Dynamical principal bundles and Kaluza–Klein models

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The Kaluza–Klein idea (T. Kaluza 1919, O. Klein 1926)

Consider a 5-dimensional pseudo-Riemannian manifold $(\mathcal{Y}, h)$. Assume that, in coordinates $(x^1, x^2, x^3, x^4, x^5)$, the metric $\mathbf{h}$ reads

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- The metric $\mathbf{h}$ satisfies the 5-dimensional generalization of the Einstein equation in vacuum

$$\text{for } 1 \leq I, J \leq 5, \quad \mathrm{Ric}(\mathbf{h})_{IJ} - \frac{1}{2}\mathrm{R}(\mathbf{h})\mathbf{h}_{IJ} = 0$$

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- Then, *if it works*, $(\mathbf{g}_{\mu\nu}, \mathbf{A}_\mu)$ are solutions of the Einstein–Maxwell system

Some inconsistency

A problem is that the two conditions

$$\frac{\partial \mathbf{g}_{\mu\nu}}{\partial x^5} = \frac{\partial \mathbf{A}_\mu}{\partial x^5} = 0$$

and

$$\text{for } 1 \leq I, J \leq 5, \quad \mathrm{Ric}(\mathbf{h})_{IJ} - \frac{1}{2}\mathrm{R}(\mathbf{h})\mathbf{h}_{IJ} = 0$$

are overdetermined together and has actually few solutions.

Hence two variants have been proposed.

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We add an extra scalar field ϕ (Thiry, Jordan–Müller, Pauli–Fierz...)

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- Then $(\mathbf{h}_{\mu\nu}, \mathbf{A}_\mu, \phi)$ are solutions of a modification of the Einstein–Maxwell system.

Second variant

No extra scalar fields so that

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where $1 \leq \mu, \nu \leq 4$. We impose that $\frac{\partial \mathbf{g}_{\mu\nu}}{\partial x^5} = \frac{\partial \mathbf{A}_\mu}{\partial x^5} = 0$ and that $\mathbf{h}$ is **a critical point of the Einstein–Hilbert action**

$$\mathscr{A}(\mathbf{h}) = \int_{\mathcal{Y}} \mathrm{R}(\mathbf{h}) d vol_{\mathbf{h}} \quad \textbf{under the constraint} \quad \frac{\partial \mathbf{g}_{\mu\nu}}{\partial x^5} = \frac{\partial \mathbf{A}_\mu}{\partial x^5} = 0$$

Then

- **h** is **not** a solution of the higher dimensional Einstein equation in vacuum

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- We will follow this approach

A further improvement (R. Kerner, 1968, D. Bleecker, 1981, answering a question by B. DeWitt, 1963)

$\mathfrak{G}$ is a compact Lie group of dimension r and $\mathfrak{g}$ is its Lie algebra. Let $\mathcal{Y}$ be a $(4+r)$-dimensional manifold which is the total space of a $\mathfrak{G}$-principal bundle over a 4-dimensional manifold $\mathcal{X}$ and let

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where $1 \leq \mu, \nu \leq 4 < i, j \leq 4 + r$ and k_{ij} is a $\mathrm{Ad}_{\mathfrak{G}}$-invariant metric on $\mathfrak{g}$. Impose

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- We present a model which answers these questions through a *dynamical* mechanism

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- $\varphi = ({\varphi^A}_B) \in so(\mathbb{R}^4 \oplus \mathfrak{g}) \otimes \Omega^1(\mathcal{F})$, a spin 'connection'

Our model: the fields

- Let $\mathfrak{G}$ be a compact, unimodular Lie group, of dimension r (examples: $U(1)$, $SU(2)$, $SU(2)\times SU(3)$), with Lie algebra $\mathfrak{g}$
- Consider a connected, oriented manifold $\mathcal{F}$ of dimension

$$\dim\mathcal{F} = 4 + r = 4 + \dim\mathfrak{G} = N,$$

- Unknown fields are triplets (θ, φ, π), with:
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- with the constraint

$$\theta^\mu \wedge \theta^\nu \wedge \pi = 0, \qquad \text{for } 1 \leq \mu, \nu \leq 4$$

The action functional

- It is the sum of $\int_{\mathcal{F}} \pi_A \wedge \left(\mathrm{d}\theta + \frac{1}{2}[\theta\wedge\theta]_{\mathfrak{g}}\right)^A$ and of the higher dimensional **Palatini action for** (θ, φ) :

$$\begin{aligned}\mathscr{A}[\theta,\varphi,\pi] \;\; = \;\; & \int_{\mathcal{F}} \pi_A \wedge \left(\mathrm{d}\theta + \frac{1}{2}[\theta\wedge\theta]_{\mathbb{R}^4\oplus\mathfrak{g}}\right)^A \\ & + \theta^{(N-2)}_{AB} \wedge \left(\mathrm{d}\varphi + \frac{1}{2}[\varphi\wedge\varphi]_{so(\mathbb{R}^4\oplus\mathfrak{g})}\right)^{AB}\end{aligned}$$

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- Could be seen as the sum of two BF theories with constraints on the B fields:

$$\int_{\mathcal{F}} \pi_A \wedge \underbrace{\Theta^A}_{\substack{\text{Cartan geometry of } \theta \text{ on } \mathcal{F} \\ \text{modelled on } (\mathbb{R}^4 \times \mathfrak{G})/\mathfrak{G}}} + \theta^{(N-2)}_{AB} \wedge \underbrace{\Phi^{AB}}_{\substack{\text{Curvature of } \varphi \text{ on} \\ T\mathcal{F} \simeq \mathcal{F}\times(\mathbb{R}^4\oplus\mathfrak{g})}}$$

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- In the following $\widehat{\mathfrak{G}}$ denotes the universal cover of $\mathfrak{G}$.

The main result

Theorem (H 2020-22-23) — Let (θ, φ, π) be a critical point of $\mathscr{A}$ such that $\operatorname{rank}\theta = N + 1$. Assume that $\widehat{\mathfrak{G}}$ is simply connected. Then

- The EDS $\theta^{\mu}|_{\mathsf{f}} = 0$, for r-dimensional submanifolds f, is completely integrable and $\mathcal{F}$ is foliated by integral leaves f and all these integral leaves f are diffeomorphic to a Lie group $\mathfrak{G}$, which is a quotient of $\widehat{\mathfrak{G}}$ by a finite subgroup.

Moreover, if $\widehat{\mathfrak{G}}$ is **compact** (e.g. $SU(2), SU(3)$),

- (θ, φ, π) endows $\mathcal{F}$ with a structure of principal bundle over a 4-dimensional manifold $\mathcal{X}$ and gives rise to a solution to the Einstein–Yang–Mills system on $\mathcal{X}$.

Some notations

We define the metric $\mathbf{h} = \eta_{\mu\nu}\theta^\mu \otimes \theta^\nu + \mathrm{k}_{jk}\theta^j \otimes \theta^k$ on $\mathcal{F}$, where $\eta_{\mu\nu}$ is the standard Minkowski metric and k_{jk} is a $\mathrm{Ad}_{\mathfrak{G}}$-invariant metric on $\mathfrak{G}$.

- We decompose $\Theta^A := \mathrm{d}\theta^A + \frac{1}{2}[\theta \wedge \theta]^A$ in the coframe $\left(\theta^A\right)_{1\leq A\leq N}$ as

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- We let $\theta^{(N)} := \theta^1 \wedge \cdots \wedge \theta^N$
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- Then $\left(\theta^{(N-2)}_{AB}\right)_{1\leq A<B\leq N}$ is a basis of $\Omega^{N-2}(\mathcal{F})$

The effect of the constraint on π and θ

Any $\pi_A \in \Omega^{N-2}(\mathcal{F})$ decomposes *a priori* as

$$\pi_A = \frac{1}{2}\pi_A{}^{\mu\nu}\theta^{(N-2)}_{\mu\nu} + \pi_A{}^{\mu k}\theta^{(N-2)}_{\mu k} + \frac{1}{2}\pi_A{}^{jk}\theta^{(N-2)}_{jk}$$

Then the constraint $\theta^\mu \wedge \theta^\nu \wedge \pi_A$ reads $\pi_A{}^{\mu\nu} = 0$, so that, actually

$$\pi_A = 0 + \pi_A{}^{\mu k}\theta^{(N-2)}_{\mu k} + \frac{1}{2}\pi_A{}^{jk}\theta^{(N-2)}_{jk}$$

Hence the Euler–Lagrange equation $\delta\mathscr{A}/\delta\pi_A = 0$, where $\mathscr{A}[\theta,\varphi,\pi] = \int_{\mathcal{F}} \theta^{(N-2)}_{AB} \wedge \left(\mathrm{d}\varphi^{AB} + \frac{1}{2}[\varphi\wedge\varphi]^{AB}\right) + \pi_A \wedge \left(\mathrm{d}\theta^A + \frac{1}{2}[\theta\wedge\theta]^A\right)$ reads:

$$\mathrm{d}\theta^A + \frac{1}{2}[\theta\wedge\theta]^A = \frac{1}{2}\Theta^A{}_{\mu\nu}\theta^\mu \wedge \theta^\nu$$

i.e. $\Theta^A{}_{\mu k} = \Theta^A{}_{jk} = 0$.

The other Euler–Lagrange equations

Recall that

$\mathscr{A}[\theta,\varphi,\pi] = \int_{\mathcal{F}} \theta^{(N-2)}_{AB} \wedge \left(\mathrm{d}\varphi^{AB} + \frac{1}{2}[\varphi\wedge\varphi]^{AB}\right) + \pi_A \wedge \left(\mathrm{d}\theta^A + \frac{1}{2}[\theta\wedge\theta]^A\right)$

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$$\mathrm{d}^\varphi \theta^{(N-2)}_{AB} = 0 \quad \Longrightarrow \quad \mathrm{d}^\varphi\theta^A = \mathrm{d}\theta^A + \varphi^A{}_B \wedge \theta^B = 0$$

i.e. φ induces a **torsion free** connection on $\mathcal{F}$, actually the Levi-Civita connection on $(\mathcal{F},\mathbf{h})$ ($\mathbf{h} = \eta_{\mu\nu}\theta^\mu \otimes \theta^\nu + \mathrm{k}_{jk}\theta^j \otimes \theta^k$)

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- The Euler–Lagrange equation $\delta\mathscr{A}/\delta\theta = 0$ reads

$$\mathrm{d}^\theta\pi_A + \underbrace{\frac{1}{2}\theta^{(N-3)}_{ABC} \wedge \Phi^{BC}}_{\text{Einstein tensor}} - \Theta^C{}_{AB}\pi_C{}^{jB}\theta^{(N)}_j = 0$$

  where $\Phi := \mathrm{d}\varphi + \frac{1}{2}[\varphi\wedge\varphi]$. It will be analyzed later on.

Summarizing the Euler–Lagrange equations

1. $\delta\mathscr{A}/\delta\pi_A = 0 \longrightarrow \mathrm{d}\theta^A + \frac{1}{2}[\theta\wedge\theta]^A_{\mathfrak{g}} = \frac{1}{2}\Theta^A{}_{\mu\nu}\theta^\mu\wedge\theta^\nu$
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3. $\delta\mathscr{A}/\delta\theta^A = 0 \longrightarrow \mathrm{d}^\theta\pi_A + \frac{1}{2}\theta^{(N-3)}_{ABC}\wedge\Phi^{BC} - \Theta^C{}_{AB}\pi_C{}^{jB}\theta^{(N-1)}_j = 0$
   $\longrightarrow$ the Einstein–Maxwell or the Einstein–Yang–Mills system

The fiber bundle structure and connections, I

Consider the Pfaffian system

$$\theta^{\mu}|_{\mathsf{f}} = 0, \quad \text{for } 1 \leq \mu \leq 4$$

where f is a connected r-dimensional submanifold of $\mathcal{F}$.

- If $\mathfrak{G} = U(1)$ (Maxwell), integral submanifolds f are just lines

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- If $\mathfrak{G} = U(1)$ (Maxwell), integral submanifolds f are just lines
- If $r > 1$, use Frobenius theorem and

$$\mathrm{d}\theta^A + \frac{1}{2}[\theta \wedge \theta]^A_{\mathfrak{g}} = \frac{1}{2}\Theta^A{}_{\mu\nu}\theta^{\mu} \wedge \theta^{\nu} \quad \Longrightarrow \quad d\theta^{\mu} = 0 \text{ mod } [\theta^{\mu}]$$

In any case, get a foliation of $\mathcal{F}$ by integral leaves f.

The fiber bundle structure and connections, II

Assume that $\widehat{\mathfrak{G}}$ is **simply connected**.

Let $\vartheta = (\vartheta^5, \cdots, \vartheta^N)$ be the left invariant **Maurer–Cartan** 1-form on $\widehat{\mathfrak{G}}$ (in particular $d\vartheta + \frac{1}{2}[\vartheta \wedge \vartheta] = 0$). Then on $\mathcal{Y} \times \widehat{\mathfrak{G}}$

$$d\theta^A + \frac{1}{2}[\theta \wedge \theta]^A_{\mathfrak{g}} = \frac{1}{2}\Theta^A{}_{\mu\nu}\theta^\mu \wedge \theta^\nu \implies d(\vartheta^j - \theta^j)|_{\mathrm{f}\times\widehat{\mathfrak{G}}} = 0 \text{ mod } [\vartheta^j - \theta^j]$$

Use (again) Frobenius theorem: each leaf f is covered by $\widehat{\mathfrak{G}}$ ($\widehat{\mathfrak{G}} \longrightarrow \mathrm{f}$) and hence and can be identified with a **quotient** $\mathfrak{G}$ of $\widehat{\mathfrak{G}}$.

As a consequence, if $\widehat{\mathfrak{G}}$ is compact, then $\mathfrak{G}$ is so.

The fiber bundle structure and connections, III

Fix any $\xi = (\xi^1, \xi^2, \xi^3, \xi^4) \in \mathbb{R}^4$ and $\zeta = (\zeta^i)_{4<i\leq N}$ and consider the vector fields X and Y on $\mathcal{F}$ such that

$$\begin{array}{rclcrcl} \theta^\mu(X) & = & \xi^\mu & \quad & \theta^\mu(Y) & = & 0 \\ \theta^i(X) & = & 0 & & \theta^i(Y) & = & \zeta^i \end{array}$$

Then $\mathrm{d}\theta^A + \frac{1}{2}[\theta\wedge\theta]^A_{\mathfrak{g}} = \frac{1}{2}\Theta^A{}_{\mu\nu}\theta^\mu\wedge\theta^\nu \Longrightarrow \mathrm{d}\theta^A(X,Y) = 0$ and, thus

$$\mathrm{d}\theta^A(X,Y) + \theta^A([X,Y]) = X\cdot\theta^A(Y) - Y\cdot\theta^A(X)$$

implies

$$[X,Y] = 0$$

We deduce that all leaves f all diffeomorphic to a same model $\mathfrak{G}$.
Hence if $\mathfrak{G}$ is **compact**, then the leaves form a fibration over $\mathcal{X} := \mathcal{F}/\{\mathrm{f}\}$: $\mathcal{X}$ **and the fibration is born!**

For instance if $\mathfrak{G} = U(1)$ we obtain a circle bundle.

The fiber bundle structure and connections, IV

Perform a local trivialization of the bundle $\mathcal{F} \longrightarrow \mathcal{X}$:

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{X} \times \mathfrak{G} \\ z & \longmapsto & (x, g) \end{array}$$

From $(\vartheta^j - \theta^j)|_{\mathfrak{f}\times\widehat{\mathfrak{G}}} = 0$, we can deduce

$$\theta^j = (\mathrm{Ad}_{g^{-1}}\mathbf{A})^j + \vartheta^j = (g^{-1}\mathbf{A}g)^j + (g^{-1}\mathrm{d}g)^j$$

Again by using $\mathrm{d}\theta^A + \frac{1}{2}[\theta \wedge \theta]^A_{\mathfrak{g}} = \frac{1}{2}\Theta^A{}_{\mu\nu}\theta^\mu \wedge \theta^\nu$ (a 4th time !) we deduce that $\mathbf{A}^j$ is constant along the fibers f.

We define $\mathbf{g} := \eta_{\mu\nu}\theta^\mu \otimes \theta^\nu$. It is invariant along the fibers and, hence, defines a metric on $\mathcal{X}$.

We thus obtained the pair of fields $(\mathbf{g}_{\mu\nu}, \mathbf{A}^j)$ on $\mathcal{X}$.

The dynamical equations, I

Now let's look at the ugly equation

$$d^\theta \pi_A + \underbrace{\frac{1}{2}\theta^{(N-3)}_{ABC} \wedge \Phi^{BC}}_{\text{Einstein tensor on } (\mathcal{F},\mathbf{h})} - \underbrace{\Theta^C{}_{AB}\ \pi_C{}^{jB}}_{\text{hybrid terms}} \theta^{(N-1)}_j = 0$$

Translate it in the trivialization (long computation):

$$d^{\mathbf{A}} p_A + \underbrace{\frac{1}{2}e^{(N-3)}_{ABC} \wedge \Omega^{BC}}_{\text{Einstein tensor on } (\mathcal{F},\mathbf{h})} - \underbrace{\mathbf{F}^C{}_{A\lambda}\ p_C{}^{j\lambda}}_{\text{some components cancel}} \ e^{(N-1)}_j = 0$$

Where here $\mathbf{F} = d\mathbf{A} + \frac{1}{2}[\mathbf{A} \wedge \mathbf{A}]$.

The dynamical equations, II

Split $\mathrm{d}^{\mathbf{A}} p_A + \frac{1}{2} e^{(N-3)}_{ABC} \wedge \Omega^{BC} - \mathbf{F}^C{}_{A\lambda} p_C{}^{j\lambda} e^{(N-1)}_j = 0$ as

$$\begin{array}{|lcl lcl|} \hline \mathrm{Ein}(\mathbf{h})_\mu{}^\nu & = & \partial_k p_\mu{}^{\nu k} & \mathrm{Ein}(\mathbf{h})_i{}^\nu & = & \partial_k p_i{}^{\nu k} \\ \mathrm{Ein}(\mathbf{h})_\mu{}^j & = & \mathrm{ugly}_\mu{}^j & \mathrm{Ein}(\mathbf{h})_i{}^j & = & \mathrm{ugly}_i{}^j \\ \hline \end{array}$$

The left hand side quantities on the first line are

$$\left\{ \begin{array}{lcl} \mathrm{Ein}(\mathbf{h})_\mu{}^\nu & = & \mathrm{Ein}(\mathbf{g})_\mu{}^\nu - \mathrm{T}(\mathbf{A})_\mu{}^\nu + \frac{1}{4}\langle B, \mathrm{k}\rangle \delta_\mu{}^\nu \\ \mathrm{Ein}(\mathbf{h})_i{}^\nu & = & \frac{1}{2} \nabla^{\mathbf{A}}_\mu \mathbf{F}_i{}^{\nu\mu} \end{array} \right.$$

where $\mathrm{T}(\mathbf{A})_\mu{}^\nu := \frac{1}{2}\mathbf{F}^i{}_{\mu\lambda}\mathbf{F}_i{}^{\nu\lambda} - \frac{1}{4}\|\mathbf{F}\|^2 \delta_\mu{}^\nu$ is the stress-energy tensor of the connection field $\mathbf{A}$ and B is the Killing form on $\mathfrak{g}$.

Problem: the right hand sides $\partial_k p_\mu{}^{\nu k}$ and $\partial_k p_i{}^{\nu k}$...

The dynamical equations, III : miraculous cancellation

Integrate, e.g., $\mathrm{Ein}(\mathbf{h})_{\mu}{}^{\nu} = \partial_k p_{\mu}{}^{\nu k}$ on a a *compact* fiber f and observe that

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- Similarly $\mathrm{Ein}(\mathbf{h})_i{}^\nu = 0$ (which implies $\mathrm{Ein}(\mathbf{h})_\mu{}^j = 0$ by symmetry)
- **Conclusion** : $(\mathbf{g}_{\mu\nu}, \mathbf{A}^j)$ are solutions of the Einstein–Yang–Mills system on $\mathcal{X}$:

$$\mathrm{Ein}(\mathbf{g})_\mu{}^\nu + \Lambda_0 \delta_\mu{}^\nu = \mathrm{T}(\mathbf{A})_\mu{}^\nu \quad \text{and} \quad \nabla_\mu^{\mathbf{A}} \mathbf{F}_i{}^{\nu\mu} = 0$$

with, assuming the signature $(-+++)$, $\Lambda_0 := \frac{1}{4}\langle B, \mathrm{k}\rangle < 0$.

Multisymplectic origin of the model

Multisymplectic formulation of Yang–Mills theory (F.H., 2015)

$$\begin{array}{ccc} \mathcal{X}, \quad x=(x^\mu) & \longrightarrow & \mathcal{P}\simeq\mathcal{X}\times\mathfrak{G}, \quad (x,g) \\ \mathrm{A} & \longrightarrow & \theta = g^{-1}\mathrm{d}g + g^{-1}\mathrm{A}g \\ \mathrm{F}=\mathrm{dA}+\frac{1}{2}[\mathrm{A}\wedge\mathrm{A}] & \longrightarrow & \Theta = \mathrm{d}\theta + \frac{1}{2}[\theta\wedge\theta] \\ \int_{\mathcal{X}}\|\mathrm{F}\|^2 & = & \frac{1}{|\mathfrak{G}|}\int_{\mathcal{P}}\|\Theta\|^2 \end{array}$$

Equivariance & normalization of θ: $z\cdot\xi \lrcorner \theta = \xi$ and $z\cdot\xi \lrcorner \Theta = 0$

Legendre transform:

$\longrightarrow (\vartheta,p,x,y)\in(\mathfrak{g}\otimes T^*\mathcal{P})\oplus_{\mathcal{P}}(\mathfrak{g}^*\otimes\Lambda^{N-2}T^*\mathcal{P})$, $\mathrm{d}x^{(n)}\wedge\vartheta^{(r)}\neq 0$

$$\omega = \mathrm{d}\left(\frac{1}{2}\|p_i{}^{\mu\nu}\|^2\mathrm{d}x^{(n)}\wedge\vartheta^{(r)} + p_i\wedge\left(\mathrm{d}\vartheta+\frac{1}{2}[\vartheta\wedge\vartheta]\right)^i\right)$$

Similar theory for gravity (F.H.–D. Vey, 2016)

A family of models, I

From topological bundles to (locally) principal bundles

Let $\mathfrak{g}$ be a Lie algebra of dimension r.

Let $\mathcal{X}^n$ be a manifold of dimension n with a coframe $(\underline{\beta}^\lambda)_{1\leq\lambda\leq n}$.

Let $P : \mathcal{F}^{n+r} \longrightarrow \mathcal{X}^n$ be *topological* bundle and $\beta^\lambda := P^*\underline{\beta}^\lambda$.

On the set of $(\theta^i, \pi_i) \in \left(\mathfrak{g}\otimes\Omega^1(\mathcal{F})\right) \oplus \left(\mathfrak{g}^*\otimes\Omega^{n+r-2}(\mathcal{F})\right)$ such that rank $(\beta^\lambda, \theta^i) = n + r$, define

$$\mathcal{A}[\theta^i, \pi_i] := \int_{\mathcal{F}} \pi_i \wedge (\mathrm{d}\theta + \frac{1}{2}[\theta\wedge\theta])^i$$

Then critical points **under the constraint** $\beta^\lambda \wedge \beta^\mu \wedge \pi_i = 0$ satisfy

$$\left\{\begin{array}{rcl} (\mathrm{d}\theta + \frac{1}{2}[\theta\wedge\theta])^i & = & \frac{1}{2}\Theta^i{}_{\lambda\mu}\beta^\lambda\wedge\beta^\mu + 0\cdot\beta^\lambda\wedge\theta^j + 0\cdot\theta^i\wedge\theta^j \\ d\pi + \mathrm{ad}^*_\theta\wedge\pi & = & 0 \end{array}\right.$$

$\longrightarrow$ **local principal** bundle structure, θ^i defines a connection on it

A family of models, II

A variant leading to Yang–Mills fields

Same fields as previously, but

We do not assume the constraint $\beta^{\lambda} \wedge \beta^{\mu} \wedge \pi_i = 0$

We add an extra term $\int_{\mathcal{F}} \frac{1}{4}|\pi_i{}^{\lambda\mu}|^2 \beta^{(n)} \wedge \theta^{(r)}$ to the action and recover:

$$\mathcal{A}[\theta^i, \pi_i] := \int_{\mathcal{F}} \frac{1}{4}|\pi_i{}^{\lambda\mu}|^2 \beta^{(n)} \wedge \theta^{(r)} + \pi_i \wedge \left(\mathrm{d}\theta + \frac{1}{2}[\theta \wedge \theta]\right)^i$$

Then critical points satisfy

$$\left\{ \begin{array}{rcl} (\mathrm{d}\theta + \frac{1}{2}[\theta \wedge \theta])^i{}_{\lambda\mu} & = & -\mathrm{k}^{ij}\eta_{\lambda\nu}\eta_{\mu\sigma}\pi_j{}^{\nu\sigma} + 0 + 0 \\ (d\pi + \mathrm{ad}^*_\theta \wedge \pi)_i & = & \frac{1}{2}|\pi_i{}^{\lambda\mu}|^2 \beta^{(n)} \wedge \theta_i^{(r-1)} \end{array} \right.$$

$\longrightarrow$ **local principal** bundle structure **and** θ^i defines a **Yang–Mills** connection on it

A family of models, III

Gravity

$\mathcal{Y}$ is an oriented 10-dimensional manifold and $\mathfrak{p} := \mathbb{R}^4 \oplus so(1,3)$ is the Poincaré Lie algebra

Fields are pairs $(\varphi^A, \pi_A) = (\varphi^\lambda, \varphi^i, \pi_\lambda, \pi_i) \in (\mathfrak{p} \otimes \Omega^1(\mathcal{Y})) \oplus (\mathfrak{p}^* \otimes \Omega^8(\mathcal{Y}))$, with the **constraints** $\varphi^\lambda \wedge \varphi^\mu \wedge \pi_i = 0$ and $\varphi^\lambda \wedge \varphi^\mu \wedge \pi_i = \kappa_i{}^{\lambda\mu}\varphi^{(10)}$

The action is

$$\mathcal{A}[\theta^A, \pi_A] := \int_{\mathcal{F}} \pi_A \wedge (\mathrm{d}\theta + \frac{1}{2}[\theta \wedge \theta])^A$$

Then critical points endow $\mathcal{Y}$ with a local principal bundle structure over a 4-dimensional manifold $\mathcal{X}$ and φ^A corresponds to a vierbein of $\mathcal{X}$ plus a connection 1-form which are solutions of the Einstein–Cartan system of equations.

One can then incorporate fermions ($\longrightarrow$ Einstein–Cartan–Dirac system of equations) : work by Jérémie **Pierard de Maujouy**

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