Positive formula for the product of conjugacy classes on the unitary group

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(joint work with Quentin François)

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- For $\theta = (\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n) \in \mathcal{H}$,

$$\mathcal{O}(\theta) := \left\{ U \underbrace{\begin{pmatrix} e^{2i\pi\theta_1} & 0 & \cdots & \\ 0 & e^{2i\pi\theta_2} & & \\ \vdots & & \ddots & \\ & & & e^{2i\pi\theta_n} \end{pmatrix}}_{e^{2i\pi\theta}} U^*, \; U \in U_n \right\}.$$

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- Group structure on $U_n \leftrightarrow$ Convolution $* : \mathcal{M}_1(\mathcal{H}) \times \mathcal{M}_1(\mathcal{H}) \to \mathcal{M}_1(\mathcal{H})$
- $\delta_\theta * \delta_{\theta'} =$ Distribution of $p(U_\theta U_{\theta'})$ where
  - U_θ (resp. $U_{\theta'}$) is sampled uniformly on $\mathcal{O}(\theta)$ (resp. $\mathcal{O}(\theta')$),
  - $p : U_n \to \mathcal{H}$ maps an element of U_n to its conjugacy class in $\mathcal{H}$.

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  - $p : U_n \to \mathcal{H}$ maps an element of U_n to its conjugacy class in $\mathcal{H}$.
- If $n \geq 3$ and $\theta, \theta' \in \mathcal{H}_{reg}$, $\delta_\theta * \delta_{\theta'}$ has a density $f_{\theta,\theta'}(\tau)$ on $\mathcal{H}$.

Objective : compute $f_{\theta,\theta'}(\tau)$.

Moduli space of flat connections on a Riemann surface

- S Riemann surface of genus g, $p_1, \ldots, p_r \in S$, $M = S \setminus \{p_1, \ldots, p_r\}$.
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- $\omega \in \Omega^1(M, \mathfrak{u}_n)$ ($\mathfrak{u}_n$-valued connection with $\mathfrak{u}_n$ Lie-algebra of U_n)

$\rightsquigarrow$ Holonomy $\omega[\gamma] \in U_n$ for all closed path $\gamma : [0,1] \to M$ based at $x \in M$.

- $\sigma \in C^\infty(M, U_n)$ acts on $\Omega^1(M, \mathfrak{u}_n)$ by $\sigma \cdot \omega(x) = Ad_{\sigma(x)}\omega(x) + \sigma(x)^{-1} d\sigma_x$.

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- Curvature form : $F : \Omega^1(M, \mathfrak{u}_n) \to \Omega^2(M, \mathfrak{u}_n)$, $F(\omega) = d\omega + \frac{1}{2}[\omega, \omega]$.
  - if $F(\omega) = 0$, $\omega[\gamma] = 1_{U_n}$ for γ contractible.
  - if $F(\omega) = 0$, $F(\sigma.\omega) = 0$ for $\sigma \in C^\infty(M, U_n)$.

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  - if $F(\omega) = 0$, $F(\sigma.\omega) = 0$ for $\sigma \in C^\infty(M, U_n)$.
- Moduli space of flat connections : $\theta^{(1)}, \ldots, \theta^{(r)} \in \mathcal{H}$, γ_i small loop around p_i.

$$\mathcal{M}(M, \theta^{(1)}, \ldots, \theta^{(r)}) = \left\{\omega \in \Omega^1(M, \mathfrak{u}_n),\ \omega[\gamma_i] \in \mathcal{O}(\theta^{(i)}),\ F(\omega) = 0\right\} / \mathcal{G}.$$

$\rightsquigarrow$ smooth finite dimensional manifold when $2g - 2 + r \geq 0$ and $\theta^{(1)}, \ldots, \theta^{(r)}$.

The volume form on moduli space

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- Algebraic view point π_1 : $\pi_1(M)$ generated by $\{a_s, b_s,\ s \leq g, c_s,\ s \leq r\}$,

$$\mathcal{M}(M,\theta^{(1)},\dots,\theta^{(r)}) \simeq \{\text{Rep } \rho : \pi_1 \to U_n, \rho(c_i) \in \mathcal{O}_{\theta_i}, 1 \leq i \leq n\}/U_n.$$

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- Symplectic view point (Atiyah and Bott, 1986) : $\Omega^1(M,\mathfrak{u}_n)$ is an infinite dimensional symplectic space quotiented by $\mathcal{G}$.

$\rightsquigarrow$ Sym. reduction theorem: $v_g(\theta^{(1)},\ldots,\theta^{(r)}) := Vol(\mathcal{M}(M,\theta^{(1)},\ldots,\theta^{(r)}))$.

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Example

If $g=0,\ r=3,\ \pi_1 = \langle c_1,c_2,c_3 | c_1c_2c_3 = 1\rangle$:

$$\begin{aligned}\mathcal{M}(M,\theta^{(1)},\theta^{(2)},\theta^{(3)}) &\simeq \{c_i \in \mathcal{O}_{\theta_i}, 1 \leq i \leq 3 | c_1c_2c_3 = 1\}/U_n \\ &\simeq \left\{g_i \in \mathcal{O}_{\theta_i}, 1 \leq i \leq 3 | c_1c_2 = c_3^{-1}\right\}/U_n\end{aligned}$$

and

$$v_0(\theta^{(1)},\theta^{(2)},\theta^{(3)}) = \frac{f_{\theta^{(1)},\theta^{(2)}}(-\theta^{(3)})}{Vol(\mathcal{O}(-\theta^{(3)}))}.$$

Approach of Witten and patching of pants

- Yang-Mills measure on (M, G) : random element of $\Omega^1(M, \mathfrak{u}_n)$ with probability

$$\frac{1}{\mathcal{Z}_t} \exp\left(-\frac{1}{t}\int_M |F(\omega)|^2\right)$$

with $\mathcal{Z}_t = \int_{\mathcal{M}(M)} \exp\left(-\frac{1}{t}\int_M |F(\omega)|^2\right) d\omega$ partition function of the theory.

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▶ Rigorous definition by random holonomies (Levy, 2000) with distribution invariant by conjugation : $p_t^{c_1,\ldots,c_r}(\theta^{(1)},\ldots,\theta^{(r)})$ (joint law of random holonomies aroung $c_1,\ldots,c_r$)

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- Pants/disks decomposition of $M \rightsquigarrow$ computation of $v_g((\theta^{(1)},\dots,\theta^{(r)})$ reduces to the ones $v_0(\theta^{(1)},\theta^{(2)},\theta^{(3)})$ and $v_0(\theta)$.

$$v_0(\theta^{(1)},\theta^{(2)},\theta^{(3)}) = \frac{1}{Vol(\mathcal{O}(-\theta^{(3)}))} f_{\theta^{(1)},\theta^{(2)}}(\theta^{(3)}) = C\sum_{\lambda Irr}\frac{1}{\dim\lambda}\prod_{i=1}^{3}\chi_\lambda(\theta^{(i)}).$$

$\rightsquigarrow$ Witten's formula : $v_g((\theta^{(1)},\dots,\theta^{(r)}) = C\sum_{\lambda Irr}\frac{1}{(\dim\lambda)^{2g-2+r}}\prod_{i=1}^{r}\chi_\lambda(\theta^{(i)})$.

Main result I

- Remark : $f_{\theta^{(1)},\theta^{(2)}}(\theta^{(3)}) \neq 0$ can only happen if

$$\sum_{i=1}^{n} \theta_i^{(1)} + \sum_{i=1}^{n} \theta_i^{(2)} - \sum_{i=1}^{n} \theta_i^{(3)} = d \in \mathbb{Z}.$$

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Theorem

For each $n \geq 3$ and $0 \leq d \leq n$, there exists a finite set $\mathcal{S}^{n,d}$ of explicit polytopes with boundaries depending linearly on $(\theta^{(1)}, \theta^{(2)}, \theta^{(3)})$ such that, when $\theta^{(1)}, \theta^{(2)}, \theta^{(3)} \in \mathcal{H}_{reg}$ and $\sum_{i=1}^{n} \theta_i^{(1)} + \sum_{i=1}^{n} \theta_i^{(2)} - \sum_{i=1}^{n} \theta_i^{(3)} = d$,

$$f_{\theta^{(1)},\theta^{(2)}}(\theta^{(3)}) = \frac{(2\pi)^{(n-1)(n-2)/2} \prod_{k=1}^{n-1} k! \Delta(e^{2i\pi\theta^{(3)}})}{n! \Delta(e^{2i\pi\theta^{(1)}}) \Delta(e^{2i\pi\theta^{(2)}})} \sum_{P \in \mathcal{S}^{n,d}} Vol(P), \qquad (1)$$

where $\Delta(e^{2i\pi\theta}) = 2^{n(n-1)/2} \prod_{i<j} \sin\left(\pi(\theta_i - \theta_j)\right)$ for $\theta \in \mathcal{H}$.

Description of $\mathcal{S}_{d,n}$: toric hive

For $0 \leq d \leq n$,

$$R_{d,n} = \left\{ (v_1, v_2) \in [\![0, n]\!]^2, d \leq v_1 + v_2 \leq n + d \right\}$$

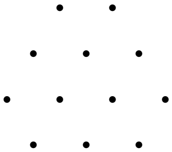

Figure: The set $R_{1,3}$ represented through the map $(v_1, v_2) \mapsto v_1 + v_2 e^{i\pi/3}$.

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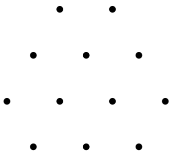

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$\mathcal{S}_{n,d}$: $P \subset \{f : R_{d,n} \to \mathbb{R}\}$ with :

1. a form of concavity with frozen region depending on P.
2. some boundary conditions depending only on $\theta^{(i)}, 1 \leq i \leq 3$,

Description of $\mathcal{S}_{d,n}$: rhombus concavity

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A rhombus of $R_{d,n}$: any sequence $(v^1, v^2, v^3, v^4) \in (R_{d,n})^4$ corresponding to

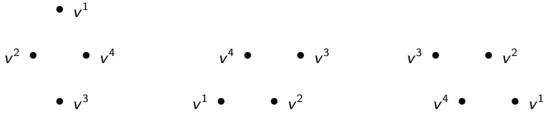

Figure: Three possible lozenges (v^1, v^2, v^3, v^4) (beware of the labeling).

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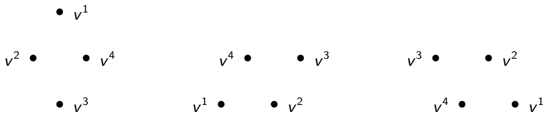

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Definition (Rhombus concavity)

A function $f : R_{d,n}$ is called rhombus concave when
$f(v_2) + f(v_4) \geq f(v_1) + f(v_3)$ on any rhombus $\ell = (v^1, v^2, v^3, v^4)$.

$\rightsquigarrow$ introduced by Knutson and Tao to address the Horn problem.

Description of $\mathcal{S}_{d,n}$: frozen region

$\mathcal{S}_{n,d} : P \subset \{f : R_{d,n} \to \mathbb{R}\}$ with :

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2. some boundary conditions depending only on $\theta^{(i)}$, $1 \leq i \leq 3$,

▶ A labeling $g : R_{d,n} \to \mathbb{Z}_3$ is regular whenever g starts at 0 at $(n-d, 0)$, decreases by 1 clockwise on the boundary and on any rhombus

$$(g(v^2) = g(v^4)) \Rightarrow \{g(v^1), g(v^3)\} = \{g(v^2) + 1, g(v^2) + 2\}.$$

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- A rhombus ℓ is called *rigid* if $(g(v^1), g(v^2), g(v^3), g(v^4)) = (a, a+1, a+2, a+1)$.

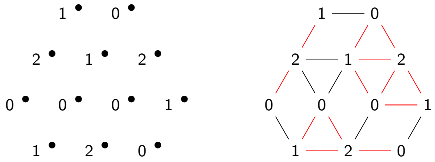

Figure: A regular labeling on $R_{d,n}$ and its colored representation

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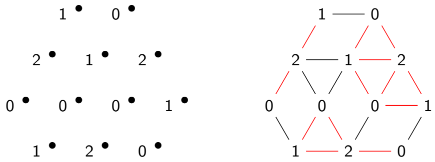

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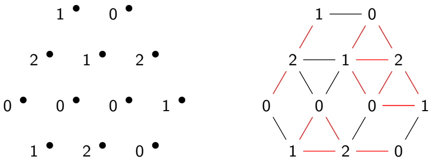

Figure: A regular labeling on $R_{d,n}$ and its colored representation

▶ A function $f : R_{d,n}$ is called rhombus concave with respect to a regular labeling $g : R_{d,n} \to \mathbb{Z}_3$ when $f(v_2) + f(v_4) \geq f(v_1) + f(v_3)$ on any lozenge $\ell = (v^1, v^2, v^3, v^4)$, with equality if ℓ is rigid with respect to g.

▶ Toric hive cone $\mathcal{C}_g$ with respect to g :

$$\mathcal{C}_g = \{f : R_{d,n} \to \mathbb{R} \text{ toric rhombus concave with respect to } g\}\,.$$

Boundary conditions

$\mathcal{S}_{n,d} : P \subset \{f : R_{n,d} \to \mathbb{R}\}$ with :

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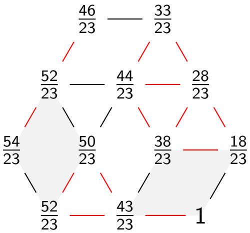

Figure: Element of $P^{g}_{\theta^{(1)},\theta^{(2)},\theta^{(3)}}$ with
$\theta^{(1)} = (\frac{13}{23} \ge \frac{6}{23} \ge \frac{2}{23})$,
$\theta^{(2)} = (\frac{18}{23} \ge \frac{10}{23} \ge \frac{5}{23})$,
$\theta^{(3)} = (\frac{20}{23} \ge \frac{9}{23} \ge \frac{2}{23})$.

Definition (Polytope $P^{g}_{\theta^{(1)},\theta^{(2)},\theta^{(3)}}$)

$P^{g}_{\theta^{(1)},\theta^{(2)},\theta^{(3)}} \subset \mathcal{C}_g$ polytope of functions $f \in \mathcal{C}_g$ with, for $0 \le i \le n$,

$$f_i^A = \sum_{s=1}^{n} \theta_s^{(2)} + \sum_{s=1}^{i} \theta_s^{(1)},\ f_i^B = (d-i)^+ + \sum_{s=1}^{i} \theta_s^{(2)},\ f_i^C = d + \sum_{s=1}^{i} \theta_s^{(3)}.$$

Description of $\mathcal{S}_{d,n}$

- For $\theta^{(1)}, \theta^{(2)}, \theta^{(3)} \in \mathcal{H}_{reg}$ and $\sum_{i=1}^n \theta_i^{(1)} + \sum_{i=1}^n \theta_i^{(2)} - \sum_{i=1}^n \theta_i^{(3)} = d$,

$$f_{\theta^{(1)},\theta(2)}(\theta^{(3)}) = \frac{(2\pi)^{(n-1)(n-2)/2} \prod_{k=1}^{n-1} k!\Delta(e^{2i\pi\theta^{(3)}})}{n!\Delta(e^{2i\pi\theta^{(1)}})\Delta(e^{2i\pi\theta^{(2)}})} \sum_{g:R_{d,n}\to\mathbb{Z}_3 \text{ regular}} Vol_g(P^g_{\theta^{(1)},\theta(2),\theta^{(3)}}), \tag{2}$$

with Vol_g adequate volume function.

- If $d=0$, unique regular labelling on $R_{0,n} \rightsquigarrow$ Knutson-Tao hive, related to Littlewood-Richardson coefficient (see Coquereaux-Zuber (2018)) .

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4. Convex reformulation of the puzzles and asymptotic of point counting in convex bodies.

Quantum Littlewood-Richardson coefficients

- Flag in $\mathbb{C}^N$: $\mathcal{F} = \{\{0\} \subset F_1 \subset \cdots \subset F_N = \mathbb{C}^N\}$, $\dim F_i = i$.
- Grassmanian space for $0 \leq n \leq N$: $G_{n,N} = \{V \subset \mathbb{C}^N, \dim V = n\}$.

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- Semi-classical limit : as $\frac{I^N}{N} \to \theta^{(1)}$, $\frac{J^N}{N} \to \theta^{(2)}$, $\frac{K^N}{N} \to \theta^{(3)}$ with $\sum_i (I_i + J_i - K_i) = nd$,

$$N^{-\frac{(n-1)(n-2)}{2}} c_{I^N J^N}^{K^N,d} \xrightarrow[n\to\infty]{} J_{\theta^{(1)},\theta^{(2)}}(\theta^{(3)}).$$

Quantum to Classical theorem of Buch-Kresch-Tamvakis

$0 \leq n_1 \leq n_2 \leq n$

- Two-step flag variety :

$D_{n_1,n_2,N} = \{V_1 \subset V_2 \subset \mathbb{C}^n\}$, dim $V_1 = n_1$, dim $V_2 = n_2\}$.

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$$f \in \#\{f \in G^d_{n,N},\ f(0) \in \Omega^{\mathcal{F}}_I, f(1) \in \Omega^{\mathcal{F}'}_J, f(\infty) \in \Omega^{\mathbb{F}''}_K) \text{ if and only if}$$

$$(\ker f, \text{span } f) \in \Omega^{\mathcal{F}}_{I^{(0)},I^{(1)}} \cap \Omega^{\mathcal{F}'}_{J^{(0)},J^{(1)}} \cap \Omega^{\mathbb{F}''}_{K^{(0)},K^{(1)}}, \text{ with}$$

- $I^{(0)}$: I with d largest integers discarded,
- $I^{(1)}$: d largest entries of I and d lowest entries of I^c.

$$\rightsquigarrow c^{\bar{K},d}_{IJ} = \#\Omega^{\mathcal{F}}_{I^{(0)},I^{(1)}} \cap \Omega^{\mathcal{F}'}_{J^{(0)},J^{(1)}} \cap \Omega^{\mathbb{F}''}_{K^{(0)},K^{(1)}} \quad (BKT, 2003).$$

Puzzles

- $I^{(0)}$: I with d largest integers discarded,
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Exemple : $N = 20, n = 5, d = 2$.
$I = \{18, 12, 11, 9, 5\} \rightsquigarrow I^{(0)} = \{11, 9, 5\}$, $I^{(1)} = \{18, 12, 2, 1\}$.

Two-step puzzles : filling of a triangle with the following pieces (which can be rotated), such that label on neighboring edges coincide.

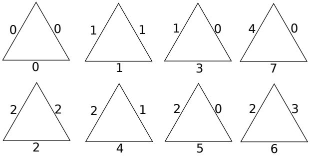

Figure: Possible pieces of the puzzle

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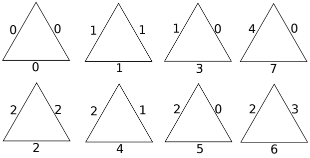

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Theorem (BKPT,2014) : $\#\Omega^{\mathcal{F}}_{I^{(0)},I^{(1)}} \cap \Omega^{\mathcal{F}'}_{J^{(0)},J^{(1)}} \cap \Omega^{\mathbb{F}''}_{K^{(0)},K^{(1)}}$ is the number of puzzles of size n with boundaries labeled $\omega_I, \omega_J, \omega_K$ where

$$\omega_I(i) = \begin{cases} 0 \text{ if } i \in I^{(0)} \\ 1 \text{ if } i \in I^{(1)} \\ 2 \text{ otherwise.} \end{cases} \qquad (\text{Exemple: } \omega_I = 11220222020122222122).$$

Convexification of puzzles

$N = 20$, $n = 5$, $d = 2$,

$$I = \{18, 12, 11, 9, 5\},\ J = \{16, 14, 7, 6, 2\},\ K = \{18, 13, 12, 10, 7\}.$$

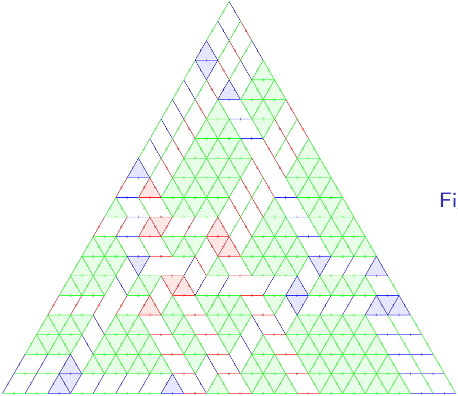

Figure: Contribution to $c_{IK}^{\bar{K},d}$

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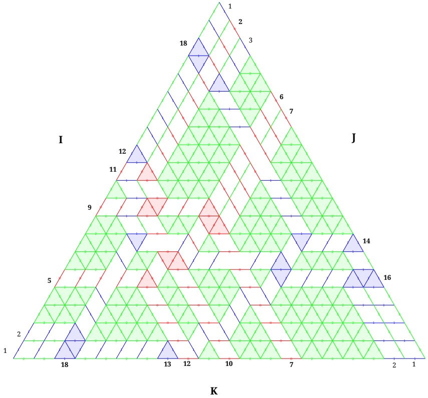

Figure: Labelling of blue/red paths

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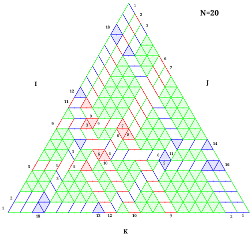

Figure: Sum equal to $n+2$ (resp $n+1$) around upward/downward triangles.

From puzzles to polytopes :

- colored path structure $\rightsquigarrow$ regular labeling,
- position of the path $\rightsquigarrow$ rhombus concave function with respect to the labeling,
- frozen region $\rightsquigarrow$ rigid crossing.

Technical part of the proof : from polytope to convex bodies.

Thank you !