Latimer - Roffelsen '24): The solution to the RHP exists and is unique.

Step 2: Properties

Recovering the usual OPs (sanity check):

When the weight function is even, one can split the EOPs into even and odd parts respectively

	$\pi_{2k}(z)$	Change of variable $\wp(z) \to z$	$p_n(z)$
Basis:	$\{1, \wp(z), \wp^2(z), \dots\}$	$\longrightarrow$	$\{1, z, z^2, \dots\}$
Weight:	$\mathsf{w}(z) = 1$	$\longrightarrow$	$\mathsf{w}(z) = ((z-a)(z-b)(z-c))^{-1/2}$
			Modified Jacobi weight

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Weight:	$\mathsf{w}(z) = 1$	$\longrightarrow$	$\mathsf{w}(z) = ((z-a)(z-b)(z-c))^{-1/2}$
			Modified Jacobi weight

Upshot:
Easier to study than Modified Jacobi weight

References:

- Carlitz 1960
- Basor- Chen- Haq '14

Step 3: Theorem (D. - Latimer - Roffelsen '24)

Both Recurrence relation and Integrability come from obtaining the linear systems for the solution of the RHP.

Difference equation: $$Y_{n+1} = R_n Y_n, \qquad R_n = \frac{1}{f_n}\begin{pmatrix} -\wp'(z)/2 & -\frac{h_n}{2\pi i} f_{n+1} \\ \frac{2\pi i}{h_n} f_n & 0 \end{pmatrix}.$$

Sketch of the proof:

- Compute the quantity $Y_{n+1} Y_n^{-1} \det Y_n$.
- The asymptotic behaviour of Y_n fixes the analytic behaviour of R_n.
- The double periodicity of Y_n and $\det Y_n$ determines the elliptic function in the 11 entry.

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3- term recurrence relation: $\pi_{n+1} = -\frac{\wp'(z)\pi_n}{2f_n} - \frac{\beta_n f_{n+1}}{f_n}\pi_{n-1}, \qquad \beta_n := \frac{h_n}{h_{n-1}}.$

Even polynomials (sanity check): $\pi_{n+2} = \left(\wp(z) - B_n\right)\pi_n - \beta_n \beta_{n-1}\pi_{n-2},$

Gives back the usual relations we know for genus 0 case.

For doubly periodic weight

Step 3: Theorem (D. - Latimer - Roffelsen '24)

Differential equation: $$Y_n' = L_n Y_n, \qquad n \geq 3,$$

$$L_n = \frac{1}{f_n}\begin{pmatrix} n\wp'(z)/2 & \frac{h_n}{2\pi i}\left((n-1)f_n + nf_{n+1}\right) \\ \frac{2\pi i}{h_{n-1}}\left((2-n)f_{n-1} + (1-n)f_n\right) & (2-n)\wp'(z)/2 \end{pmatrix}.$$

Second order differential equation for EOPs:

$$\pi_n'' = \left(\frac{\wp'}{f_n} + n\left(\frac{f_{n+1}}{f_n}\right)'\left((n-1) + n\frac{f_{n+1}}{f_n}\right)^{-1}\right)\pi_n'$$
$$+\left(\left(\frac{n\wp'}{2f_n}\right)' - n\left(\frac{f_{n+1}}{f_n}\right)'\frac{n\wp'}{2\left((n-1)f_n + nf_{n+1}\right)} - \det L_n\right)\pi_n\,.$$

For constant weight

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Differential equation: $Y_n' = L_n Y_n, \qquad n \geq 3,$

$$L_n = \frac{1}{f_n} \begin{pmatrix} n\wp'(z)/2 & \frac{h_n}{2\pi i}\left((n-1)f_n + nf_{n+1}\right) \\ \frac{2\pi i}{h_{n-1}}\left((2-n)f_{n-1} + (1-n)f_n\right) & (2-n)\wp'(z)/2 \end{pmatrix}.$$

Even polynomials (sanity check):

1. We can obtain a linear system w.r.t 'time' τ : $\partial_\tau Y_{2n} = M_{2n} Y_{2n}$.

2. The above two equations form the Lax pair for the elliptic form of Painlevé VI equation (Hitchin case).

3. Consequently, the solutions and tau-function of elliptic form of Painlevé VI equation can be written in terms of Hankel determinants of moments

For constant weight

Step 3: Theorem (D. - Latimer - Roffelsen '24)

Compatibility condition and the differential-difference equation:

$$R_n' - L_{n+1}R_n + R_nL_n = 0,$$

⬇

$$\beta_n = \frac{g_3 - g_2\alpha_n + 4\alpha_n^3}{4(\alpha_{n-1} - \alpha_n)(\alpha_n - \alpha_{n+1})},$$

$$\alpha_{n+1} = \frac{(1-n)\alpha_n\left(4\alpha_n^3 - 3g_2\alpha_n + 4g_3\right) - \alpha_{n-1}\left(4(n-2)\alpha_n^3 + ng_2\alpha_n - (2n-1)g_3\right)}{4n\alpha_n^3 + (n-1)\alpha_{n-1}\left(g_2 - 12\alpha_n^2\right) + g_2(n-2)\alpha_n - g_2(2n-3)}.$$

Task for the future: Find some structure in above equations and find some nice equations

Step 3: Theorem (D. - Latimer - Roffelsen '24)

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C'est terrible ! It's awful!

C'est terrible ! It's great!

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Step 3: Theorem (D. - Latimer - Roffelsen '24)

Compatibility condition and the differential-difference equation:

$$R_n' - L_{n+1}R_n + R_nL_n = 0,$$

$$\Downarrow$$

$$\beta_n = \frac{g_3 - g_2\alpha_n + 4\alpha_n^3}{4(\alpha_{n-1} - \alpha_n)(\alpha_n - \alpha_{n+1})},$$

$$\alpha_{n+1} = \frac{(1-n)\alpha_n\left(4\alpha_n^3 - 3g_2\alpha_n + 4g_3\right) - \alpha_{n-1}\left(4(n-2)\alpha_n^3 + ng_2\alpha_n - (2n-1)g_3\right)}{4n\alpha_n^3 + (n-1)\alpha_{n-1}\left(g_2 - 12\alpha_n^2\right) + g_2(n-2)\alpha_n - g_2(2n-3)}.$$

Even polynomials (sanity check): It gives some discrete Painlevé equation.

Task for the future: Find some structure in above equations and find some nice equations

Conclusion

Takeaways:

- Systematic study of EOPs using Riemann-Hilbert problems
- Recurrence relations and distribution of zeros of EOPs
- Integrable structures underlying EOPs

Open problems:

- Asymptotic analysis of EOPs
- Systematic classifiction of EOPs with complicated weights
- Higher geus extensions
- Applications to Random Matrices
- Possible universality results (something may be in the works)…

Thanks for listening!