Calogero-Ruijsenaars family of classical integrable systems

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Journées Physique-Mathématique Dijon-Lyon-Metz
25/09/2024

Plan for the talk

1. **Models in the RS family**
2. Geometric point of view
3. Going further...

JOURNAL OF THE PHYSICAL SOCIETY OF JAPAN, Vol. 22, No. 2, FEBRUARY, 1967

Vibration of a Chain with Nonlinear Interaction

Morikazu TODA

Department of Physics, Faculty of Science, Tokyo University of Education, Tokyo

(Received September 27, 1966)

Vibration of a chain of particles interacting by nonlinear force is investigated. Using a transformation exact solutions to the equation of motion are aimed at. For a special type of interaction potential of the form

$$\phi(r)=\frac{a}{b}e^{-br}+ar+\text{const.},\ (a,\ b>0)$$

exact solutions are actually obtained in terms of the Jacobian elliptic functions. It is shown that the system has N "normal modes". Expansion due to vibration or "thermal expansion" of the chain is also discussed.

$$H=\frac{1}{2m}\sum_{j=1}^{n}p_j^2+\sum_{j=1}^{n-1}\phi(q_{j+1}-q_j)$$

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 10, NUMBER 12 DECEMBER 1969

Ground State of a One-Dimensional N-Body System

F. CALOGERO
Istituto di Fisica, Università di Roma and Istituto Nazionale di Fisica Nucleare
Sezione di Roma, Rome, Italy

(Received 14 March 1969)

The problem of N quantum-mechanical equal particles interacting pairwise by inverse-cube forces ("centrifugal potential") in addition to linear forces ("harmonical potential") is considered in a one-dimensional space. An explicit expression for the ground-state energy and for the corresponding wave-function is exhibited. A class of excited states is similarly displayed.

$$\left[-\frac{1}{2m}\sum_{j=1}^{n}\frac{\partial^2}{\partial q_j^2}+\frac{1}{4}m\,\omega^2\sum_{i<j}(q_i-q_j)^2+g\sum_{i<j}\frac{1}{(q_i-q_j)^2}-E\right]\psi=0$$

PHYSICAL REVIEW A VOLUME 4, NUMBER 5 NOVEMBER 1971

Exact Results for a Quantum Many-Body Problem in One Dimension*

Bill Sutherland
Physics Department, University of California, Berkeley, California 94720
(Received 12 April 1971)

We investigate exactly a system of either fermions or bosons interacting in one dimension by a two-body potential $V(r)=g/r^2$ with periodic boundary conditions. In addition to rederiving known results for correlation functions and thermodynamics in the thermodynamic limit, we present expressions for the one-particle density matrix at zero temperature and particular (nontrivial) values of the coupling constant g, as a determinant of order $N\times N$. These concise expressions allow a discussion of the momentum distribution in the thermodynamic limit. In particular, for a case of repulsive bosons, the determinant is evaluated explicitly, exhibiting a weak (logarithmic) singularity at zero momentum, and vanishing outside of a "Fermi" surface.

$$\left[-\frac{1}{2}\sum_{j=1}^{n}\frac{\partial^2}{\partial q_j^2}+\frac{g\pi^2}{L^2}\sum_{i<j}\frac{1}{\sin^2\frac{\pi(q_i-q_j)}{L}}-E\right]\psi=0$$

Periodic: $\psi(x_1,\ldots,x_j+L,\ldots,x_n)=\psi(x_1,\ldots,x_j,\ldots,x_n),\ \forall j$

ADVANCES IN MATHEMATICS 16, 197–220 (1975)

Three Integrable Hamiltonian Systems Connected with Isospectral Deformations*

J. MOSER

Courant Institute of Mathematical Sciences, New York University, New York, New York 10012

DEDICATED TO STAN ULAM

"Therefore it seems an anachronismus to discuss these exceptional integrable systems nowadays."

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- Proves classical Liouville integrability of previous 3 examples
- Promotes Lax pair for Liouville integrability
- $\omega = 0$ case for classical Calogero, but mentions Adler for $\omega \neq 0$

The Lax pair fo the Calogero system by Moser:

$$L = Y + \mathrm{i}\, Z_1, \qquad B = \mathrm{i}\, D_2 - \mathrm{i}\, Z_2$$

for:

$$Z_\alpha = ((1-\delta_{kl})\,(q_k - q_l)^{-\alpha})_{lk},$$

$$Y = \mathrm{diag}(p_1, \ldots, p_n),$$

$$D_\alpha = \mathrm{diag}(\textstyle\sum_{j\neq k}(q_k - q_j)^{-\alpha})$$

First "family direction" : the potential

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + g^2 \sum_{i<j} V(q_i - q_j)$$

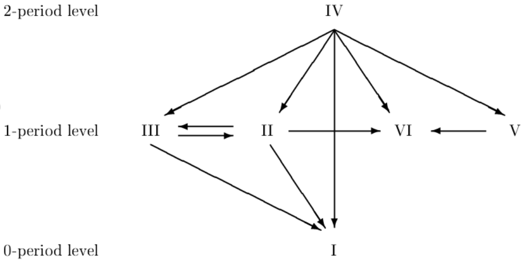
- I. (rational) $V(q) = q^{-2}$ [Calogero,'69]
- III. (trigo.) $V(q) = \sin(q)^{-2}$ [Sutherland,'71]
- II. (hyperbo.) $V(q) = \sinh(q)^{-2}$ [Calogero-Marchioro-Ragnisco,'75]
- IV. (elliptic) $V(q) = \wp(q)$ [Calogero,'75]

First "family direction" : the potential (bis)

In fact...

To conclude this section, we detail similar limit relations between the Lax matrices for the six types of systems introduced above. These can be encoded in the following hierarchy:

(2.104)



[Ruijsenaars, *Systems of Calogero-Moser type* at 1994 Banff Summer School, '99]

Inventiones math. 37, 93 – 108 (1976)

Inventiones mathematicae

Completely Integrable Hamiltonian Systems Connected with Semisimple Lie Algebras

M. A. Olshanetsky* and A. M. Perelomov
Institute for Theoretical and Experimental Physics, 117259 Moscow, USSR

1. Introduction

The classical one-dimensional n-body system is described by the Hamiltonian[1]

$$H = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + U(q_1, \ldots, q_n) \tag{1.1}$$

where q_k are coordinates, p_k are momenta and potential U is of the form

$$U = g^2 \sum_{k<l} V(q_k - q_l). \tag{1.2}$$

In his recent paper [2] Moser considered such systems with potential $V(q) = q^{-2}$ (Calogero model [3]) and with $V(q) = a^2 \sinh^{-2} aq$ (Sutherland model [4]). Using the Lax method [5] Moser has found in explicit form n independent integrals of motion. For the Calogero model he has also shown that these integrals are in involution (i.e. the Poisson brackets of any two integrals equal zero) and consequently according to Liouville theorem the system is completely integrable.

The Moser's method was used to investigate analogous systems with potential $V(q) = a^2 \sinh^{-2} aq$ [6] and $V(q) = a^2 \wp(aq)$ [7] ($\wp(aq)$ is the Weierstrass function). In paper [6] the corresponding quantum systems are also considered.

In this paper we investigate the systems of the type (1.1) with potential

$$U(q) = g^2 \sum_{k<l} [V(q_k - q_l) + \varepsilon V(q_k + q_l)] + g_1^2 \sum_{k=1}^{n} V(q_k) + g_2^2 \sum_{k=1}^{n} V(2q_k) \tag{1.3}$$

where function $V(q)$ has the form

I $V(q) = q^{-2}$, **II** $V(q) = a^2 \sinh^{-2} aq$, **III** $V(q) = a^2 \sin^{-2} aq$,

IV $V(q) = a^2 \wp(aq)$.

Second "family direction" : root systems

$$\mathbf{V}(q) = g^2 \sum_{i<j} \Big[V(q_i - q_j) + \epsilon\, V(q_i + q_j) \Big] + g_1^2 \sum_{j=1}^{n} V(q_j) + g_2^2 \sum_{j=1}^{n} V(2q_j)$$

- A_{n-1} for $\epsilon = 0,\ g_1 = 0,\ g_2 = 0$
- B_n for $\epsilon = 1,\ g_2 = 0$
- C_n for $\epsilon = 1,\ g_1 = 0$
- D_n for $\epsilon = 1,\ g_1 = 0,\ g_2 = 0$
- BC_n for $\epsilon = 1$

ANNALS OF PHYSICS 170, 370–405 (1986)

A New Class of Integrable Systems and Its Relation to Solitons

S. N. M. RUIJSENAARS AND H. SCHNEIDER

Mathematics Department, Tübingen University,
Tübingen, Federal Republic of Germany

Received November 26, 1985

We present and study a class of finite-dimensional integrable systems that may be viewed as relativistic generalizations of the Calogero–Moser systems. For special values of the coupling constants we obtain N-particle systems that are intimately related to the N-soliton solutions of the sine–Gordon and Korteweg–de Vries equations, among other ones.

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- (CM) H, $P = \sum_j p_j$, $B = \sum_j q_j$ yields
  $\{H, P\} = 0$, $\{H, B\} = P$, $\{P, B\} = n$ (Galilean Lie algebra)

- (rel.CM) $H = c^2 \sum_j \cosh(p_j/c)$, $P = c \sum_j \sinh(p_j/c)$, $B = \sum_j q_j$ yields
  $\{H, P\} = 0$, $\{H, B\} = P$, $\{P, B\} = H/c^2$ (Poincaré Lie algebra)

Third "family direction" : relativistic case

Other constructions from [Ruijsenaars-Schneider,'86] (and later):

- Lax matrix with $L_{rel} = \mathrm{Id}_n + c^{-1} L_{nr} + O(c^{-2})$ as $c \to \infty$
- Degeneration $H_{rel} = n\, c^2 + H_{nr} + O(c^{-2})$ as $c \to \infty$

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Important interpretation:

from "rational momentum" $H_{nr} = \frac{1}{2} \sum_j p_j^2 + \ldots$
to "trigo/hyperbo momentum" $H_{rel} = c^2 \sum_j \cosh(p_j/c)$

Physica 11D (1984) 337–348
North-Holland, Amsterdam

A GENERALISATION OF THE CALOGERO–MOSER SYSTEM

John GIBBONS* and Theo HERMSEN
Istituto di Fisica, Università di Roma, 00185 Rome, Italy
and
Istituto Nazionale di Fisica Nucleare, sezione di Roma, Italy

Received 4 January 1983

A generalised Calogero–Moser many-body system, in which the particles possess extra internal degrees of freedom, is introduced and solved by the method of Olshanetsky and Perelomov.
Further, the hierarchy of flows commuting with this system is obtained from a hierarchy of linear systems possessing SL(N) symmetry, by the usual method of reduction; thus they are shown to be integrable.

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{i<j} \frac{\langle f_i|e_j\rangle \langle f_j|e_i\rangle}{(q_i - q_j)^2}$$

$\rightsquigarrow$ add canonically conjugate (co)vectors: $\langle f_j| \in (\mathbb{C}^d)^*$, $|e_k\rangle \in \mathbb{C}^d$, $\ell \geq 2$

This is a complex analytic system !

Volume 111A, number 3 PHYSICS LETTERS 2 September 1985

AN INTEGRABLE MARRIAGE OF THE EULER EQUATIONS WITH THE CALOGERO–MOSER SYSTEM

S. WOJCIECHOWSKI [1]

Department of Mathematics, UMIST, P.O. Box 88, Manchester M60 1QD, UK

Received 3 June 1985; accepted for publication 10 June 1985

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{i<j} \frac{l_{jk}l_{kj}}{(q_i - q_j)^2}$$

$\rightsquigarrow$ add matrix-valued spins $l_{jk} \in \mathfrak{o}(n)$

- Real system, but slightly different nature
- Superintegrability

Fourth "family direction" : spin case

- $H = \frac{1}{2}\sum_j p_j^2 + \sum_{i<j} \frac{\langle f_i | e_j \rangle \langle f_j | e_i \rangle}{(q_i - q_j)^2}$
  integrable under constraint $\langle f_j | e_j \rangle = g \in \mathbb{C}^\times$ (or matrix version)

- Degeneration $d = 1 \rightsquigarrow H = \frac{1}{2}\sum_j p_j^2 + g^2 \sum_{i<j} \frac{1}{(q_i - q_j)^2}$

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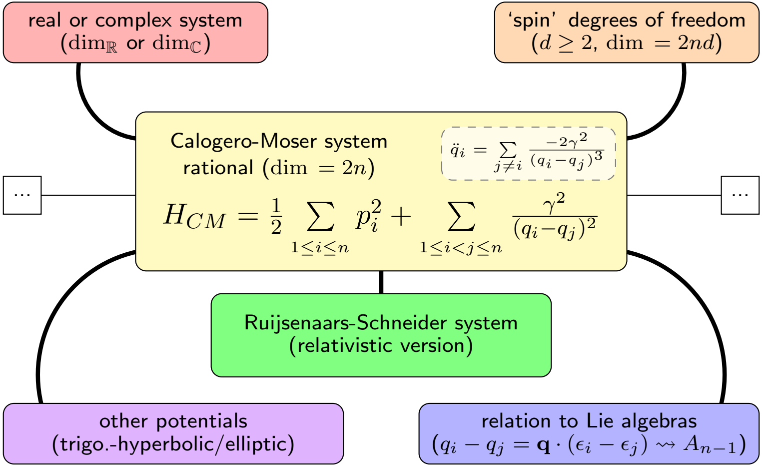
Bonus "family direction": Complex vs. Real system

The Calogero-Ruijsenaars family



Intersections of family directions

Some important works:

1. Elliptic + Lie algebras (5+2 parameters) [Inozemtsev,'89] , ...
2. (Real) Relativistic + Elliptic + Lie algebras [van Diejen,'95] , ...
3. (complex) Elliptic + spins [Krichever-Babelon-Billey-Talon,'95] , ...
   ... + relativistic [Krichever-Zabrodin,'95] , ...
4. ...

Plan for the talk

1. Models in the RS family
2. **Geometric point of view**
3. Going further...

Communications on Pure and Applied Mathematics, Vol. XXXI, 481–507 (1978)

0010-3640/78/0031-0481$01.00

Hamiltonian Group Actions and Dynamical Systems of Calogero Type

D. KAZHDAN, *Harvard University*, B. KOSTANT *Massachusetts Institute of Technology* AND S. STERNBERG *Harvard University*

Hamiltonian reduction of Calogero (rat CM) system from $T^*\mathfrak{u}(n)$

Hamiltonian reduction of Sutherland (trigo CM) system from $T^*U(n)$

The first reduction...

Hamiltonian reduction of Calogero system from [KKS,'78]

1. $T^*\mathfrak{u}(n) \simeq \{(A,B) \mid A, B \text{ selfadjoint}\} \overset{conj.}{\curvearrowleft} U(n)$
   Moment map: $\mu = -\mathrm{i}[A,B]$

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3. Consider symplectic manifold $\mu^{-1}(\mathcal{O}_v)/U(n)$ $(\lambda = -1)$

4. Consider slice with $A = \mathrm{diag}(q_1, \ldots, q_n)$, $q_1 \leq \ldots \leq q_n$. $\leq \rightsquigarrow <$

   $\rightsquigarrow$ B is the Lax matrix

Reduction in the "potential family"

1. Sutherland (type III) system from $T^*U(n)$ [KKS,'78]
2. (See [Wilson'98] for complex case)
3. Elliptic type IV system :
   only *infinite*-dimensional Hamiltonian reduction
   cf. [Nekrasov,'99]
4. There is finite-dim version! [F.-Chalykh,202x]

Reduction in the “root system family”

Basically play with $T^*\mathfrak{g}$ for I. (and T^*G for III.)

[many, many people...]

Reduction in the “relativistic family”

Poisson-Lie reduction from $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$

[Fock & Rosly, Poisson structures on moduli [...], '99] from '92 seminar

The Poisson structure on $\mathcal{A}^\delta$ and $\mathcal{G}^\delta$. — The next step is to define the Poisson structure. The idea is to construct Poisson structures both on $\mathcal{A}^\delta$ and $\mathcal{G}^\delta$, compatible in the sense that

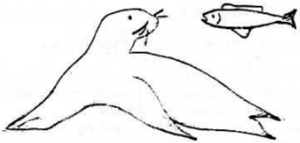


Figure 8: the Poisson structure considered by Fock

the action of the gauge group $\mathcal{G}^\delta$ is a Poisson action, that is:

- $\mathcal{G}^\delta$ is a Poisson Lie group
- $\mathcal{A}^\delta$ is a Poisson manifold
- $\mathcal{G}^\delta \times \mathcal{A}^\delta \to \mathcal{A}^\delta$ is a Poisson mapping.

[Audin,'97]

Reduction in the "relativistic family"

Alternative approach...

Using the Hamiltonian reduction on the combinatorial model of the space of flat connections on the torus without a point, the authors of [13] prove that the Poisson structure $\{\cdot,\cdot\}_{FR}$ has a holomorphic extension from $\mathbf{U}$ to the whole CM_τ, and this Poisson structure is nondegenerate (i.e. CM_τ is a symplectic variety). Another way to see this Poisson structure is to use Quasi-Poisson reduction [14]. In this picture the Poisson structure is the result of the reduction of the natural Quasi-Poisson structure on the product $GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$ and it is immediate that this Poisson structure is symplectic.

[Oblomkov, DAHA and CM spaces,'04]

$\rightsquigarrow$ quasi-Poisson reduction and Poisson-Lie reduction give the same[1] result!

[1] In this case only...

Reduction in the "spin family"

Two possible points of view:

- "Collective spins" : reduction at higher dim. coadjoint orbit
  see [Reshetikhin,'02], [Li-Xu,'02], ...
- "Vector spins" : Extend phase space by $T^*\mathbb{C}^d$ (or similar)
  see [Wilson,'09], ...
- Can also do both, cf. recent works of Fehér

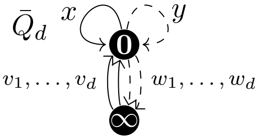
Some recent results...

Personal point of view on recent advances:

Some recent results...

Personal point of view on recent advances:

[Wilson,∼'98;Bielawski-Pidstrygach,'10; Tacchella,'15; Chalykh-Silantyev,'17]



$d \geq 2$. $\mathcal{M} = \mathrm{Rep}\left(\mathbb{C}\bar{Q}_d, (1,n)\right)$

space parametrised by:

$X, Y \in \mathfrak{gl}_n$

$V_\alpha \in \mathrm{Mat}_{1\times n}, W_\alpha \in \mathrm{Mat}_{n\times 1}$

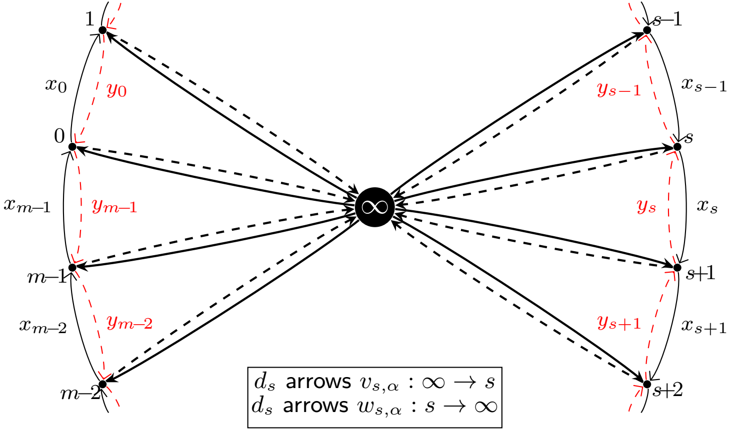
$$\mathcal{C}_{n,d} := \left\{ [X,Y] - \sum_{1\leq\alpha\leq d} W_\alpha V_\alpha = \lambda_0 \,\mathrm{Id}_n \right\} // \,\mathrm{GL}_n \qquad (\lambda_0 \neq 0)$$

Poisson structure at NC level [Van den Bergh,'08]

Some recent results...

Extended cyclic quiver ⇒ system of rat. CM type on QV

[Chalykh-Silantyev,'17] + [F.-Görbe,'21]



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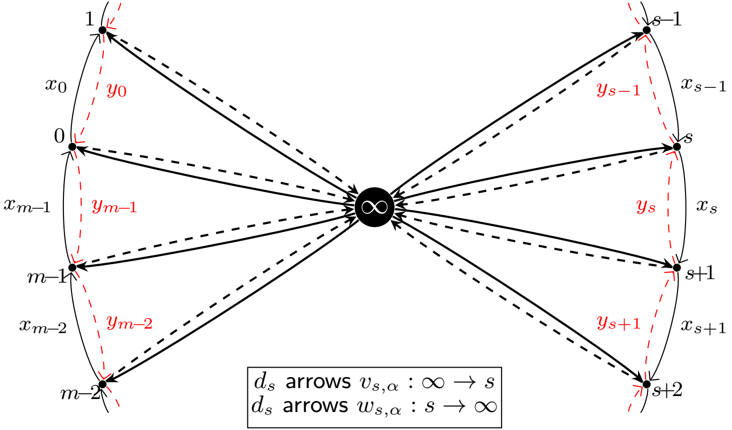

For trigo. RS ⇝ use multiplicative QV [Chalykh-F.,'17,'20], [F.,arXiv:2108.02496]

Some recent results...

- In [Arutyunov-Olivucci,'20] : spin trigo RS from Poisson-Lie theory
  Interesting/Annoying: different Poisson structure than [CF,'20]

  $\rightsquigarrow$ The two are compatible ! [F.,arXiv:2310.18751]

- spin trigo RS in real case :
  PL and qP reduction are **very** different!
  [F.-Feher-Marshall,'20] vs. [F.-Feher,'23]

  ($\hookrightarrow$ partial picture)

- [Feher,'22–'24] : superintegrability of "generic" collective spins

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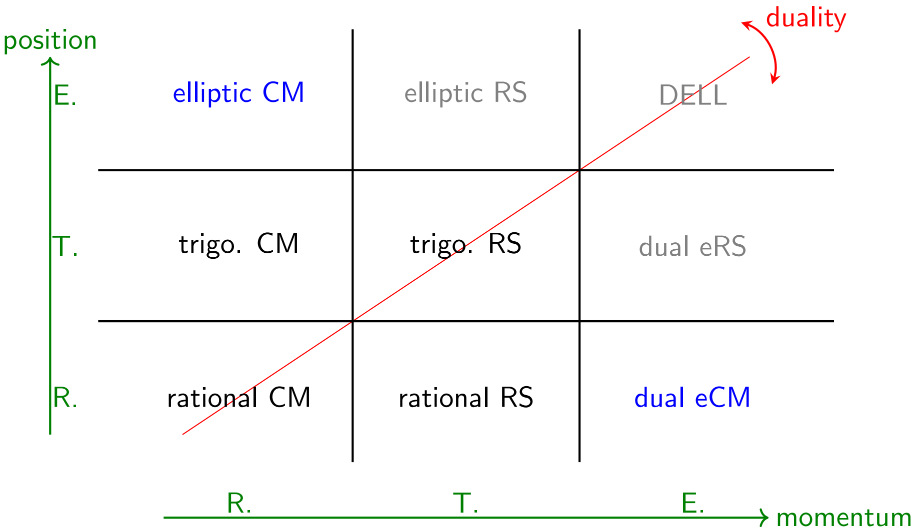
Ruijsenaars duality

[Fock-Gorsky-Nekrasov-Rubtsov,'00] and [Gorsky-Rubtsov,'01] $(n = 2)$



Going further

- Collective spins outside generic/lowest case

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- Integrability for *any* (M)QV

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- Integrability for *any* (M)QV
- Reduction of supersymmetric version
  In progress with T. Kimura

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- Collective spins outside generic/lowest case
- Integrability for *any* (M)QV
- Reduction of supersymmetric version
  In progress with T. Kimura
- Relation to integrable hierarchies
- ...

Thank you for your attention !

Maxime Fairon
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mfairon.perso.math.cnrs.fr