Lie-Poisson Equations:
From Quantum Liquids to Gravity

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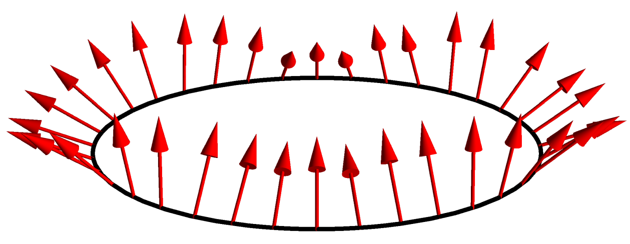
Plan of talk

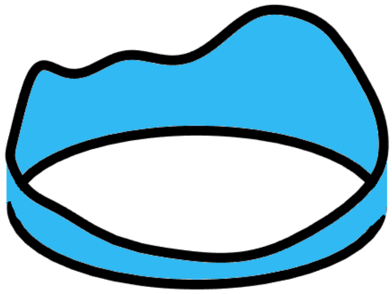
1. Lie-Poisson equations in general
2. Examples: mechanics, spin chains, fluids, gravity
3. Berry phases in Lie-Poisson equations

Rigid body configs = rotations

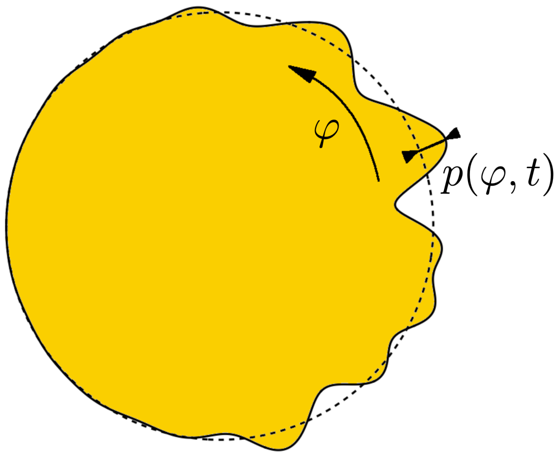
Spin chain configs = local rotations



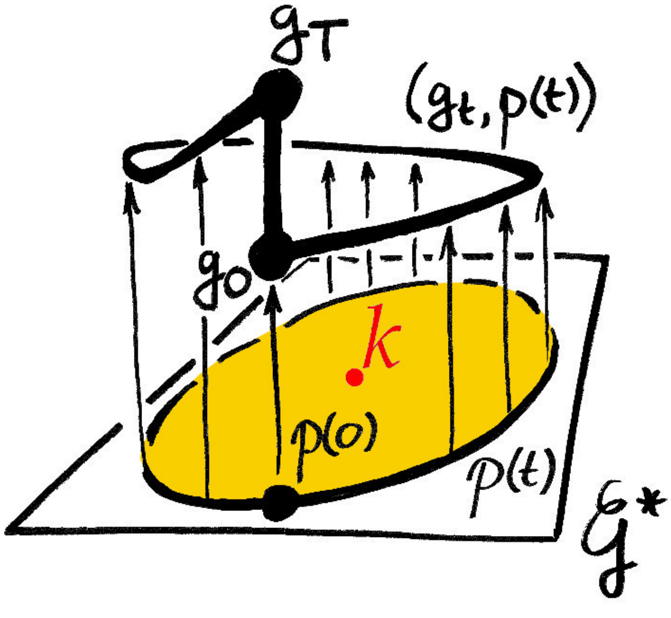
Shallow water configs = circle diffeomorphisms



Density waves on the edge of quantum fluids



Berry phases from periodic momenta



LIE-POISSON EQUATIONS: FROM QUANTUM LIQUIDS TO GRAVITY

Refs: 2002.01780 w/ Kozyreff, 2408.03991 w/ Beauvillain & Petropoulos

Hello, thx for invite. Pleasure, hope you'll enjoy.

Motivate: Geometry of physics. Useful coz:
- Understand structure of phys theories; – Make new predictions.

Here LP eqns, blend Lie groups & symplectic (Poisson) geometry.
Goal of talk: Argue LP eqns are everywhere, explain some of my work

Plan: 1. LP eqns in general, 2. Examples, 3. Berry phases.

1. LP eqns in general: A. Lie groups

Let G Lie group, elems $f, g, \ldots$, $\mathfrak{g}$ Lie algebra, elems $X, Y, \ldots$

$\mathfrak{g}^*$ dual vector space of $\mathfrak{g}$, elems p, k,

each a linear map $p: \mathfrak{g} \to \mathbb{R}: X \mapsto \langle p, X \rangle$

Defs: Adjoint rep of G given by $\mathrm{Ad}_g(X) \equiv \partial_t|_0 (g e^{tX} g^{-1})$

— — — $\mathfrak{g}$ — — $\mathrm{ad}_X(Y) \equiv \partial_t|_0 \mathrm{Ad}_{e^{tX}}(Y) = [X, Y]$ Lie bracket

Coadjoint rep of G — — $\langle \mathrm{Ad}^*_g(p), X \rangle \equiv \langle p, \mathrm{Ad}_{g^{-1}}(X) \rangle$

— — — $\mathfrak{g}$ — — $\mathrm{ad}^*_X(p) \equiv \partial_t|_0 \mathrm{Ad}^*_{e^{tX}}(p) = -p \circ \mathrm{ad}_X$

Write $\mathrm{Ad}^*_g(p) \equiv g \cdot p$ for brevity. Coadjoint orbit of k is $\mathcal{O}_k \equiv \{ g \cdot k \mid g \in G \}$

B. Phase spaces from Lie groups.

Cotangent bundle of any mf is symplectic

→ Consider phase space T^*G. Claim: $\cong G \times \mathfrak{g}^*$ as bundle,

with symplectic form $\Omega \equiv -dA$, $A_{(g,p)} \equiv \langle p, dg\, g^{-1} \rangle$. Keep till end

Here $dg\, g^{-1} \equiv$ right Maurer-Cartan form on G,

i.e. one-form valued in $\mathfrak{g}$ s.t., $\forall$ path γ in G s.t. $\gamma(0) = g$,

$$(dg\, g^{-1})(\partial_t|_0 \gamma) \equiv \partial_t|_0 (\gamma_t \cdot g^{-1}) \quad = \mathrm{id} \ @ \ t=0 \Rightarrow \in T_{\mathrm{id}} G = \mathfrak{g} \ ✓.$$

Example: $G = \mathbb{R}$ gives $A = p\,dx$, $\Omega = dx \wedge dp$.

→ Interpret G = space of posit°s / configurat°s
(more generally) $\mathfrak{g}$ = ______ velocities
$\mathfrak{g}^*$ = ______ momenta

C. LP eqns. Reminder: Given Hamiltonian fct H on symp mf, eom = flow of vect field ξ s.t. $\iota_\xi \Omega = dH$.

Exple: $G = \mathbb{R}$, get $\dot p = -\partial H/\partial x$, $\dot x = \partial H / \partial p$.

Consider $T^*G = G \times \mathfrak{g}^*$, let $H = H(p)$ translat°-invariant.

→ Corresp eom are LP eqns:

$$\dot p = \mathrm{ad}^*_{dH_p}(p) \quad \text{"force"}, \qquad \dot g g^{-1} = dH_p \quad \text{"velocity"}.$$ Keep till end

Here this is legal coz $H: \mathfrak{g}^* \to \mathbb{R} \Rightarrow dH_p : T_p\mathfrak{g}^* = \mathfrak{g}^* \to \mathbb{R}$ linear, $dH \in (\mathfrak{g}^*)^* =$

Note: × Translat° symmetry $(g,p) \mapsto (gf, p)$ $\forall f \in G$. If $\dim G = \infty$, huge!

× If g_t solves velocity eqn, then $p_t = g_t \cdot k$ solves momentum eq. keep till end

⇒ Any solut° of LP lives in a single orbit $\mathcal{O}_k$!

→ Conservat° law due to translat° symmetry.

Important special case: $H(p)$ quadratic

→ Write $H(p) = \frac{1}{2}\langle p, I^{-1}(p)\rangle$ with $I: \mathfrak{g} \to \mathfrak{g}^*$ inertia operator Keep for exples
(linear, invertible, positive, symmetric)

→ Velocity eqn $\dot g g^{-1} = I^{-1}(p)$

→ LP eqn = Euler-Arnold eqn
= geodesic in G for invariant metric.

2. Examples : A. Mechanics.

× $G = \mathbb{R}$, $I(X) = mX$ → $H = p^2/2m$, LP eq = free massive particle.

× $G = SO(3)$, choose I = inertia tensor.

→ Angular momentum eq: $\dot{p} = ad^*_{I^{-1}(p)}(p) \underset{\downarrow}{\cong} ad_{I^{-1}(p)}(p) = [I^{-1}(p), p] \cong I^{-1}(p) \wedge p$ vector prod in $\mathbb{R}^3$

→ Euler eqn for free rigid body! $ad \cong ad^*$

→ g_t s.t. $\dot{g}g^{-1} = I^{-1}(p)$ gives time-dep orientat° of body.

B. Spin chains

G = loop group of $SO(3)$ = {maps $S^1 \to SO(3) : \varphi \mapsto g(\varphi)$}

→ $\mathfrak{g}^*$ = {maps $S^1 \to \mathbb{R}^3 : \varphi \mapsto p(\varphi)$} local spin distribut°s,

with pairing $\langle p, X \rangle = \int d\varphi \, \langle p(\varphi), X(\varphi) \rangle$ keep for KacMoody (restrict to smooth dual)

Choose inertia $I^{-1}(p) \equiv -\partial_\varphi^2 p(\varphi) \equiv -p''$

→ LP eqn = Landau-Lifschitz eqn $\dot{p} = p \wedge p''$, $\dot{g}g^{-1} = -p''$.

Notes: × This is a continuous Heisenberg spin chain, $g(\varphi) \cong$ local wave fct

× Equivalent to nonlinear Schrödinger eqn, ubiquitous in nlin waves

× Also describes vortex filaments in hydrodynamics.

C. Classical & quantum liquids

General idea : fluid in domain M has configs ∈ Diff M, velocities ∈ Vect M

→ Hydrodynamics = LP eqn on $G \leq$ Diff(M).

→ Ideal fluid for G = SDiff(M), shallow water on S^1 for G = Diff S^1

Now consider seemingly unrelated $G = \widehat{C^\infty(S^1)} \ni (f, \alpha)$, $(f\alpha) * (g\beta) = (f+g, \alpha+\beta + \int d\varphi\, f g')$

→ $U(1)$ Kac-Moody group, dual vectors $p(\varphi)$ s.t. $\langle p, X \rangle = \int d\varphi\, p(\varphi) X(\varphi)$.

→ Choose $H[p]$, get LP eqns $\dot{p} = (\delta H/\delta p)'$, $\dot{g} = \delta H/\delta p$

→ Chiral density waves p, with fermion phase g (from bosonizat°)

= eom of edge modes of incompressible quantum liquids, ie topological insulators

→ $p(\varphi)$ stems from area-preserving diffeomo in 2D.

D. A trivial example: quantum gravity

Let $H(p) = \langle p, X\rangle$ for some fixed $X \in \mathfrak{g}$. Solut° of LP: $g_t = e^{tX} \cdot g_0$

→ Happens in (2+1)D gravity, where G = asymptotic symmetry group, typically $\supseteq \mathrm{Diff}\, S^1$

→ In quantum theory, initially coherent states remain coherent

⇒ Semiclassical analysis boils down to group theory.

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3. Berry phases in LP eqns

LP eqns have nice structure, but useful for phys? Predict anyth. new?

→ My work since 2017: Berry phases. Let me explain.

A. Periodic momenta. Consider mom. eqn $\dot{p} = \mathrm{ad}^*_{dH(p)}$.

Suppose its solut° is periodic: $p(t+T) = p(t)$, period T.

→ Implicat° for g_t? In general, $g_T \neq g_0$ but $p_T = g_T \cdot k = g_0 \cdot k = p_0$

⇒ $g_0^{-1} g_T$ stabilizes k, describes "rotat°" during one period.

Angle of that rotat°? Let's assume Stab_k Abelian

→ Universal cover of Stab is vector group ⇒ Write $g_0^{-1} g_T = \Delta\phi \in \mathfrak{g}$.

→ Integrate symplectic potential A along closure of g_t:

$$\oint A = \int_{\supset} A + \int_{|} A = \underbrace{\int_0^T dt\, \langle p, dH_p\rangle}_{} - \langle k, \Delta\phi\rangle$$

Rearrange this to find $\Delta\phi = \underbrace{\text{Berry phase}}_{\propto \oint A \,=\, \text{symplectic flux in } \mathcal{O}_k} + \text{Dynamical phase}$

⇒ $\Delta\phi$ is observable that probes geometry of $\mathcal{O}_k$!

B. Exples: × Spin chains: $\Delta\phi$ = rotat° of wave fct after one period

× Hydro: $\Delta\phi / T$ = drift velocity of fluid parcels

× Quantum liquids: $\Delta\phi$ = phase of fermion after period (QH interfer.)

→ All very much observable in experiments!