Universal central extensions of the Lie algebra of unimodular vector fields

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References

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Plan

Lie algebras

Definition (Lie algebra)

- $\mathbb{R}$-vector space $\mathfrak{g}$
- bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ s.th. $\forall x, y, z \in \mathfrak{g}$

  $[x, y] = -[y, x]$ (skew-symmetry)

  $[x, [y, z]] = [[x, y], z] + [y, [x, z]]$ (Jacobi identity)

Example

- $\mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}^{n \times n}$, all matrices
- $\mathfrak{so}(n, \mathbb{R})$, $\mathfrak{sl}(n, \mathbb{R})$, subalgebras of $\mathfrak{gl}(n, \mathbb{R})$
- $\mathfrak{X}(M)$, vector fields on a smooth manifold M
- $\mathfrak{X}(M, \omega)$, vector fields preserving a volume form $\omega \in \Omega^n(M)$

Lie algebra cohomology

Definition (Lie algebra coholomology)

$$\mathbb{R} \xrightarrow{\delta^0=0} \mathfrak{g}^* \xrightarrow{\delta^1} \Lambda^2\mathfrak{g}^* \xrightarrow{\delta^2} \Lambda^3\mathfrak{g}^* \xrightarrow{\delta^3} \Lambda^4\mathfrak{g}^* \xrightarrow{\delta^4}$$

$$\delta^k(\alpha)(v_1, \ldots, v_{k+1}) = \sum_{1\leq i<j\leq k+1} (-1)^{i+j}\alpha([v_i, v_j], v_1, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_{k+1})$$

$$H^k(\mathfrak{g}) = \frac{\ker \delta^k}{\mathrm{Image}\ \delta^{k-1}}$$

Example

- $\mathfrak{g}$ abelian (i.e. $[\cdot, \cdot] = 0$) $\implies H^k(\mathfrak{g}) = \Lambda^k\mathfrak{g}^*$
- $\mathfrak{g}$ Lie algebra of G (connected, compact) $\implies H(\mathfrak{g}) = H_{\mathrm{deRham}}(G)$

Perfectness

- always $H^0(\mathfrak{g}) = \mathbb{R}$.
- $H^1(\mathfrak{g}) = \ker(\delta^1) = ker([\cdot,\cdot]^*) = \{\alpha \in \mathfrak{g}^* \mid \alpha([\mathfrak{g},\mathfrak{g}]) = 0\}$.
  I.e. $H^1(\mathfrak{g}) = 0 \Longleftrightarrow [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

Definition (perfectness)

$\mathfrak{g}$ perfect $:\Longleftrightarrow [\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

- $H^2(\mathfrak{g})$ classifies central extensions by $\mathbb{R}$...

Central extensions

Definition (Central extension)

Central extension of $\mathfrak{g}$
$:\Longleftrightarrow$ Lie algebra surjection $p : \tilde{\mathfrak{g}} \to \mathfrak{g}$ such that $[\ker(p), \tilde{\mathfrak{g}}] = 0$.
Call it a central extension by $\mathfrak{c} = \ker(p)$.

Example

- trivial central extension by $\mathfrak{c} = \mathbb{R}^n$: $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{c}$ (with brackets extending trivially)
- $\mathfrak{g} = \mathbb{R}^2 = \langle\langle a, b\rangle\rangle$ (with trivial bracket), $\mathfrak{c} = \mathbb{R} = \langle\langle c\rangle\rangle$,
bracket on $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{c}$ by $[a, b] = c$.
(Heisenberg algebra as extension of $\mathbb{R}^2$.)
- finite-dimensional semisimple Lie algebras are perfect have no non-trivial central extensions.

Universal central extensions

Definition (Universal central extension)

$\hat{q} : \hat{\mathfrak{g}} \to \mathfrak{g}$ universal central extension
$:\Leftrightarrow$ for all $p : \tilde{\mathfrak{g}} \to \mathfrak{g}$ exists unique $f : \hat{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ such that $\hat{q} = pf$.

$$\begin{array}{ccc} \hat{\mathfrak{g}} & \xrightarrow{\hat{q}} & \mathfrak{g} \\ {\scriptstyle \exists! f}\downarrow & & \downarrow{\scriptstyle id} \\ \tilde{\mathfrak{g}} & \xrightarrow{p} & \mathfrak{g} \end{array}$$

Theorem

- *A universal central extension exists iff* $H^1(\mathfrak{g}) = 0$.
- *It is characterized by* $H^1(\hat{\mathfrak{g}}) = 0$, $H^2(\hat{\mathfrak{g}}) = 0$.

Infinite dimensions and topologies

- Consider now $\mathfrak{X}(M,\omega)$, where $\omega \in \Omega^m(M)$ is a volume.
  - $\mathfrak{X}(M,\omega)$ inherits a Fréchet topology from $\mathfrak{X}(M)$.
  - The Lie bracket is continuous
  - $[\mathfrak{X}(M,\omega), \mathfrak{X}(M,\omega)] = \mathfrak{X}_{ex}(M,\omega) = \{X \mid \iota_X\omega \text{ exact}\}$

- Modify previous slides with the topology
  - Replace $H(\mathfrak{g})$ by $H_{cont}(\mathfrak{g})$, which only admits continuous maps $\Lambda^k\mathfrak{g} \to \mathbb{R}$ as cochains.
  - require $\tilde{\mathfrak{g}} \to \mathfrak{g}$ to be continuous and the existence of a linear split $\mathfrak{g} \to \tilde{\mathfrak{g}}$ in the definition of central extension.
  - $\mathfrak{g}$ perfect and $H^2_{cont}(\mathfrak{g})$ finite-dimensional implies the existence of a universal central extension.

Main result

Theorem (JRV, Conjecture by Claude Roger 1993)

Let $m > 2$.
$\hat{\mathfrak{g}} = \frac{\Omega^{m-2}(M)}{d\Omega^{m-3}(M)}$ *is the universal central extension of* $\mathfrak{g} = \mathfrak{X}_{ex}(M, \omega)$ *(in the category of linearly split extensions by locally convex spaces), with*

- $\hat{q}(\bar{\alpha}) := X_\alpha$ *for the unique field with* $\iota_{X_\alpha}\omega = d\alpha$.
- $[\bar{\alpha}, \bar{\beta}] := \overline{L_{X_\alpha}\beta}$

As a diagram the extension reads: $H^{m-2}_{deRham}(M) \to \hat{\mathfrak{g}} \to \mathfrak{X}_{ex}(M, \omega)$.

By a theorem of Neeb we need to show

- $\hat{g}$ is perfect
- $H^2_{cont}(\hat{\mathfrak{g}}) = 0$

The Leibniz algebra

Definition (left Leibniz algebra)

$\mathfrak{L}$, $[\cdot,\cdot] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ such that $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$.

Example

$\mathfrak{L} = \Omega^{m-2}(M)$ with $[\alpha, \beta] = L_{X_\alpha}\beta$ is a Leibniz algebra,
not a Lie algebra.

Theorem (JRV)

$[\mathfrak{L}, \mathfrak{L}] = \mathfrak{L}$.

Corollary: $\hat{\mathfrak{g}}$ and $\mathfrak{g}$ are perfect.

Local perfectness

Lemma

Let $\mathfrak{L}_{loc} = \Omega_c^{m-2}(U)$, $U \subset \mathbb{R}^m$ *connected,* $\omega = dx^1 \wedge ... \wedge dx^m$.
$\forall x \in \mathfrak{L}_{loc}$ *exist* $y_i, z_i \in \mathfrak{L}_{loc}$ *such that*

$$x = \sum_{i=1}^{(m+1)^3} [y_i, z_i]$$

proof: Coordinate computation and Poincare Lemma

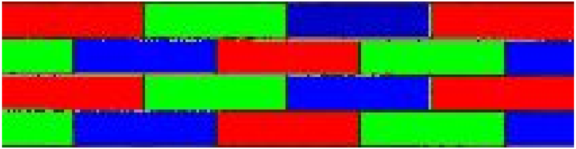
Ostrands theorem and pavings

Theorem (Ostrand, Brouwer-Lebesgue)

Let $\mathcal{U}$ an open cover of M. Then there exist open sets $V_{i,\alpha}$, $i \in \{1, ..., m+1\}$, $\alpha \in \mathbb{N}$ with the following properties:

- *Each $V_{i,\alpha}$ is a connected open subset of an element in $\mathcal{U}$.*
- *For fixed i and $\alpha \neq \beta$, $V_{i,\alpha} \cap V_{i,\beta} = \emptyset$*
- *$V_{i,\alpha}$ cover M.*

In particular $W_i = \bigsqcup_\alpha V_{i,\alpha}$ gives an open cover of M by $m+1$ sets.



proof of global perfectness of $\mathfrak{L}$

Proof.

Pick an open cover $\mathcal{U} = \{U_\beta\}$ of M such that for each U_β there are coordinates $\phi_\beta : U_\beta \to \mathbb{R}^m$ with $\omega|_{U_\beta} = \phi_\beta^*(dx^1 \wedge ... \wedge dx^m)$.

Apply Ostrands theorem and pick a partition of unity for $V_{i,\alpha}$

- Use local Lemma to get solutions on $V_{i,\alpha}$ (with $(m+1)^3$ terms each)
- patch them together to solutions on W_i (with $(m+1)^3$ terms each)
- Put all of them together (with $(m+1)^4$ terms in total)

□

Summary and plan

So far we know:

- $\mathfrak{L} = \Omega^{m-2}(M)$ is a perfect Leibniz algebra.
- $\hat{\mathfrak{g}} = \frac{\Omega^{m-2}(M)}{d\Omega^{m-3}(M)}$ is perfect.
- $\mathfrak{g} = \mathfrak{X}_{ex}(M, \omega)$ is perfect.

We want to show $H^2_{cont}(\hat{\mathfrak{g}}) = 0$ as follows:

- Pick a cocycle $\Psi : \Lambda^2 \hat{\mathfrak{g}} \to \mathbb{R}$
- Consider the induced map $\hat{\Psi} : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}'$.
  Use $\pi : \mathfrak{L} \to \hat{\mathfrak{g}}$ to obtain $\Phi = \pi' \hat{\Psi} \pi : \mathfrak{L} \to \mathfrak{L}'$
- Show it has a good local coordinate expression (Peetre's theorem)
- Show it is exact locally (using representation theory of $\mathfrak{sl}(m, \mathbb{R})$)
- Glue the local solutions

Peetre's theorem

Definition (support-decreasing)

Let E, F be vector bundles over M, $\Phi : \Gamma_c(E) \to \Gamma_c(F)'$ is called *support-decreasing* if for all f:

$$\mathrm{supp}(\Phi(f)) \subset \mathrm{supp}(f)$$

Theorem (Peetre's theorem)

If Φ is support-decreasing

- *The set of discontinuity of Φ is discrete.*
- *Outside of it Φ can be locally written as:*

$$\Phi(f) = \sum_{|I| \leq k} a_I D_I f$$

where $I = (i_1, ..., i_m)$ are multiindices, $D_I = (\partial_{x_1})^{i_1} ... (\partial_{x_m})^{i_m}$ and a_I are distributions.

The perfectness trick

Let Ψ be a cocycle, and $\hat{\Psi}$, Φ its induced maps.

- Φ support-decreasing
  $\iff$ If $\alpha, \beta \in \mathfrak{L}$ have disjoint supports, then $\Psi(\bar{\alpha}, \bar{\beta}) = 0$.
- Let $\mathrm{supp}(\alpha) \subset \mathrm{supp}(\beta)^c$
  $\exists \gamma_i, \delta_i$ (with supports in $\mathrm{supp}(\beta)^c$) such that $\alpha = \sum_{i=1}^N [\gamma_i, \delta_i]$.

$$\begin{aligned}\Psi(\bar{\alpha}, \bar{\beta}) &= \sum \Psi([\bar{\gamma}_i, \bar{\delta}_i], \bar{\beta}) \\ &= \sum \Psi([\bar{\gamma}_i, \bar{\beta}], \bar{\delta}_i) - \sum \Psi([\bar{\delta}_i, \bar{\beta}], \bar{\gamma}_i) = 0\end{aligned}$$

  Since $[\bar{\epsilon}, \bar{\beta}] = \overline{L_{X_\epsilon} \beta}$ vanishes for support-disjoint β, ϵ.
- Peetre's theorem applies to Φ. Moreover Φ is continuous
  $\implies$ It is a distribution-valued diff. operator near <u>any</u> point.

Remainder of the proof

- Consider $\Phi|_U$ for small contractible sets U Use the fact that $\Phi|_U$ is a Diff. Op. vanishing on $\Omega_{cl}^{m-2}(M)$ to push it to $\Omega^{m-1}(M)$.
- Diff. Ops. are determined by their values on Polynomials. Use this to find local potentials of Ψ inductively.
- Use sheaf properties of $\mathfrak{L}$ to glue the local potentials.

$\Longrightarrow H^2_{cont}(\hat{\mathfrak{g}}) = 0.$

Theorem

$\hat{\mathfrak{g}} = \frac{\Omega^{m-2}(M)}{d\Omega^{m-3}(M)}$ *is the universal central extension of* $\mathfrak{g} = \mathfrak{X}_{ex}(M, \omega)$.

Further results

- $H^2_{cont}(\mathfrak{X}_{ex}(M,\omega)) = H^{n-2}_{deRham}(M)'$. Moreover $H^2_{cont}(\mathfrak{X}(M,\omega)) = H^{n-2}_{deRham}(M)' \oplus \Lambda^2 H^{n-1}_{deRham}(M)'$.
- $\hat{\mathfrak{g}} = \frac{\Omega_c^{m-2}(M)}{d\Omega_c^{m-3}(M)}$ is also the universal central extension of $\mathfrak{g} = \{X_\alpha \mid \alpha \in \Omega_c^{m-2}(M)\}$ (among finite-dimensional extensions)
- $d\Omega^{m-3}(M)$ in $\Omega^{m-2}(M)$ is exactly the ideal of squares, i.e. consists of finite sums of elements of the type $[x, x]$.
- For $m = 3$ and M compact, we can also find a universal central extension of Fréchet Lie groups.