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F. Gieres:

Different approaches to the inverse problem of variational calculus

This talk: not focused on
- generality
- abstraction
- math. rigor

but rather puts forward concreteness + simplicity

The notion of inverse problem:

Reminder 1: Books on class. non-relat. mech.:

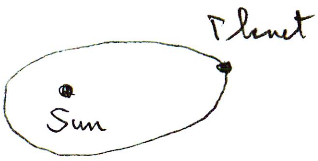
Find the solution(s) $t \longmapsto \vec{x}(t) \in \mathbb{R}^3$ of the e.o.m.
(syst. of SODE's)

$$m \ddot{\vec{x}} = \underbrace{\vec{F}}_{\text{given}}(\vec{x}, \dot{\vec{x}}, t) \quad \hat{=} \quad \text{Direct problem}$$

e.g. $F_{grav} \propto \frac{1}{r^2}$ with $r \equiv \|\vec{x}\|$.

Inverse problem:

Deduce $\vec{F}$ from the solut. of the e.o.m.



Other contexts: Particle physics, math, ...

Newton's solution: Consider a circular orbit in the xy-plane

Kepler 3

$$\left.\begin{array}{l} \dfrac{r^3}{T^2} = \text{const} \\ \omega = \dfrac{2\pi}{T} \end{array}\right\} \Rightarrow r^3 \omega^2 = \text{const} \quad \Rightarrow r\omega^2 \sim \frac{1}{r^2}$$

Kepler 2:

$$\vec{x}(t) = (r \cos \omega t)\, \vec{e}_x + (r \sin \omega t)\, \vec{e}_y$$

$$\Rightarrow \ddot{\vec{x}} = - \omega^2 \vec{x} \quad \Rightarrow \boxed{\|\ddot{\vec{x}}\| = \omega^2 r \sim \frac{1}{r^2}}$$

Reminder 2: [Lagrange 1788]

For $\vec{F}(\vec{x}) = -\vec{\nabla} \underbrace{V(\vec{x})}_{\text{pot.}}$, on defines the

Lagr. fct $L(\vec{x}, \dot{\vec{x}}, \cancel{t}) \equiv \frac{m}{2} \dot{\vec{x}}^2 - V(\vec{x})$

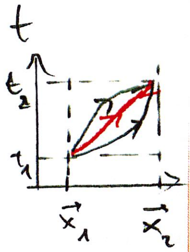
Action fctl $S: \{t \mapsto \vec{x}(t)\} \longrightarrow \mathbb{R}$

$$\vec{x} \longmapsto S[\vec{x}] \equiv \int_{t_1}^{t_2} dt\, L(\vec{x}(t), \dot{\vec{x}}(t), t)$$

Extremal pts of S:

$0 = \delta S[\vec{x}]$ for variations $\vec{x}(t) \rightsquigarrow \vec{x}(t) + \delta \vec{x}(t)$

with $\delta \vec{x}(t_{1,2}) = \vec{0}$



$$\Leftrightarrow 0 = -\frac{\delta S}{\delta x_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} \quad \left(= (m \dot{x}_i)' + \partial_i V\right)$$

Direct problem: S given $\rightsquigarrow$ e.o.m.

Inverse —: $S \leftsquigarrow$ e.o.m.

Questions : existence , uniqueness , construct. of S

Interest : Noether's theorem , quantization , statist. mechanics etc.

⇝ General problem (for mech.) : For $i \in \{1, \dots, n\}$

Given SODE's $\quad 0 = E_i(t, \vec{x}, \dot{\vec{x}}, \ddot{\vec{x}}) \quad (= \ddot{x}_i - f_i(t, \vec{x}, \dot{\vec{x}}))$

$$\exists ? \text{ fctl } S[\vec{x}] \quad / \quad E_i = - \frac{\delta S}{\delta x_i}$$

History :

Analyt/geom approach

1887 Helmholtz

1941 J. Douglas $(n=2)$

1984 Crampin, Prince, Thompson

2024 $\quad n=3$ almost solved

Fct'l approach

18~~8~~87 V. Volterra

1913 V. Volterra

1954 M. Vainberg

1969 : E. Tonti

Reminder 3:

Vector analysis in $\mathbb{R}^3$: $\qquad \vec F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

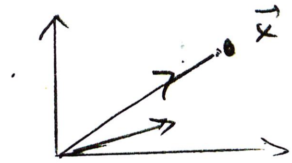
$$\vec F = -\vec\nabla V \qquad \Longleftrightarrow \qquad \mathrm{rot}\,\vec F = \vec 0$$

$$F_i = -\partial_i V \qquad \underset{[\partial_i,\partial_j]=0}{\Longleftrightarrow} \qquad \partial_i F_j - \partial_j F_i = 0$$

(Integrability condit for $\vec F = -\vec\nabla V$)

For $\mathrm{rot}\,\vec F = 0$; consider

$$s \longmapsto \vec r(s) = s\,\vec x \qquad \text{with } s\in[0,1]$$



and choose $V(\vec 0) = 0$. Then

$$V(\vec x) = -\int_0^{\vec x} \vec F(\vec r)\cdot \underbrace{d\vec r}_{\substack{=\frac{d\vec r}{ds}ds \\ = \vec x\, ds}} \qquad \Rightarrow \qquad V(\vec x) = -\int_0^1 ds\, \vec x\cdot\vec F(s\vec x)$$

Volterra (1887, 1913). $\vec E \equiv (E_i) : \{t \longmapsto \vec x(t)\} \longrightarrow \{t \longmapsto \vec y(t)\}$

$$E_i[\vec x] = y_i = 0$$

$$E_i = -\frac{\delta S}{\delta x_i} \qquad \underset{\left[\frac{\delta}{\delta x_i},\frac{\delta}{\delta x_j}\right]=0}{\Longleftrightarrow} \qquad \boxed{\frac{\delta E_i}{\delta x_j} - \frac{\delta E_j}{\delta x_i} = 0}$$

(Integrability condit for $E_i = -\frac{\delta S}{\delta x_i}$)

~ Lagr. fct: VVT-Lagr

$$L[\vec x] \equiv -\int_0^1 ds\, x^i E_i(t, s\vec x, s\dot{\vec x}, s\ddot{\vec x})$$: depends on $\dddot{\vec x}$

— — $\ddot{\vec x}$

Gauge transf. $L \rightsquigarrow L + \frac{d}{dt} g(\vec x, \dot{\vec x}, t)$

Tonti 1969 :

$$\frac{\delta E_i(t)}{\delta x_j(t')} = \frac{\partial E_i}{\partial x_k}(t) \underbrace{\frac{\partial x_k(t)}{\partial x_j(t')}}_{= \delta_{jk}\,\delta(t-t')} + \frac{\partial E_i}{\partial \dot{x}_k} \ldots + \ldots$$

The integrability condit. reads [Helmholtz]

$$0 = O_{ij}(t)\,\delta(t-t') + I_{ij}(t)\,\frac{d}{dt}\delta(t-t') + II_{ij}(t)\,\frac{d^2}{dt^2}\delta(t-t')$$

$\Leftrightarrow$ (1), (2), (3)

(1) $$\frac{\partial E_i}{\partial x^j} - \frac{\partial E_j}{\partial x^i} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial E_i}{\partial \dot{x}^j} - \frac{\partial E_j}{\partial \dot{x}^i}\right) = 0$$

(2) $$\frac{\partial E_i}{\partial \dot{x}^j} + \frac{\partial E_j}{\partial \dot{x}^i} - \frac{d}{dt}\left(\frac{\partial E_i}{\partial \ddot{x}^j} + \frac{\partial E_j}{\partial \ddot{x}^i}\right) = 0$$

(3) $$\frac{\partial E_i}{\partial \ddot{x}^j} - \frac{\partial E_j}{\partial \ddot{x}^i} = 0$$

Syst. of PDE's

Solution for $n = 1, 2, 3, \ldots$: ?

For $n = 1$; $x_1 \equiv x$.

Ex : $$0 = E(x, \dot{x}, \ddot{x}) \equiv \ddot{x} + \underbrace{\omega^2}_{>0} x + \underbrace{\gamma}_{>0} \dot{x}$$

(2) $$0 = \frac{\partial E}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial E}{\partial \ddot{x}}\right) = \gamma \quad \Rightarrow \quad \nexists \text{ Lagr.}$$

Equivalent e.o.m. :

$$\underbrace{e^{\gamma t}}_{\text{multiplier}}(\ddot{x} + \omega^2 x + \gamma x) = 0 \quad \rightsquigarrow \quad L = e^{\gamma t}\left(\frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2\right)$$

$\rightsquigarrow$ Question for general $n \geq 1$:

$\exists ?\ \underbrace{(g_{ij}(t, \vec{x}, \dot{\vec{x}}))}_{\text{invertible matrix}}$ / $\boxed{g_{ij}\,(\ddot{x}^j - f^j) = -\frac{\delta S}{\delta x^i}}$

"Multiplier inverse problem"
(Helmholtz condit $\rightsquigarrow$ condit for g_{ij})
[*]

Answer :

For $n = 1$: Yes

– $n = 2$: J. Douglas : 4 cases with numerous subcases

(Case 1 : ϕ = multiple of id,
– 2 : $\nabla\phi$ = lin. combinat of ϕ and id
Subcases $\hat{=}$ $\phi \overset{?}{=}$ diag'ble, properties of ~~eigenspaces~~)

For $n = 3$: No complete solut.

Problem : the analytic approach of Douglas is complicated.

[*] $$-\frac{\delta S}{\delta x^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \ldots = \underbrace{\boxed{\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}}}_{= g_{ij}(\vec{x}, \dot{\vec{x}}, t)} \ddot{x}^j + \ldots$$

"Geometrization":

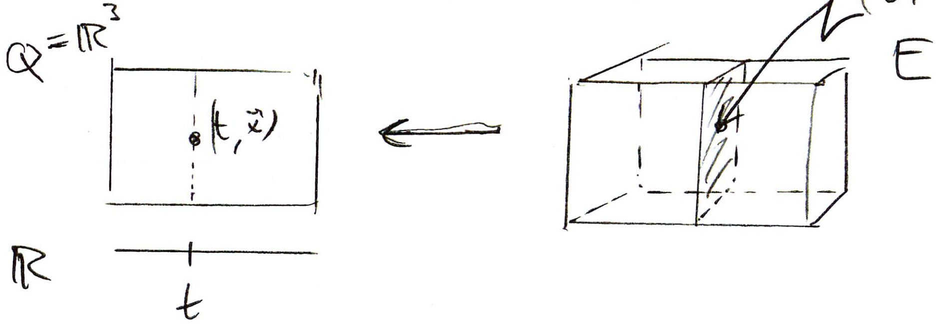
F. Gallissot 1951 : Rewrite Newton's e.o.m in terms of diff. forms
(→ E. Cartan, A. Lichnerowicz)

$$m\ddot{\vec{x}} = \vec{F}(\vec{x}) \iff \begin{cases} u^i = \dot{x}^i \\ m\dot{u}^i = F^i \end{cases}$$

$$\boxed{\omega \equiv F_i\, dx^i \wedge dt - m u_i\, du^i \wedge dt - m\, dx^i \wedge du_i}$$

= 2-form on evolution space $\left(Q \equiv \mathbb{R}^3 \ni \vec{x}\right)$

$$E \equiv \underbrace{\mathbb{R}}_{t} \times \underbrace{TQ}_{\vec{x},\,\vec{u}} \cong J^1(\mathbb{R}\times Q \longrightarrow \mathbb{R})$$



$\frac{\partial}{\partial x^i}, \ \frac{\partial}{\partial u^i}$ = characterist v.f. of ω:

$$) = i_{\partial/\partial x^i}\,\omega = F^i dt - m\, du^i = (F^i - m\dot{u}^i)\, dt$$

$$) = i_{\partial/\partial u^i}\,\omega = -m u_i\, dt + m\, dx^i = -m(u^i - \dot{x}^i)\, dt$$

"eqs associated with ω"

We have

$$d\omega = dF_j \wedge dx^j \wedge dt = \tfrac{1}{2}(\partial_i F_j - \partial_j F_i)\, dx^i \wedge dx^j \wedge dt$$

i.e. $d\omega = 0 \iff \vec{F} = -\vec{\nabla} V$

Then $\boxed{\omega = d\vartheta_L}$ with $\boxed{\vartheta_L = \frac{\partial L}{\partial u^i}\theta^i + L\, dt}$ Poincaré - Cartan 1 - form

where $L \equiv \frac{m}{2}\vec{u}^2 - V(\vec{x})$, $\theta^i \equiv dx^i - u^i dt$ (contact 1-form)

$\left(\vartheta_L = p_i dx^i - H dt \equiv \vartheta_H \right)$

⇝ The multiplier inverse problem amounts to:

Given $\vec{f}(\vec{x}, \dot{\vec{x}}, t)$, i.e. ~~the~~ SODE v.f. $\Gamma \equiv \frac{\partial}{\partial t} + u^i \frac{\partial}{\partial x^i} + f^i \frac{\partial}{\partial u^i}$, the existence of a Lagr., i.e. the existence of an invertible matrix $(g_{ij}(\vec{x}, \dot{\vec{x}}, t))$ satisfying the Helmholtz condit.$^{(HC)}$ is equivalent to the existence of a 2-form ω on E satisfying

$d\omega = 0 \quad (\iff HC)$

ω = horizontal (i.e. $\omega\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = 0$)

ω has max. rank

$i_\Gamma \omega = 0$

Expression of ω :

$$\omega = g_{ij}\, \psi^i \wedge \theta^j \quad \text{with} \begin{cases} \theta^i = dx^i - u^i\, dt \\ \psi^i = du^i - f^i\, dt + \underbrace{A^i_j}_{= -\frac{1}{2}\frac{\partial f^i}{\partial x^j}}\, \theta^j \end{cases}$$

[#]

Other approaches : $\vec{\xi} = (\xi^I) \equiv (x^1, \dots, x^n, u^1, \dots, u^n)$

Newton's eqs $\Longleftrightarrow$ $\dot{\xi}^I = F^I(\vec{\xi}, t)$

$\rightsquigarrow$ Lagr. $L(\vec{\xi}, \dot{\vec{\xi}}, t) = a_I(\vec{\xi}, t)\, \dot{\xi}^I - H(\vec{\xi}, t)$

with a_I, H satisfying certain condit.

$\boxed{\text{Case of field theory}}$:

$\vec{x}(t) \rightsquigarrow \vec{\varphi}(\vec{x}, t)$

SODE $\rightsquigarrow$ SOPDE

$\omega = d\vartheta_L \rightsquigarrow$ exact multisymplect. form $(n+1)$-form

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[#] Approach of EDS theory:

Find all closed, max. rank 2-forms

in $\text{Span}\{\psi^i \wedge \theta^j\} \subset \Omega^2(E)$

1 Summary

Inverse problem of Newtonian or Lagrangian mechanics: Given the

$$\text{equation of motion functions}: \qquad E_i(t, \vec{x}, \dot{\vec{x}}, \ddot{\vec{x}}) \qquad \text{with } i \in \{1, \dots, n\}, \tag{1.1}$$

we have

$$E_i = -\delta S/\delta x^i \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}\, \ddot{x}^j + \frac{\partial^2 L}{\partial \dot{x}^i \partial x^j}\, \dot{x}^j + \frac{\partial^2 L}{\partial \dot{x}^i \partial t} - \frac{\partial L}{\partial x^i}, \tag{1.2}$$

if the functions E_i satisfy the following set of integrability conditions (**Helmholtz conditions**):

$$0 = \frac{\partial E_i}{\partial \ddot{x}^j} - \frac{\partial E_j}{\partial \ddot{x}^i} \tag{1.3a}$$

$$0 = \frac{\partial E_i}{\partial \dot{x}^j} + \frac{\partial E_j}{\partial \dot{x}^i} - \frac{d}{dt}\left(\frac{\partial E_i}{\partial \ddot{x}^j} + \frac{\partial E_j}{\partial \ddot{x}^i}\right) \tag{1.3b}$$

$$0 = \frac{\partial E_i}{\partial x^j} - \frac{\partial E_j}{\partial x^i} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial E_i}{\partial \dot{x}^j} - \frac{\partial E_j}{\partial \dot{x}^i}\right). \tag{1.3c}$$

Multiplier inverse problem: Let a force field $\vec{f}(t, \vec{x}, \dot{\vec{x}})$ be given. To study the problem

$$g_{ij}(\ddot{x}^j - f^j) = -\frac{\delta S}{\delta x^i} \qquad \text{for } i \in \{1, \dots, n\}, \tag{1.4}$$

one introduces the quantities ($\vec{u} \doteq \dot{\vec{x}}$)

$$\Gamma \equiv \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + f^i(t, \vec{x}, \dot{\vec{x}}) \frac{\partial}{\partial \dot{x}^i} \qquad \text{(SODE vector field)}$$

$$A^i_j \equiv -\frac{1}{2}\frac{\partial f^i}{\partial u^j}, \qquad \Phi^i_j \equiv -\frac{\partial f^i}{\partial x^j} - A^i_k A^k_j - \Gamma(A^i_j). \tag{1.5}$$

The **Helmholtz conditions** then read

$$\boxed{g_{ij} = g_{ji}, \qquad \frac{\partial g_{ij}}{\partial u^k} = \frac{\partial g_{ik}}{\partial u^j}, \qquad g_{ik}\Phi^k_j = g_{jk}\Phi^k_i, \qquad \Gamma(g_{ij}) = g_{ik}A^k_j + g_{jk}A^k_i}\,. \tag{1.6}$$

If a solution g_{ij} exists, then a (non-degenerate) *Lagrangian function* $L(t, \vec{x}, \dot{\vec{x}})$ is obtained by integrating the relation

$$\boxed{\frac{\partial^2 L}{\partial \dot{x}^i \, \partial \dot{x}^i} = g_{ij}}\,. \tag{1.7}$$

Geometric reformulation of the multiplier inverse problem: Given a SODE vector field Γ on E, *the existence of a non-singular matrix* (g_{ij}) *satisfying the Helmholtz conditions* (1.6) *is equivalent to the existence of a 2-form* ϖ *on* E *which has the following properties:*

$$\boxed{d\varpi = 0\,, \qquad i_\Gamma \varpi = 0\,, \qquad \varpi \text{ is horizontal and has maximal rank}}\,. \tag{1.8}$$

Procedure: One writes

$$\boxed{\varpi \equiv g_{ij}\,\psi^i \wedge \theta^j} \qquad \text{with} \quad \begin{cases} \det(g_{ij}) \neq 0 \\ \theta^i \equiv dx^i - u^i dt \\ \psi^i \equiv du^i - f^i\,dt + A^i_j\,\theta^j\,. \end{cases} \tag{1.9}$$

Then, one has $i_\Gamma \varpi = 0$ and

$$\varpi^n \equiv \underbrace{\varpi \wedge \cdots \wedge \varpi}_{n \text{ factors}} = \det(g_{ij})\,\psi^1 \wedge \cdots \wedge \psi^n \wedge \theta^1 \wedge \cdots \wedge \theta^n \neq 0\,,$$

as well as

$$\varpi(\partial/\partial u^k, \partial/\partial u^l) = 0\,.$$

Furthermore, *the relation* $d\varpi = 0$ *is equivalent to the validity of the Helmholtz conditions* (1.6).