Form factors and overlaps for the spin chains

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The XXZ spin-1/2 Heisenberg chain

1. Periodic chain.

Hamiltonian

$$H_{\text{bulk}} = \sum_{m=1}^{L} \left(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta\,(\sigma_m^z \sigma_{m+1}^z - 1)\right)$$

$\Delta = \cosh\zeta$ - anisotropy Periodic boundary conditions: $\sigma_{L+1} = \sigma_1$.

2. Open chain.

$$H = \sum_{m=1}^{L-1} \left(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta\,(\sigma_m^z \sigma_{m+1}^z - 1)\right) + h_- \sigma_1^z + h_+ \sigma_L^z$$

$h_\pm$ - boundary fields.

We consider $\Delta > 1$ - massive antiferromagnetic regime, $\Delta = \cosh\zeta$

Form Factors

The main question: systematic computation of the form-factors in the thermodynamic limit from the **Algebraic Bethe ansatz**

Form factors: matrix elements of **local fields**, local spin operators σ_m^a, $a = x, y, z$

$|\Psi_g\rangle$ the ground state of the model $|\Psi_e\rangle$ - an excited state

$$|\mathcal{F}_a(\Psi_e)|^2 = \frac{\langle \Psi_g | \sigma_m^a | \Psi_e \rangle \langle \Psi_e | \sigma_m^a | \Psi_g \rangle}{\langle \Psi_g | \Psi_g \rangle \langle \Psi_e | \Psi_e \rangle}$$

Then more advanced questions can be studied like matrix elements of **currents**

- Integrable QFT - F. Smirnov 1992 **bootstrap approach**
- Massive XXZ, M. Jimbo and T. Miwa 1995 q-**vertex operator** approach
- General XXZ, N.K, J.M. Maillet, V. Terras, 1999 **Algebraic Bethe ansatz** approach

- Dynamical correlation functions **at zero temperature**:

$$f_a(m,t) = \langle \sigma^a_{m+1}(t)\sigma^a_1(0)\rangle = \sum_{\Psi_e} \exp(it\Delta E_e - im\Delta p_e)\,|\mathcal{F}_a(\Psi_e)|^2$$

Turns out to be an excellent tool of **asymptotic analysis**.

- Dynamical structure factors:

$$S(k,\omega) = \int_{-\infty}^{\infty} dt \sum_{m=-\infty}^{\infty} f_a(m,t)\exp(imk - it\omega)$$

Experimentally mesurable quantity : can be computed **numerically** from the form factors (J.S. Caux et al.) and **asymptotically (edge exponents)**.

Boundary overlaps

Quench: dynamics of a system after abrupt change of one parameter. We change **one** boundary field $h_- \longrightarrow \widetilde{h}_-$. Local change, but can drastically modify the ground state (globally).

$|\Psi\rangle$ the ground state before the change of field $|\widetilde{\Psi}\rangle$ - ground state after the change of field.

The most basic overlap: scalar product of ground states.

$$|\mathcal{F}|^2 = \frac{\langle \Psi \,|\widetilde{\Psi}\rangle \, \langle \widetilde{\Psi} \,|\Psi\rangle}{\langle \Psi \,|\Psi\rangle \, \langle \widetilde{\Psi} \,|\widetilde{\Psi}\rangle}$$

Gives for example the dominant term for the Loschmidt echo (dynamics of the initial state after the change of the boundary magnetic field:

$$\mathcal{L}(t) = \left| \langle \Psi| \, e^{-i\widetilde{H}t} \, |\Psi\rangle \right|^2$$

XXZ chain: Algebraic Bethe ansatz

L.D. Faddeev, E.K. Sklyanin, L.A. Takhtajan (1979). Main object: quantum monodromy matrix:

$$T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_a .$$

- Diagonal elements $\longrightarrow$ commuting conserved charges: transfer matrix

$$\mathcal{T}(\lambda) = \mathrm{tr}_a\, T_a(\lambda) = A(\lambda) + D(\lambda), \qquad [\mathcal{T}(\lambda), \mathcal{T}(\mu)] = 0$$

- Hamiltonian:

$$H = c\, \frac{\partial}{\partial \lambda} \log \mathcal{T}(\lambda) \Big|_{\lambda = \frac{i\zeta}{2}}, \qquad [H, \mathcal{T}(\lambda)] = 0$$

- Non-diagonal elements $\longrightarrow$ creation/annihilation operators.

Bethe states

Ferromagnetic state: $|0\rangle = |\uparrow\uparrow \dots \uparrow\rangle$, $A(\lambda)\,|0\rangle = a(\lambda)\,|0\rangle$, $D(\lambda)\,|0\rangle = d(\lambda)\,|0\rangle$.
Off-shell Bethe states: $|\Psi(\{\lambda_1,\dots,\lambda_N\})\rangle = B(\lambda_1)\dots B(\lambda_N)\,|0\rangle$.
For any Bethe state we define **Baxter polynomial** and **exponential counting function**

$$q(\lambda) = \prod_{j=1}^{N} \sin(\lambda - \lambda_j), \qquad \mathfrak{a}(\lambda) = \frac{a(\lambda)}{d(\lambda)}\frac{q(\lambda + i\zeta)}{q(\lambda - i\zeta)}.$$

if the **Bethe equations** are satisfied (on-shell Bethe state)

$$\mathfrak{a}(\lambda_j) + 1 = 0, \qquad j = 1, \dots N$$

then it is an eigenstate of the **transfer matrix** and the Hamiltonian

$$\mathcal{T}(\mu)\,|\Psi(\{\lambda\})\rangle = \tau(\mu)\,|\Psi(\{\lambda\})\rangle\,, \qquad \tau(\mu) = \big(\mathfrak{a}(\mu) + 1\big)\frac{q(\mu - i\zeta)}{q(\mu)}.$$

Open spin chain, Algeraic Bethe ansatz

Boundary matrices satisfying reflection equation (Cherednik 1984)

$$R_{12}(\lambda-\mu)\,K_1(\lambda)\,R_{12}(\lambda+\mu)\,K_2(\mu)=K_2(\mu)\,R_{12}(\lambda+\mu)\,K_1(\lambda)\,R_{12}(\lambda-\mu).$$

We consider only diagonal solution: $K(\lambda)=\begin{pmatrix}\sinh(\lambda+\xi-\imath\zeta/2) & 0\\ 0 & \sinh(\xi-\lambda-i\zeta/2)\end{pmatrix},$

Algebraic Bethe Ansatz, Sklyanin 1988, Double row monodromy matrices:

$T(\lambda)$ -usual monodromy matrix, $\widehat{T}(\lambda)=\sigma_0^y\,T^{t_0}(-\lambda)\,\sigma_0^y$ returned monodromy matrix.

$$\mathcal{U}_-(\lambda)=T(\lambda)\,K_-(\lambda)\,\widehat{T}(\lambda)=\begin{pmatrix}\mathcal{A}_-(\lambda) & \mathcal{B}_-(\lambda)\\ \mathcal{C}_-(\lambda) & \mathcal{D}_-(\lambda)\end{pmatrix},$$

$$\mathcal{U}_+^{t_0}(\lambda)=T^{t_0}(\lambda)\,K_+^{t_0}(\lambda)\,\widehat{T}^{t_0}\lambda)=\begin{pmatrix}\mathcal{A}_+(\lambda) & \mathcal{C}_+(\lambda)\\ \mathcal{B}_+(\lambda) & \mathcal{D}_+(\lambda)\end{pmatrix},$$

Algebraic Bethe Ansatz, open chain

1. Transfer matrix:

$$\mathcal{T}(\lambda) = \mathrm{tr}_0\{K_+(\lambda)\,\mathcal{U}_-(\lambda)\} = \mathrm{tr}_0\{K_-(\lambda)\,\mathcal{U}_+(\lambda)\}.$$

$$[\mathcal{T}(\lambda), \mathcal{T}(\mu)] = 0$$

2. Hamiltonian:

$$H = c\frac{d}{d\lambda}\mathcal{T}(\lambda)_{|\lambda=-i\zeta/2} + \text{constant.}$$

$$h_\pm = -\sinh\zeta\,\coth\xi_\pm$$

3. Bethe states, Baxter polynomials:

$$|\psi_+(\{\lambda\})\rangle = \prod_{k=1}^{N}\mathcal{B}_+(\lambda_j)\,|0\rangle\,, \qquad \mathcal{Q}(\lambda) = \prod_{j=1}^{N}\sin(\lambda-\lambda_j)\sin(\lambda+\lambda_j)$$

Note: operators $\mathcal{B}_+(\lambda)$ don't depend on h_-.

Bethe equations

Counting function

$$\mathfrak{A}(\lambda) = \frac{a(\lambda)d(-\lambda)}{d(\lambda)a(-\lambda)} \frac{\sin(\lambda + \imath\xi_+ + i\zeta/2 \sin(\lambda + \imath\xi_- + i\zeta/2)}{\sin(\lambda - \imath\xi_+ - i\zeta/2 \sin(\lambda - \imath\xi_- - i\zeta/2)} \frac{\mathcal{Q}(\lambda + i\zeta)}{\mathcal{Q}(\lambda - i\zeta)}$$

if the parameters λ satisfy the Bethe equations:

$$\mathfrak{A}(\lambda_j) = 1$$

$|\psi_+(\{\lambda\})\rangle$ is an eigenstate of the transfer matrix $\mathcal{T}(\mu)$:

$$\mathcal{T}(\mu) \, |\psi_+(\{\lambda\})\rangle = \tau(\mu, \{\lambda_j\}) \, |\psi_+(\{\lambda\})\rangle \, ,$$

$$\tau(\mu) = \left(\mathfrak{A}(\mu) \frac{\sin(2\mu + i\zeta)}{\sin(2\mu - i\zeta)} + 1 \right) \frac{\mathcal{Q}(\mu - i\zeta)}{\mathcal{Q}(\mu)}.$$

Scalar products and norms, periodic case

N. Slavnov, 1989: $\{\lambda_1,\dots\lambda_N\}$ - solution of Bethe equations, $\{\mu_1,\dots\mu_N\}$ - generic

$$\langle\Psi(\{\mu\})\,|\Psi(\{\lambda\})\rangle = \frac{\prod_{k=1}^{N} q(\mu_k - i\zeta)}{\prod_{j>k}\sin(\lambda_j-\lambda_k)\sin(\mu_k-\mu_j)}\det_N \mathcal{M}(\{\lambda\}|\{\mu\}),$$

$$\mathcal{M}_{j,k}(\{\lambda\}|\{\mu\}) = \mathfrak{a}(\mu_k)t(\lambda_j-\mu_k) - t(\mu_k-\lambda_j), \quad t(\lambda) = \frac{i\sinh\zeta}{\sin\lambda\sin(\lambda-i\zeta)}.$$

Norms of the on-shell Bethe states are given by the Gaudin formula

$$\langle\Psi(\{\lambda\})\,|\Psi(\{\lambda\})\rangle = (-1)^N\frac{\prod_{j=1}^{N} q(\lambda_j - i\zeta)}{\prod_{j\neq k}\sin(\lambda_j-\lambda_k)}\det\mathcal{N}(\{\lambda\}),$$

$$\mathcal{N}_{j,k}(\{\lambda\}) = \mathfrak{a}'(\lambda_j)\delta_{j,k} - K(\lambda_j-\lambda_k), \quad K(\lambda) = t(\lambda)+t(-\lambda).$$

Computation of determinants

N.K. Maillet Terras '99: **quantum inverse problem**, we know that the computation of form factors can be reduced to the scalar products.

$$S(\{\lambda\}|\{\mu\}) \equiv \frac{\langle\Psi(\{\mu\})\,|\Psi(\{\lambda\})\rangle\langle\Psi(\{\lambda\})\,|\Psi(\{\mu\})\rangle}{\langle\Psi(\{\lambda\})\,|\Psi(\{\lambda\})\rangle\langle\Psi(\{\mu\})\,|\Psi(\{\mu\})\rangle}$$

$$= \prod_{j=1}^{N} \frac{q_\lambda(\mu_j)q_\mu(\lambda_j)}{q_\lambda(\lambda_j)q_\mu(\mu_j)} \cdot \frac{\det\mathcal{M}(\{\lambda\}|\{\mu\})\det\mathcal{M}(\{\mu\}|\{\lambda\})}{\det\mathcal{N}(\{\lambda\})\det\mathcal{N}(\{\mu\})}.$$

The main idea is extremely simple: we compute the following matrices from a system of linear equations

$$F_\lambda = \mathcal{N}^{-1}(\{\lambda\})\mathcal{M}\left(\{\lambda\}|\{\mu\}\right), \quad F_\mu = \mathcal{N}^{-1}(\{\mu\})\mathcal{M}\left(\{\mu\}|\{\lambda\}\right),$$

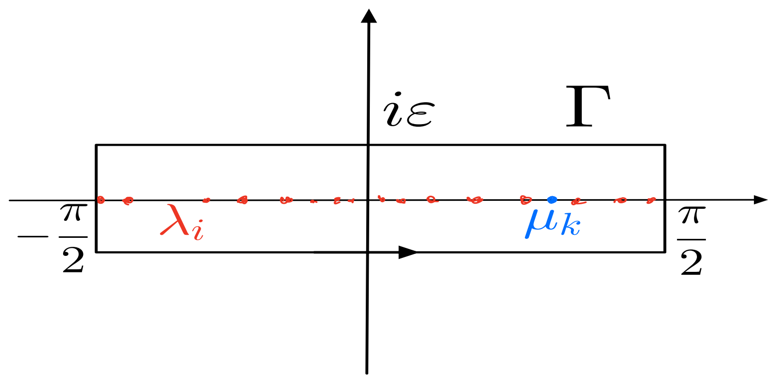
$$\mathfrak{a}'_\lambda(\lambda_j)F_{\lambda j,k} - \sum_{a=1}^{N} K(\lambda_j - \lambda_a)F_{\lambda a,k} = \mathfrak{a}_\lambda(\mu_k)t(\lambda_j - \mu_k) - t(\mu_k - \lambda_j).$$

We set

$$\mathfrak{a}'_\lambda(\lambda_j) F_{\lambda_{j,k}} = G_\lambda(\lambda_j; \mu_k)$$

Linear equations $\longrightarrow$ Contour integral equation for a meromorphic function $G_\lambda(\lambda;\mu)$

$$G_\lambda(\lambda;\mu_k) - \frac{1}{2\pi i}\oint_\Gamma d\nu\, K(\lambda-\nu)\frac{G_\lambda(\nu;\mu_k)}{1+\mathfrak{a}_\lambda(\nu)} = (\mathfrak{a}_\lambda(\mu_k)+1)t(\lambda-\mu_k),$$



We set

$$G_\lambda(\lambda;\mu) = \big(1 + \mathfrak{a}_\lambda(\mu)\big)\rho_\lambda(\lambda;\mu)$$

Thermodynamic limit $\longrightarrow$ Integral equation

$$\rho_\lambda(\lambda;\mu) + \frac{1}{2\pi i}\int_{-\pi/2+i0}^{\pi/2+i0} d\nu\, K(\lambda-\nu)\rho_\lambda(\nu;\mu) = t(\lambda-\mu).$$

Lieb equation for the density of Bethe roots! $\longrightarrow$ **elliptic Cauchy determinant**

$$F_{\lambda_{j,k}} = \frac{\mathfrak{a}_\lambda(\mu_k)+1}{\mathfrak{a}'_\lambda(\lambda_j)} \cdot \frac{(q^2,q^2)_\infty}{(-q^2,q^2)_\infty} \cdot \frac{\vartheta_2(\mu_k-\lambda_j,q)}{\vartheta_1(\mu_k-\lambda_j,q)} + O(L^{-\infty}), \quad q = e^{-\zeta}$$

XXX case: 2-spinon form factor

N.K. G. Kulkarni '19: Matrix element of σ_z between the ground state of the XXX chain and a state with 2 holes (spinons) μ_{h_1} and μ_{h_2}

Final result for the form factor:

$$|\mathcal{Y}(\mu_{h_1}-\mu_{h_2})|^2 = \lim_{L\to\infty} L^2|\mathcal{F}_z|^2 = \frac{2}{G^4\left(\frac{1}{2}\right)}\left|\frac{G\left(\frac{\mu_{h_1}-\mu_{h_2}}{2i}\right)G\left(1+\frac{\mu_{h_1}-\mu_{h_2}}{2i}\right)}{G\left(\frac{1}{2}+\frac{\mu_{h_1}-\mu_{h_2}}{2i}\right)G\left(\frac{3}{2}+\frac{\mu_{h_1}-\mu_{h_2}}{2i}\right)}\right|^2.$$

Where $G(z)$ iz the Barnes G-function (related to the double Γ-function).

$$G(z+1) = \Gamma(z)G(z), \qquad G(1) = 1.$$

This reproduces the result for the two-spinon form factor obtained in the q-vertex operator framework from the M. Jimbo and T. Miwa multiple integral formulas by A. H. Bougourzi, M. Couture and M. Kacir

Open chain: scalar products and norms

N.K. K. Kozlowski, J.M. Maillet, G. Niccoli, N. Slavnov, V. Terras '07

$$S(\{\lambda\}|\{\mu\}) \equiv \frac{\langle\Psi(\{\mu\})\,|\Psi(\{\lambda\})\rangle\langle\Psi(\{\lambda\})\,|\Psi(\{\mu\})\rangle}{\langle\Psi(\{\lambda\})\,|\Psi(\{\lambda\})\rangle\langle\Psi(\{\mu\})\,|\Psi(\{\mu\})\rangle}$$

$$= \prod_{j=1}^{N} \frac{\mathcal{Q}_\lambda(\mu_j)\mathcal{Q}_\mu(\lambda_j)}{\mathcal{Q}_\lambda(\lambda_j)\mathcal{Q}_\mu(\mu_j)} \cdot \frac{\det\mathcal{M}(\{\lambda\}|\{\mu\})\det\mathcal{M}(\{\mu\}|\{\lambda\})}{\det\mathcal{N}(\{\lambda\}),\det\mathcal{N}(\{\mu\})}.$$

Slavnov matrix;

$$\mathcal{M}_{j,k}(\{\lambda\}|\{\mu\}) = \mathfrak{A}_\lambda(\mu_k)t\big((-\mu_k+\lambda_j)-t(-\mu_k-\lambda_j)\big)+t(\mu_k-\lambda_j)-t(\mu_k+\lambda_j),$$

Gaudin matrix

$$\mathcal{N}_{j,k}(\{\lambda\}) = \mathfrak{A}'_\lambda(\lambda_j)\delta_{j,k} - K(\lambda_j-\lambda_k) + K(\lambda_j+\lambda_k)$$

Computation of determinants: open case

Same idea as in the periodic case

$$F_\lambda = \mathcal{N}^{-1}(\{\lambda\})\mathcal{M}\left(\{\lambda\}|\{\mu\}\right), \quad F_\mu = \mathcal{N}^{-1}(\{\mu\})\mathcal{M}\left(\{\mu\}|\{\lambda\}\right),$$

Linear equations $\longrightarrow$ Contour integral equation $\longrightarrow$ Linear integral equation

$$\rho_\lambda(\lambda;\mu) + \frac{1}{2\pi i}\int_{-\pi/2+i0}^{\pi/2+i0} d\nu\, K(\lambda-\nu)\rho_\lambda(\nu;\mu) = t(\lambda-\mu) + t(\lambda+\mu).$$

Solution:

$$F_{\lambda_{j,k}} = \frac{\mathfrak{A}_\lambda(\mu_k)-1}{\mathfrak{A}'_\lambda(\lambda_j)} \cdot \frac{(q^2,q^2)_\infty}{(-q^2,q^2)_\infty}\left(\frac{\vartheta_2(\lambda_j-\mu_k,q)}{\vartheta_1(\lambda_j-\mu_k,q)} + \frac{\vartheta_2(\mu_k+\lambda_j,q)}{\vartheta_1(\mu_k+\lambda_j,q)}\right) + O(L^{-\infty})$$

Once again **Cauchy determinant**

Cauchy determinant: open case

We use the following notations:

- ratio of the transfer matrix eigenvalues

$$\chi(\lambda) = \frac{\tau(\lambda, \{\mu_j\})}{\tau(\lambda, \{\lambda_j\})}$$

- and the following function

$$\varphi(\lambda, q) = \frac{\vartheta_1(\lambda, q)}{\sin \lambda}$$

Then we express the overlap as follows

$$S(\{\lambda\}|\{\mu\}) = \prod_{j=1}^{N} \frac{\chi(\lambda_j)}{\chi(\mu_j)} \prod_{j,k=1}^{N} \frac{\varphi(\lambda_j - \lambda_k, q)\varphi(\mu_j - \mu_k, q)\varphi(\lambda_j + \lambda_k, q)\varphi(\mu_j + \mu_k, q)}{\varphi^2(\lambda_j - \mu_k)\varphi^2(\lambda_j + \mu_k)}$$

It remains to fix the two states and compute products in the **thermodynamic limit**.

Ground states

Configurations of the Bethe roots in the ground state depends on the boundary magnetic fields: $h_- = -\sinh\zeta\coth\xi_-$ (first site) and $h_+ = -\sinh\zeta\coth\xi_+$ (last site). There are several cases leading to different structures of the ground state (S. Grijalva, J. Di Nardis, V. Terras '19).

We consider 3 most important situations. We limit our analysis to the case $h_- > h_+$.

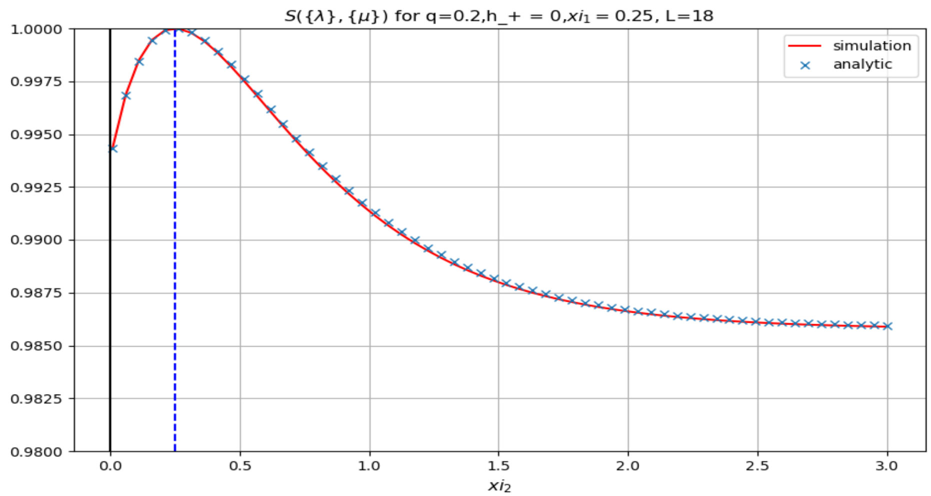
- $\Delta - 1 < h_- < \Delta + 1$: All $L/2$ the roots are real distributed with a density given by the Lieb equation.
- $0 < h_- < \Delta - 1$. $L/2 - 1$ real roots and a **boundary root** $\lambda_{\mathrm{BR}} = -i(\zeta/2 + \zeta_-) + O(L^{-\infty})$
- $h_+ < \Delta - 1$, $\Delta + 1 < h_-$: $L/2 - 1$ real roots and a **boundary root** λ_{BR}

We change one field $h_- \longrightarrow \widetilde{h}_-$, $\xi_- \longrightarrow \widetilde{\xi}_-$.

Final result: only real roots

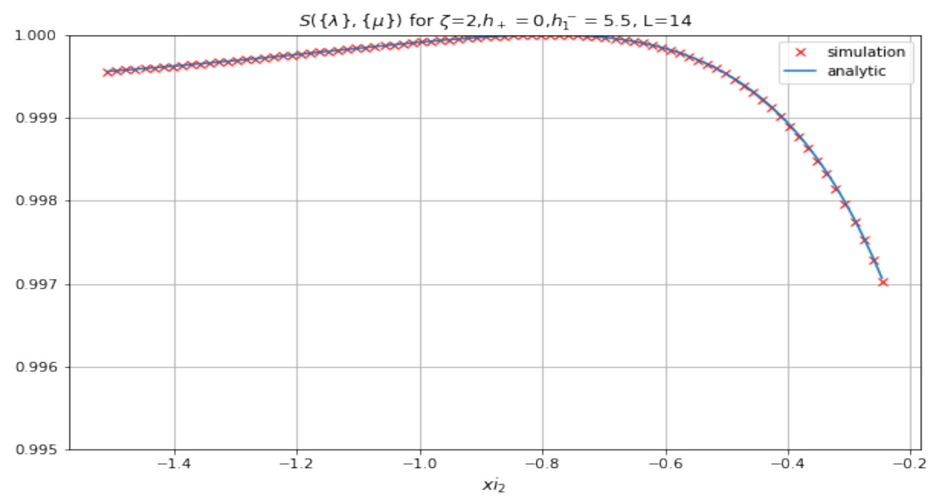
Notations: $q = e^{-\zeta}$, $p = e^{-2\xi_-}$, $\widetilde{p} = e^{-2\widetilde{\xi}_-}$.

$$S(\{\lambda\}|\{\mu\}) = \frac{F^2(q^4 p\widetilde{p})}{F(q^4p^2)F(q^4\widetilde{p}^2)}, \qquad F(u) = \frac{(uq^4, q^4, q^4)_\infty}{(uq^2, q^4, q^4)_\infty}.$$



Final result: one boundary complex root

$$S(\{\lambda\}|\{\mu\}) = \frac{F^2(p^{-1}\widetilde{p}^{-1})}{F(p^{-2})F(\widetilde{p}^{-2})}$$



q-vertex operator approach: same results by R. Weston

Conclusion and outlook

Advantages of the new approach:

- Explicit results, no Fredholm determinants.
- We know how to deal with complex roots
- Possibility to apply in a systematic way for all the regimes of the XXZ chain, periodic case, open case etc.

Open problems:

- Can we apply this method far from the **ground state**?
- Impurities, non-local quenches?