# Poisson algebra bundles and covariant field theory

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based on https://arxiv.org/abs/2407.15287

and forthcoming next part

**JPM DiLyMe à Lyon**
September 23-26, 2024

# Context: field theory in a nutshell

**Data:** vector bundle $E \to M$ over spacetime manifold
plus Lagrangian $\mathcal{L} : JE \to \mathrm{Dens}_M$

**Fields:** $\mathcal{E}(M,E) = \Gamma^\infty(M,E) = \{\text{smooth sections } \varphi : M \to E\}$

**Solutions of eq. of motion:** $\mathcal{E}_\mathcal{L}(M,E) = \{\varphi \mid \text{Euler-Lagrange eq. } EL(\varphi) = 0\}$

**Observables:** on-shell $C^\infty(\mathcal{E}_\mathcal{L}(M,E)) = \{F : \mathcal{E}_\mathcal{L}(M,E) \to \mathbb{R} \text{ smooth}\}$
start with off-shell $C^\infty(\mathcal{E}(M,E)) = \{F : \mathcal{E}(M,E) \to \mathbb{R} \text{ smooth}\}$
then $C^\infty(\mathcal{E}_\mathcal{L}(M,E)) = C^\infty(\mathcal{E}(M,E))/I(\mathcal{E}_\mathcal{L}(M,E))$
both algebras with multiplication $(F_1 \cdot F_2)(\varphi) = F_1(\varphi)F_2(\varphi)$
but restrictions for Poisson bracket $\{F_1, F_2\}_\mathcal{L}(\varphi) = F_1'(\varphi)\Delta_\mathcal{L} F_2'(\varphi)$
where $\Delta_\mathcal{L}$ is the causal propagator det. by $\mathcal{L}$ (Peierls bracket)

**QFT:** deformation quantization of Poisson algebra $\big(\mathcal{A}_0 \subset C^\infty(\mathcal{E}_\mathcal{L}(M,E)), \cdot, \{\ ,\ \}_\mathcal{L}\big)$
plus time-ordered product = different deformation using Feynman propagator
plus renormalization to fix some ambiguities.

# Motivation: bundle description of observables in field theory

**Local observables:** $F_f(\varphi) = \int_M \langle f(x), (j\varphi)^n(x)\rangle\, dx$

represented by (generalized) sections $f : M \to S^n(JE^*)$

**Multilocal observables:** product of local

$$(F_{f_1} \cdot F_{f_2})(\varphi) = \int_{M\times M} \langle f_1(x) \otimes f_2(y), (j\varphi)^{n_1}(x) \otimes (j\varphi)^{n_2}(y)\rangle\, dx\, dy$$

rep. by $f_1 \boxtimes f_2 : M \times M \to S^{n_1}(JE^*) \boxtimes S^{n_2}(JE^*)$

where $\boxtimes$ is the external tensor product of vector bundles on $M$

**Problem:** $F_{f_1} \cdot F_{f_2}$ is commutative,

but $f_1 \boxtimes f_2$ is not commutative, because $(x, y) \neq (y, x)$ in $M \times M$.

**Aim:** find a consistent bundle description of polynomial (multilocal) observables.

## Main result and plan

**Claim [FKR 2024]:** there is an injective Poisson algebra map

$$F : \mathcal{D}'_{Pois}(\mathrm{UConf}(M), \mathrm{P}_{\mathcal{L}}(E)) \hookrightarrow C^{\infty}(\mathcal{E}(M,E))$$

sending a **distribution** $T$ to the **function** $F_T : \mathcal{E}(M,E) \to \mathbb{R}$

$$\varphi \mapsto F_T(\varphi) = \langle T, e(\varphi) \rangle = \int_{\mathrm{UConf}(M)} \langle T(\underline{x}), e(\varphi)(\underline{x}) \rangle$$

$\mathrm{UConf}(M)$ = space of unordered configurations replacing $M \times M \times \cdots \times M$ (cf. Olga's talk)

**Plan of the talk:** explain the other ingredients

$\mathrm{P}_{\mathcal{L}}(E)$ is a suitable Poisson algebra bundle on $\mathrm{UConf}(M)$ (cf. Olga's talk)

$e(\varphi)$ is a canonical section of a bundle dual to $\mathrm{P}_{\mathcal{L}}(E)$

$\mathcal{D}'_{Pois}$ is a suitable space of distributions $T$ closed for the induced Poisson algebra structure and which makes the value of $F_T$ finite.

# Setup: vector bundles over configuration spaces

- **Many-points manifolds / orbifolds of "non-pure dimension":**

tuples manifold $\qquad\qquad$ multi-configurations orbifold

$$\begin{array}{ccc} \bigsqcup_{k\geqslant 0} M^k & \xrightarrow{\quad S_k \quad} & \bigsqcup_{k\geqslant 0} M^k/S_k \\ \uparrow & & \uparrow \\ \mathrm{OConf}(M) = \bigsqcup_{k\geqslant 0} M^k\backslash\Delta^k & \xrightarrow{S_k} & \mathrm{UConf}(M) = \bigsqcup_{k\geqslant 0}(M^k\backslash\Delta^k)/S_k \end{array}$$

ordered configurations mfd $\qquad\qquad$ unordered configurations mfd

- **Vector bundles over** $\mathrm{UConf}(M)$**:** $\mathsf{V} = \bigsqcup_{\underline{x}\in\mathrm{UConf}(M)} \mathsf{V}_{\underline{x}} = \bigsqcup_{k\geqslant 0} \mathsf{V}_k$

- **Symmetric** 2**-monoidal category:** $\Big(VB(\mathrm{UConf}(M)), \otimes, I_{\otimes}, \boxtimes, I_{\boxtimes}\Big)$

Hadamard: $(\mathsf{V}\otimes\mathsf{W})_{\underline{x}} = \mathsf{V}_{\underline{x}} \otimes \mathsf{W}_{\underline{x}}$ with $\mathsf{I}_{\otimes} = \mathrm{UConf}(M) \times \mathbb{K}$

Cauchy: $(\mathsf{V}\boxtimes\mathsf{W})_{\underline{x}} = \bigoplus_{\underline{x}=\underline{x}'\sqcup\underline{x}''} \mathsf{V}_{\underline{x}'} \otimes \mathsf{W}_{\underline{x}''}$ with $\mathsf{I}_{\boxtimes} = \begin{cases} \mathbb{K} \to \{\varnothing\} & k = 0 \\ \{0\} \to \mathrm{UConf}_k(M) & k \geqslant 1 \end{cases}$

- **Theorem:** $\boxtimes$ on $\mathrm{UConf}(M)$ is a symmetrized version of $\boxtimes$ on $\mathrm{OConf}(M)$

# Cauchy-Hadamard 2-algebra bundles

- **2-algebra bundle:** $A \to \mathrm{UConf}(M)$ with

$$m_\otimes : A \otimes A \to A \qquad u_\otimes : \mathsf{I}_\otimes \to A$$
$$m_\boxtimes : A \boxtimes A \to A \qquad u_\boxtimes : \mathsf{I}_\boxtimes \to A$$

plus compatibility using $\mu$, $\delta$, $\iota$, $\mathrm{sh}$.

**2-coalgebra bundle:** $C \to \mathrm{UConf}(M)$ with dual maps $\Delta_\otimes$, $\varepsilon_\otimes$, $\Delta_\boxtimes$, $\varepsilon_\boxtimes$ and dual relations

$\Longrightarrow$ $C^*$ is a 2-algebra bundle.

- $\otimes$ **and** $\boxtimes$**-tensor bundles:** from $E \to M$ with dual $E^* \to M$:

$$\begin{array}{ccc}
T^{a,\boxtimes}(E^*) = \bigoplus_n (E^*)^{\boxtimes n},\ \boxtimes & \xleftarrow{\ gr*\ } & T^{c,\boxtimes}(E) = \bigoplus_n E^{\boxtimes n},\ \Delta_\boxtimes \\
\text{algebra} \downarrow & & \uparrow \text{coalgebra} \\
S^{\boxtimes}(E^*) = \bigoplus_n (E^*)^{\boxtimes n}/S_n,\ \boxdot & \xleftarrow[\ gr*\ ]{} & \Sigma^{\boxtimes}(E) = \bigoplus_n (E^{\boxtimes n})^{S_n},\ \Delta_\boxdot
\end{array}$$

(same for $\otimes$)

- **Theorem:** $\boxed{\mathrm{Dens}_{\mathrm{UConf}(M)} \cong S^{\boxtimes}(\mathrm{Dens}_M)}$ commutative $\boxtimes$-algebra bundle with $\boxdot$.

- **Bundle of observables:** $\boxed{\mathsf{P}_{\mathcal{L}}(E) := S^{\boxtimes} S^{\otimes}(JE)^* \otimes \mathrm{Dens}_{\mathrm{UConf}(M)} \cong \big(\hat{\Sigma}^{\boxtimes}\hat{\Sigma}^{\otimes}(JE)\big)^\vee}$

$\langle\ ,\ \rangle$ canonical pairing between sections $T = \Phi \otimes \nu$ of $\mathsf{P}_{\mathcal{L}}(E)$ and

exponential section $\boxed{e(\varphi) = \sum_{n_1,\ldots,n_k} \frac{1}{n_1! \cdots n_k!} (j\varphi)^{\otimes n_1} \boxtimes \cdots \boxtimes (j\varphi)^{\otimes n_1}}$ of $\hat{\Sigma}^{\boxtimes}\hat{\Sigma}^{\otimes}(JE)$.

# Poisson $\boxtimes$-algebra bundles

- **Poisson 2-algebra bundle:** $(\mathrm{P}, \bullet_\otimes, 1_\otimes, \bullet_\boxtimes, 1_\boxtimes)$ commutative 2-algebra bundle with Poisson bracket $\boxed{\{\ ,\ \} : \mathrm{P} \boxtimes \mathrm{P} \to \mathrm{P}}$ such that

  antisymmetry: $\{a_{\underline{x}}, b_{\underline{y}}\} = -\{b_{\underline{y}}, a_{\underline{x}}\}$

  Jacobi identity: $\{\{a_{\underline{x}}, b_{\underline{y}}\}, c_{\underline{z}}\} + \{\{b_{\underline{y}}, c_{\underline{z}}\}, a_{\underline{x}}\} + \{\{c_{\underline{z}}, a_{\underline{x}}\}, b_{\underline{y}}\} = 0$

  $m_\boxtimes$-Leibniz rule: $\{a_{\underline{x}}, b_{\underline{y}} \bullet_\boxtimes c_{\underline{z}}\} = \{a_{\underline{x}}, b_{\underline{y}}\} \bullet_\boxtimes c_{\underline{z}} + b_{\underline{y}} \bullet_\boxtimes \{a_{\underline{x}}, c_{\underline{z}}\}$

  $m_\otimes$-Leibniz rule: $\{a_{\underline{x}}, b_{\underline{y}} \bullet_\otimes c_{\underline{y}}\} = \{a_{\underline{x}}, b_{\underline{y}}\} \bullet_\otimes (1_{\underline{x}} \bullet_\boxtimes c_{\underline{y}}) + (1_{\underline{x}} \bullet_\boxtimes b_{\underline{y}}) \bullet_\otimes \{a_{\underline{x}}, c_{\underline{y}}\}$

  where we omit sums over disjoint configurations coming from repeated splits.

- **Theorem:** Any antisymmetric bundle map $k : (JE)^* \boxtimes (JE)^* \to \mathrm{I}_\otimes$ over $\mathrm{UConf}_2(M)$ is the kernel of a Poisson bracket which makes

  $S^\boxtimes S^\otimes (JE)^*$ a Poisson 2-algebra bundle

  $\boxed{\mathrm{P}_{\mathcal{L}}(E) = S^\boxtimes S^\otimes (JE)^* \otimes \mathrm{Dens}_{\mathrm{UConf}(M)}}$ a Poisson $\boxtimes$-algebra bundle.

  *Proof:* extend $k$ to symmetric powers using Leibniz rules and prove Jacobi.

- **N.B.** For field theory, the kernel $k$ is determined by the Lagrangian $\mathcal{L}$.

## (Regular) Poisson algebras of sections and distributions

- **Sections and distributions:** given a vector bundle $\mathsf{V} \to \mathrm{UConf}(M)$

sections: $\phi : \mathrm{UConf}(M) \to \mathsf{V}$ as usual $= \{\phi_k : \mathrm{UConf}_k(M) \to \mathsf{V}_k\}$

$$\mathcal{E}(\mathrm{UConf}(M), \mathsf{V}) \text{ smooth } \supset \mathcal{D}(\mathrm{UConf}(M), \mathsf{V}) \text{ compact support}$$

distributions: $\mathcal{D}(\mathrm{UConf}(M), \mathsf{V})' =: \mathcal{D}'(\mathrm{UConf}(M), \mathsf{V}^\vee)$

$$\mathcal{E}(\mathrm{UConf}(M), \mathsf{V})' =: \mathcal{E}'(\mathrm{UConf}(M), \mathsf{V}^\vee)$$

where $\mathsf{V}^\vee = \mathsf{V}^* \otimes \mathrm{Dens}_{\mathrm{UConf}(M)}$ is the functional dual such that $(V^\vee)^\vee \cong V$.

$$\begin{array}{lccc} \text{regular distributions} & \mathcal{D}(\mathrm{UConf}(M), \mathsf{V}^\vee) & \hookrightarrow & \mathcal{E}(\mathrm{UConf}(M), \mathsf{V}^\vee) \\ & \downarrow & & \downarrow \\ \text{all distributions} & \mathcal{E}'(\mathrm{UConf}(M), \mathsf{V}^\vee) & \hookrightarrow & \mathcal{D}'(\mathrm{UConf}(M), \mathsf{V}^\vee) \end{array}$$

- **Theorem:** If P is a Poisson $\boxtimes$-algebra bundle on $\mathrm{UConf}(M)$, then

$\mathcal{E}(\mathrm{UConf}(M), \mathsf{P})$ and $\mathcal{D}'(\mathrm{UConf}(M), \mathsf{P})$ are Poisson algebras

- **Corollary:** $\mathcal{D}'(\mathrm{UConf}(M), \mathsf{P}_{\mathcal{L}}(E))$ is a (regular) Poisson algebra (*) for any choice of a kernel bundle map $k : (JE)^* \boxtimes (JE)^* \to \mathsf{I}_\otimes$ over $\mathrm{UConf}_2(M)$ i.e. a smooth section $k \in \mathcal{E}(\mathrm{UConf}_2(M), \Lambda^{\boxtimes 2}(JE))$ (a regular distribution).

(*) in the cat. of bornological locally convex modules over the Frechet algebra $\mathcal{E}(\mathrm{UConf}(M))$

# Poisson algebras of observables in field theory

- **Lemma:** The causal propagator $\Delta_{\mathcal{L}} \in \mathcal{D}'(M \times M, E \boxtimes E)$ determines a singular kernel $k_{\mathcal{L}} \in \mathcal{D}'(\mathrm{UConf}_2(M), \Lambda^{\boxtimes 2}(E))$ for a Poisson bracket on $\mathcal{D}'(\mathrm{UConf}(M), \mathrm{P}_{\mathcal{L}}(E))$.

- **Work in progress:**

  Problem 1: distributions with no support restrictions do not give finite integrals, need support restrictions.

  Problem 2: Singular kernels require restrictions on the WF sets of distributions, cf. microcausal functionals as in [Brunetti-Fredenhagen-Ribero 2012].

  Problem 3: $\mathcal{D}(\mathrm{UConf}(M), \mathrm{P}) \subset \mathcal{E}(\mathrm{UConf}(M), \mathrm{P})$ not a subalgebra!

- **Idea:**

$$
\begin{array}{ccccc}
\mathcal{D}(\mathrm{UConf}(M), \mathrm{P}_{\mathcal{L}}(E)) & \hookrightarrow & \mathcal{E}_{Pois}(\mathrm{UConf}(M), \mathrm{P}_{\mathcal{L}}(E)) & \hookrightarrow & \mathcal{E}(\mathrm{UConf}(M), \mathrm{P}_{\mathcal{L}}(E)) \\
\text{not algebra} \downarrow & & \text{algebra} \downarrow & & \text{algebra} \downarrow \\
\mathcal{E}'(\mathrm{UConf}(M), \mathrm{P}_{\mathcal{L}}(E)) & \hookrightarrow & \mathcal{D}'_{Pois}(\mathrm{UConf}(M), \mathrm{P}_{\mathcal{L}}(E)) & \hookrightarrow & \mathcal{D}'(\mathrm{UConf}(M), \mathrm{P}_{\mathcal{L}}(E)) \\
& \searrow^{F} & \downarrow F & \swarrow^{\nexists F} & \\
& & C^\infty(\mathcal{E}(M, E), \mathbb{R}) & &
\end{array}
$$

# Outlook

- Covariant QFT: Poisson $\boxtimes$-bundles are compatible with deformation quantization (operator product) and with Laplace pairing deformation (time-ordered product) described on polynomial observables in [C. Brouder, B. Fauser, A.F., R. Oekl 2004].
- Lift the whole construction to the orbifold of multi-configurations $\bigsqcup_k M^k/S_k$ to describe renormalization in QFT [PhD project for Hai Châu Nguyên].
- Extend to graded symmetric tensors to include fermions.
- Study structure group and connections of $\mathrm{P}_{\mathcal{L}}(E)$ and extend to gauge fields.
- Include the multisymplectic description of dynamics and compute $C^\infty(\mathcal{E}_{\mathcal{L}}(M,E))$!
- Direct links with species, types of algebras, operads...

**Thank you for the attention!**