# Putting chiral fermions on tensor networks as a generalized eigenvalue problem

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## Quantum problems are solving the linear equation

$$\hat{H}|\Psi\rangle = E|\Psi\rangle$$

## Generalized eigenvalue problem

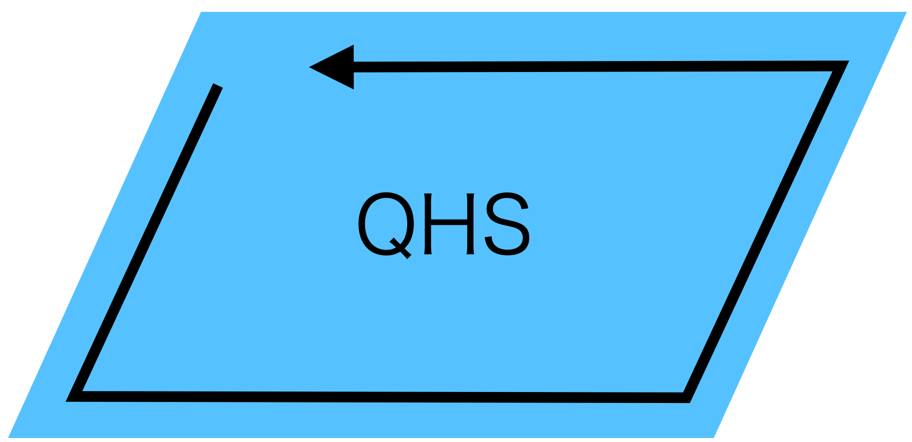
$\hat{N} = \hat{I}$ is the original problem

$$\hat{H}|\Psi\rangle = E\hat{N}|\Psi\rangle$$

## What can we do with this?

We can solve fermion-doubling and simulate lattice chiral fermions

# Fermion doubling



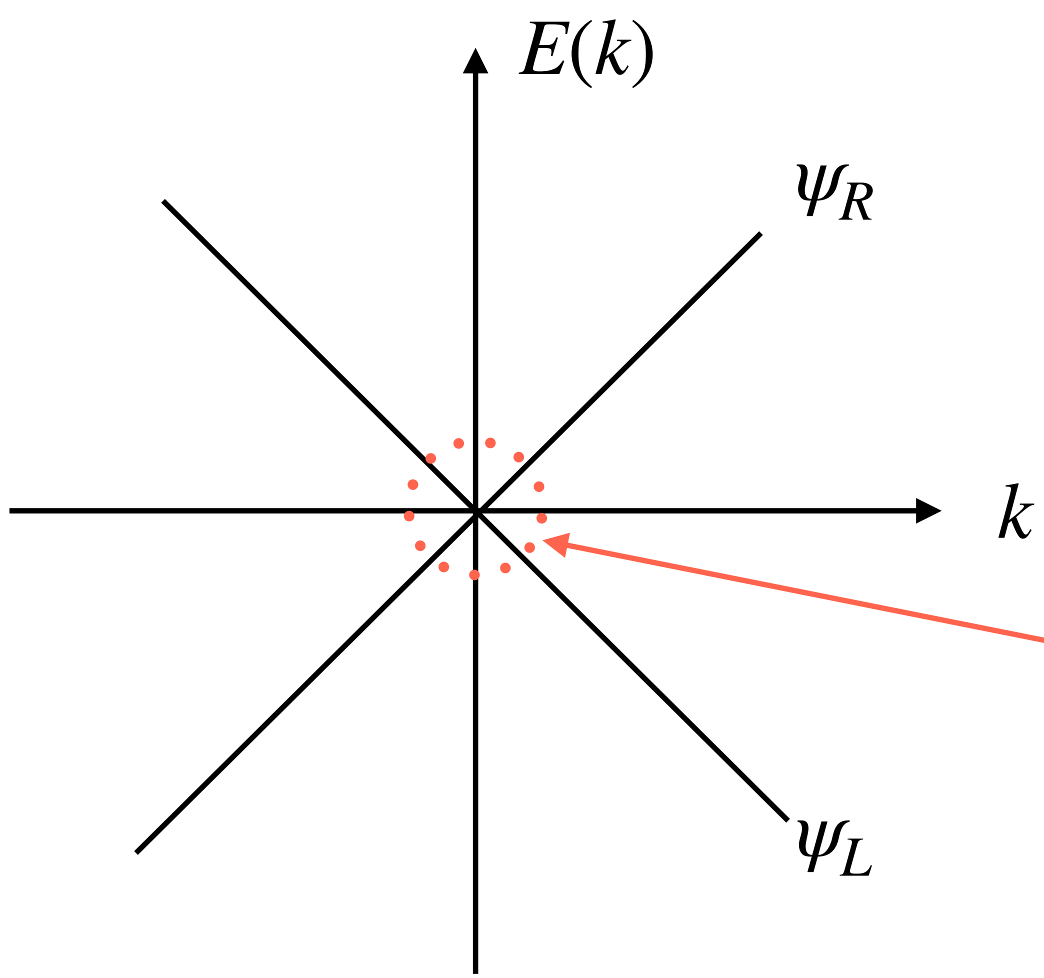

Dirac equation

$$(i\gamma^{\mu}\partial_{\mu} - mc)\psi = 0$$

$$\downarrow$$

$$i\partial_t\psi_R = -i\partial_x\psi_R$$

$$i\partial_t\psi_L = +i\partial_x\psi_L$$

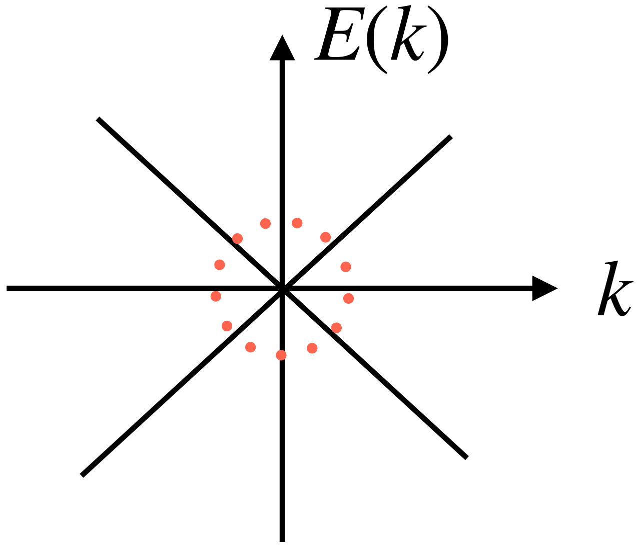



Dirac point protected by

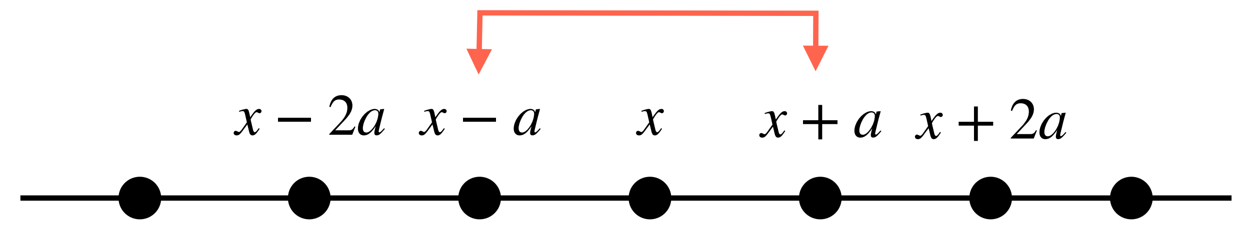
Chiral and time-reversal symmetry

# Fermion doubling

Continuous theory

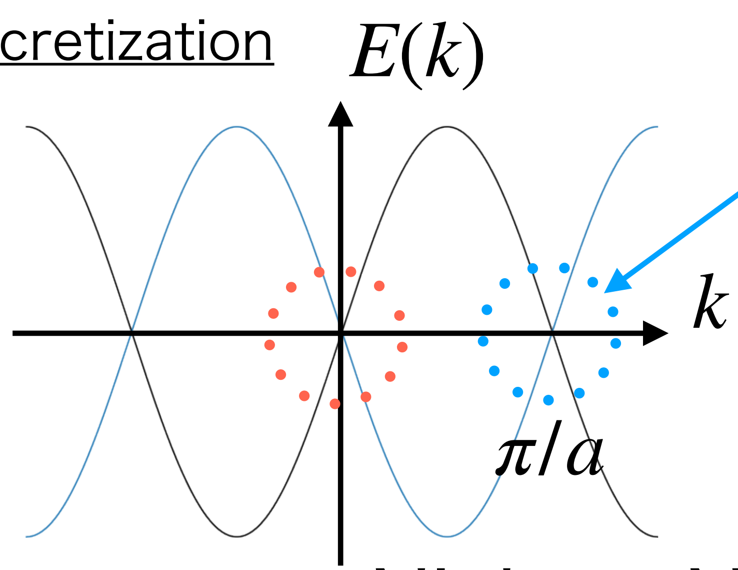



$$i\partial_x\psi \rightarrow \frac{i}{2a}(\psi(x+a)-\psi(x-a))$$




$$i\partial_t\psi = -i\partial_x\psi \longrightarrow H\vec{\psi} = \begin{pmatrix} 0 & \frac{i}{2} & 0 & \cdots & -\frac{i}{2} \\ -\frac{i}{2} & 0 & \ddots & \ddots & \cdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{i}{2} \\ \frac{i}{2} & \cdots & 0 & -\frac{i}{2} & 0 \end{pmatrix} \vec{\psi}$$

Discretization




Unwanted Dirac point

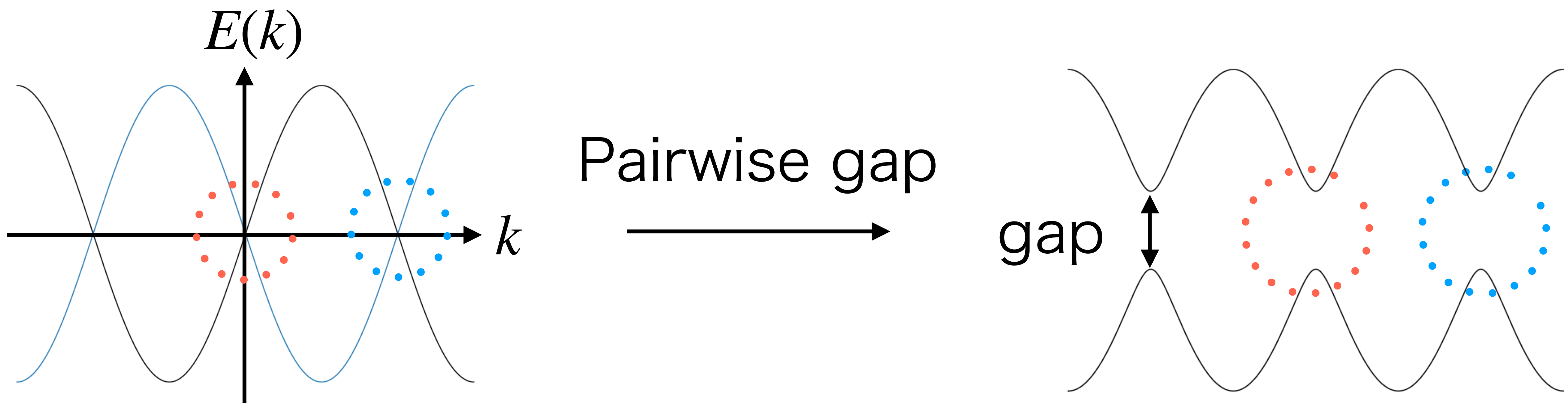
$$\longleftarrow \quad E(k) = \sin(ka)$$

$$k \sim k + 2\pi$$
$$-\pi \sim \pi$$

Nielsen-Ninomiya theorem(Fermion doubling)

**Local** and **symmetry-preserving** discretization of fermions always introduce another unnecessary Dirac points

# Fermion doubling is terrible!




Fermion doubling causes different results from field theory

How do we avoid this?

Nielsen-Ninomiya theorem

**Local** and **symmetry-preserving** discretization of fermions always introduce another unnecessary Dirac points

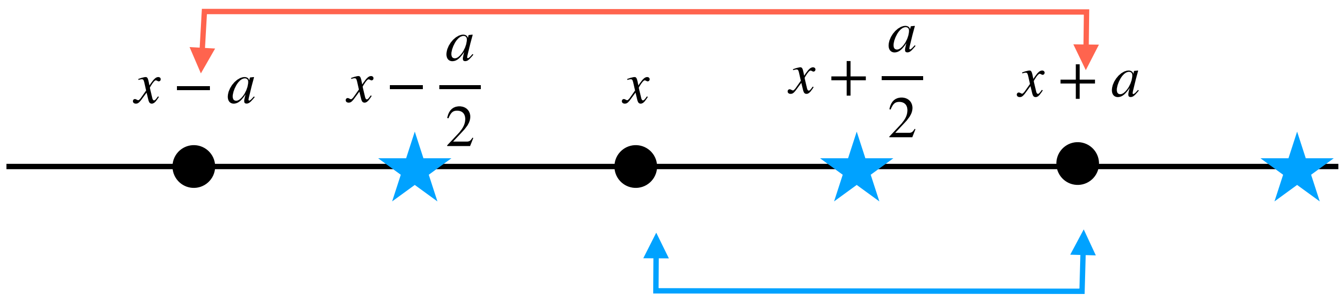
# Stacey’s fermion

Eliminating lattice fermion doubling

Richard Stacey

Phys. Rev. D **26**, 468 – Published 15 July 1982

$$i\partial_x\psi \to \frac{i}{2a}(\psi(x+a)-\psi(x-a))$$




$$\phi(x+\frac{a}{2}) \to \frac{1}{2}\left[\psi(x+a)+\psi(x)\right]$$

$$i\partial_x\phi(x+\frac{a}{2}) \to \frac{i}{a}\left[\psi(x+a)-\psi(x)\right]$$

$$\frac{1}{2}\left[\phi(x-\frac{a}{2})+\phi(x+\frac{a}{2})\right]$$

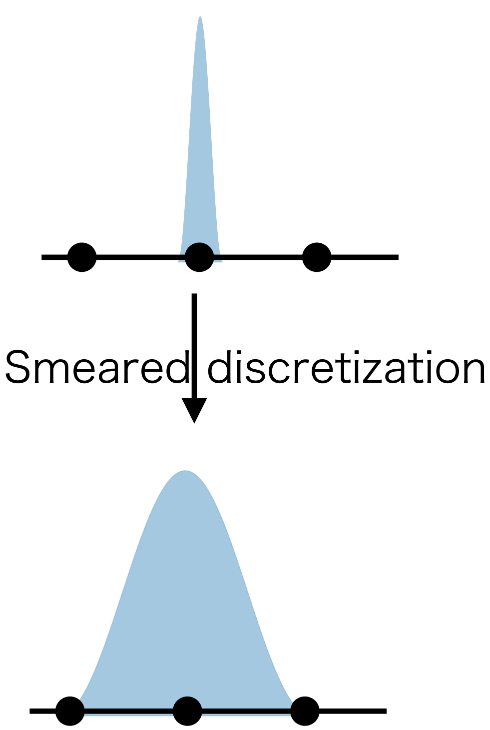
$$\to \frac{1}{4}\left[\psi(x-a)+2\psi(x)+\psi(x+a)\right]$$

$$\frac{i}{2}\left[\partial_x\phi(x+\frac{a}{2})+\partial_x\phi(x-\frac{a}{2})\right]$$

$$\to \frac{i}{2a}\left[\psi(x+a)-\psi(x-a)\right]$$

$$i\partial_t\psi(x) = -i\partial_x\psi(x)$$

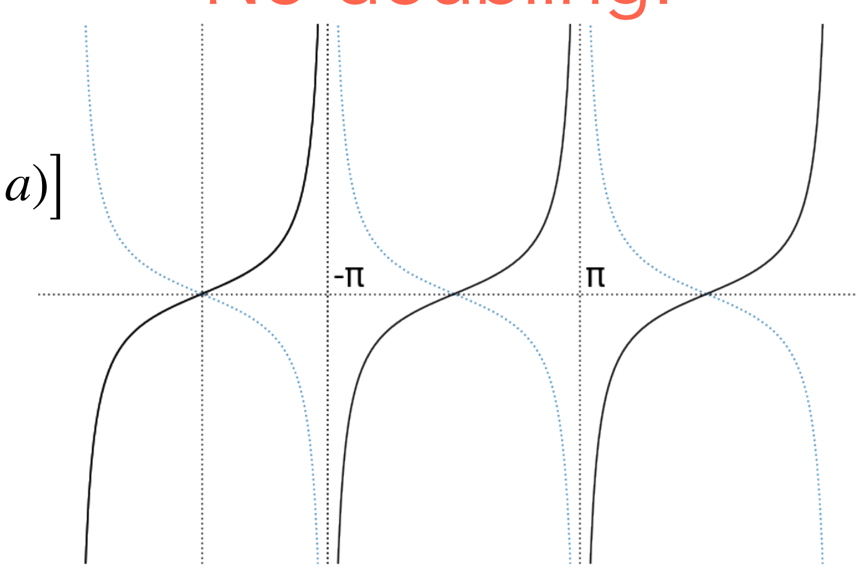
No doubling!




$$\cos^2\left(\frac{ka}{2}\right) \qquad\qquad \sin(ka)$$

$$i\partial_t\frac{1}{4}\left[\psi(x-a)+2\psi(x)+\psi(x+a)\right] = \frac{i}{2a}\left[\psi(x+a)-\psi(x-a)\right]$$

$$E(k) = \frac{\sin(ka)}{a\cos^2(\frac{ka}{2})} = \frac{2}{a}\tan\left(\frac{ka}{2}\right)$$



# Stacey's fermion is generalized eigenvalue

$$i\partial_t \frac{1}{4}\left[\psi(x-a)+2\psi(x)+\psi(x+a)\right] = \frac{i}{2a}\left[\psi(x+a)-\psi(x-a)\right]$$

$$H\overrightarrow{\psi} = EN\overrightarrow{\psi}$$

$$H = \begin{pmatrix} 0 & \frac{i}{2} & 0 & \cdots & -\frac{i}{2} \\ -\frac{i}{2} & 0 & \ddots & \ddots & \cdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{i}{2} \\ \frac{i}{2} & \cdots & 0 & -\frac{i}{2} & 0 \end{pmatrix} N = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \cdots & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{1}{4} \\ \frac{1}{4} & \cdots & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

This Stacey's fermion has been revisited ⋯

M. J. Pacholski (2021)
A. Donis Vela (2022)
C.W.J Beenaker (2023)
V. A. Zakharov (2024)
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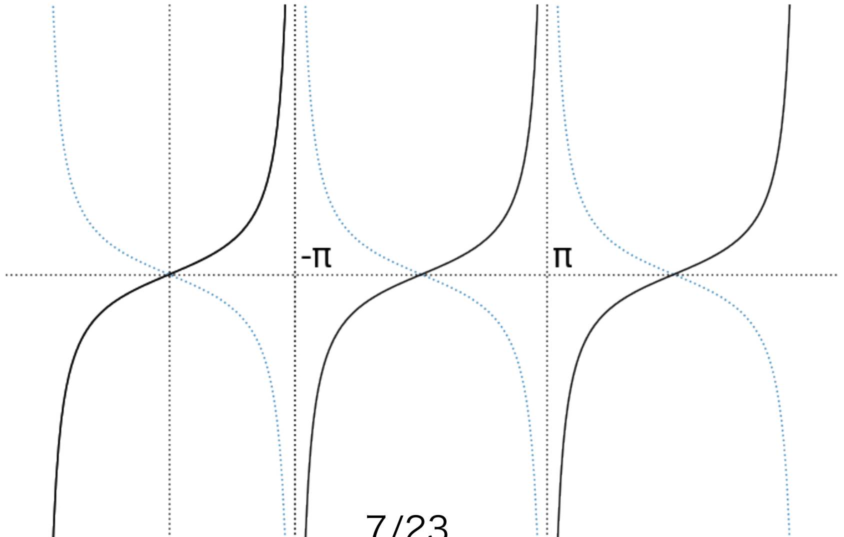



Symmetry protected Dirac point

# Our goal

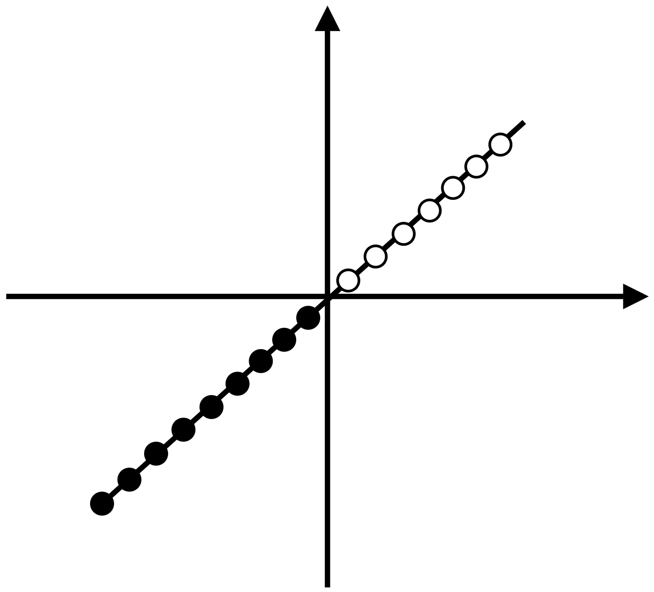
## Stacey's fermion → one-particle physics

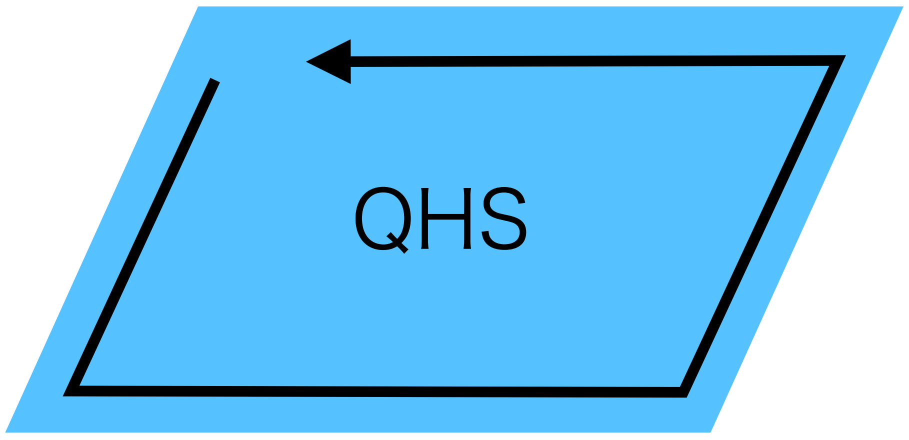
Can we construct many-body interacting physics of Stacey fermion?

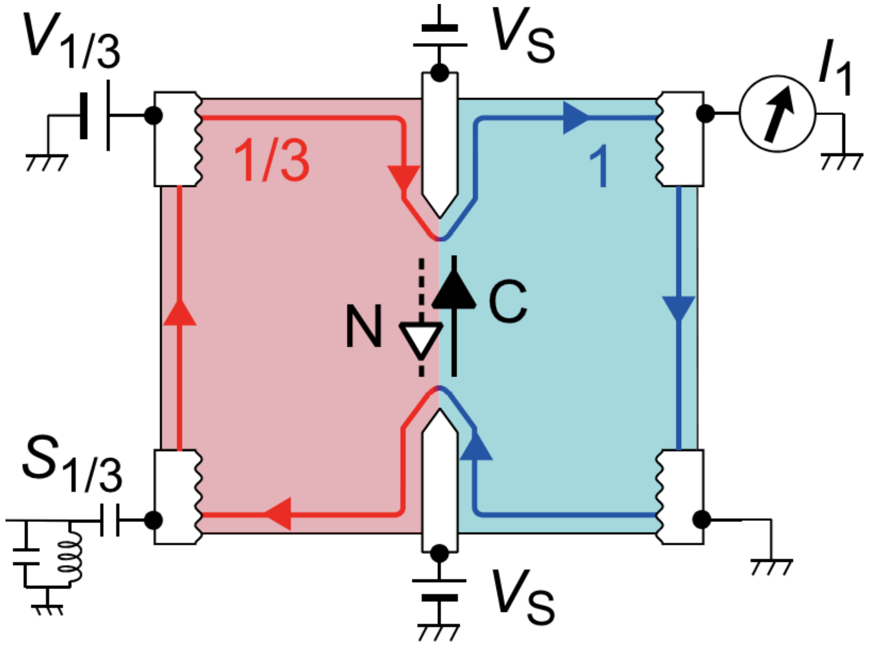
We can construct the second quantization of Stacey fermion with "locality" on tensor network

# Our goal

We want to construct a purely one-dimensional lattice model that has **a chiral spectrum.**






M.Hashisaka PRX(2023)

# Our collaborators

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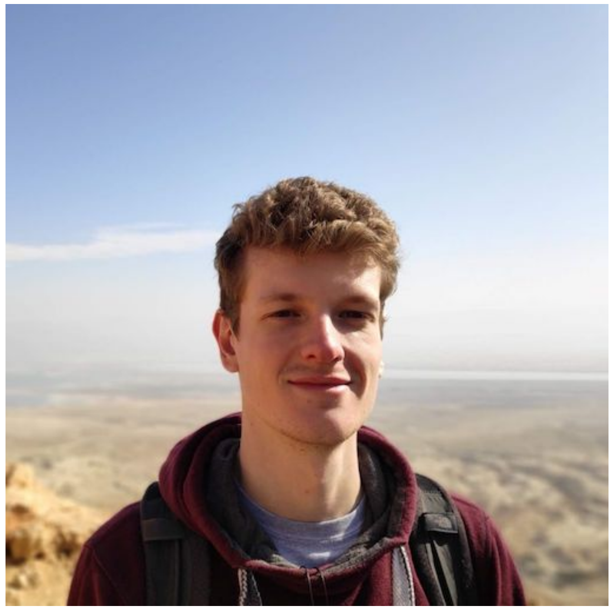

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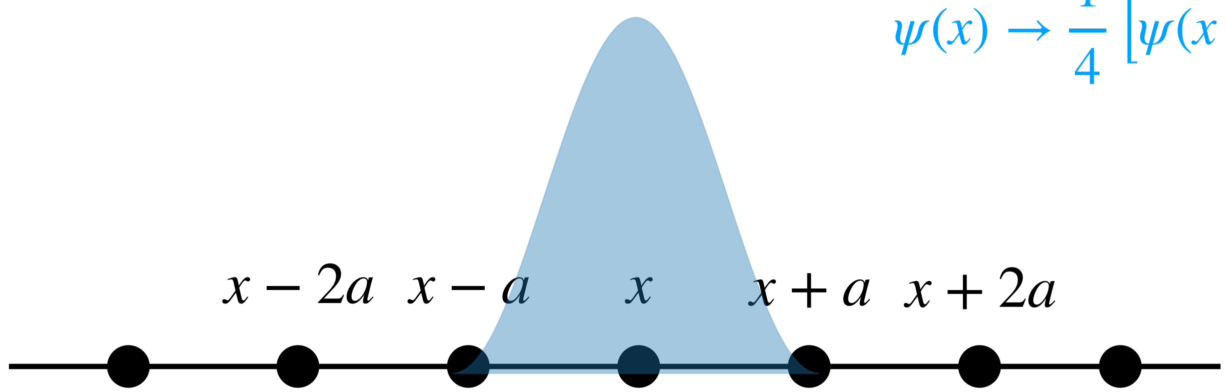
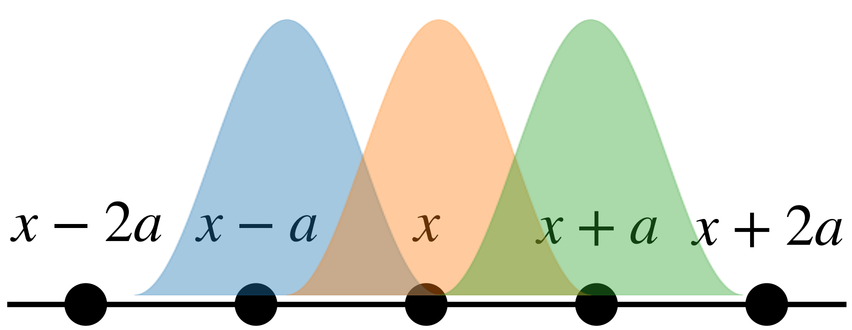
# Non-orthogonal orbitals

The key idea of Stacey's formalism is discretization of fields with a certain width instead of $\delta$ function.

$$\psi(x) \to \frac{1}{4}\left[\psi(x-a) + 2\psi(x) + \psi(x+a)\right]$$




This tails of "orbitals" allow us to annihilate the particle of neighboring sites.




<u>Conventional fermion</u> $c_i$

$$\{c_i^\dagger, c_j\} = \delta_{i,j}$$

$$\langle \Omega | c_i c_{i+1}^\dagger | \Omega \rangle = 0$$

<u>Generalized fermion</u> $a_i$

$$\{a_i^\dagger, a_j\} = N_{i,j}$$

$$\langle \Omega | a_i a_{i+1}^\dagger | \Omega \rangle \neq 0$$

# What is the "smeared" operator $a_i$ ?

$$a_i = \frac{1}{2}(c_{i-\frac{1}{2}} + c_{i+\frac{1}{2}}) \qquad a_i^\dagger = \frac{1}{2}(c^\dagger_{i-\frac{1}{2}} + c^\dagger_{i+\frac{1}{2}})$$

$$\vec{a} = D\vec{c}$$

$$\{a_i^\dagger, a_j\} = (D^\dagger D)_{ij} = N_{ij}$$

$$\{a_i^\dagger, a_i\} = 1/2 \qquad \{a_i^\dagger, a_{i\pm1}\} = 1/4$$

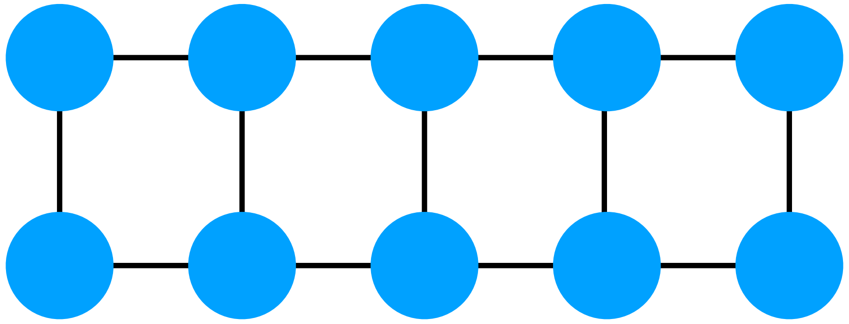
$$N = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \cdots & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{1}{4} \\ \frac{1}{4} & \cdots & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

# Many-body norm is MPO

$$|n_1, n_2, \ldots, n_L\rangle = (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\ldots(a_L^\dagger)^{n_L}|\Omega\rangle$$

$$|\psi\rangle = \sum_{n_1 n_2 \cdots} \underbrace{\mathrm{Tr}\left(A_1^{n_1}A_2^{n_2}\ldots A_L^{n_L}\right)}_{|\phi\rangle} |n_1, n_2, \ldots, n_L\rangle$$

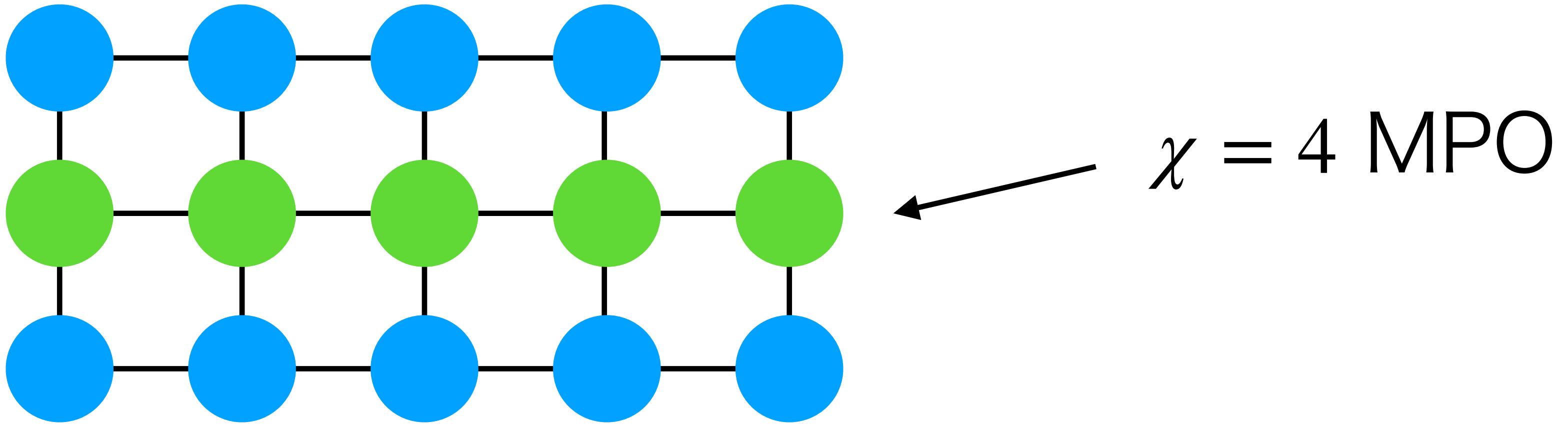
$\langle\psi|\psi\rangle \neq$



$\langle\Omega|a_i a_{i+1}^\dagger|\Omega\rangle \neq 0$

$$\langle\psi|\psi\rangle = \mathrm{Tr}\left(A_1^{n_1}A_2^{n_2}\ldots A_L^{n_L}\right)\tilde{N}_{n_1\ldots n_L;n_1'\ldots n_L'}\mathrm{Tr}\left(A_1^{n_1'}A_2^{n_2'}\ldots A_L^{n_L'}\right)$$

$$\tilde{N}_{n_1\ldots n_L;n_1'\ldots n_L'} = \langle\Omega|(a_L)^{n_L}\ldots(a_1)^{n_1}(a_1^\dagger)^{n_1'}\ldots(a_L^\dagger)^{n_L'}|\Omega\rangle$$



# Hamiltonian

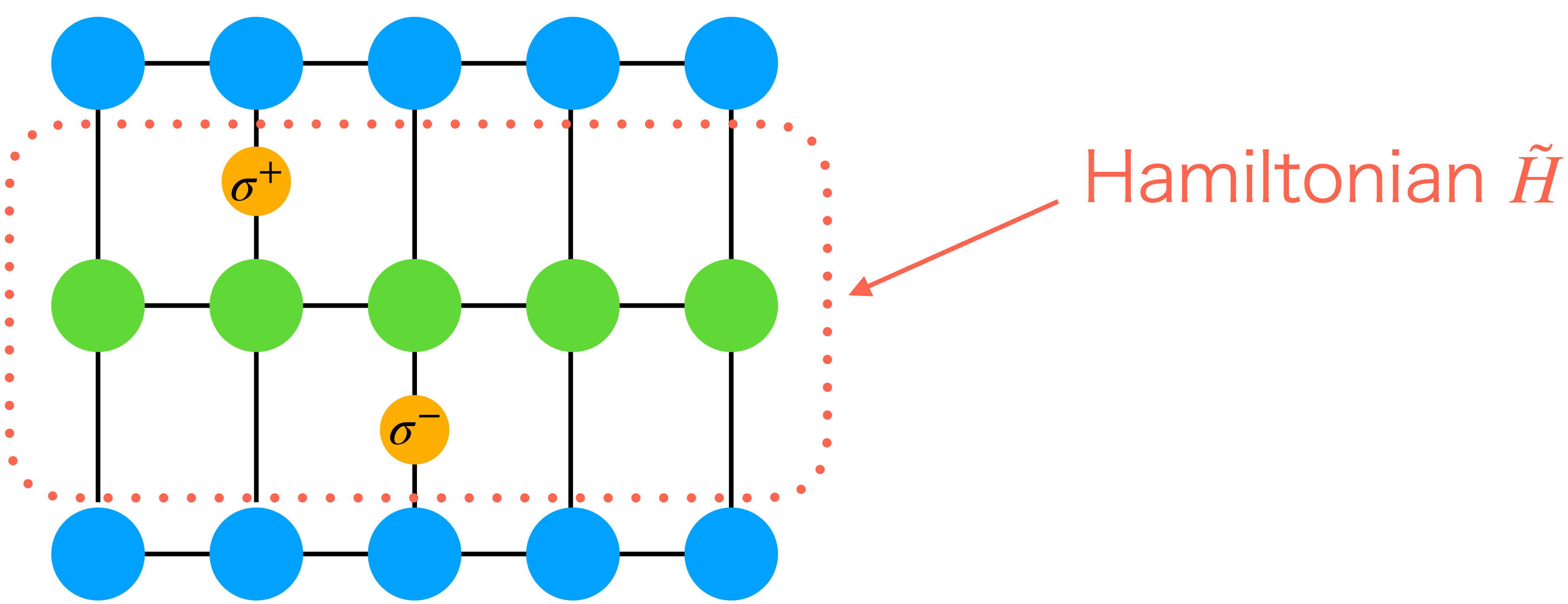
We define the Hamiltonian with $b_i$ (the conjugate of $a_i$)

$$\{b_i, a_j^\dagger\} = \delta_{ij}$$

$$\hat{H} = J \sum_n \left( i b_n^\dagger b_{n+1} + h.c. \right) + U \sum_n b_n^\dagger b_{n+1}^\dagger b_{n+1} b_n$$

This Hamiltonian "locally" modifies the bra and ket states.

$$\langle \psi | b_n^\dagger b_{n+1} | \psi \rangle =$$



# Many-body generalized eigenvalue problem

Many-body states

$$\tilde{H}|\phi\rangle = E\tilde{N}|\phi\rangle$$

Many-body Hamiltonian

Many-body norm tensor

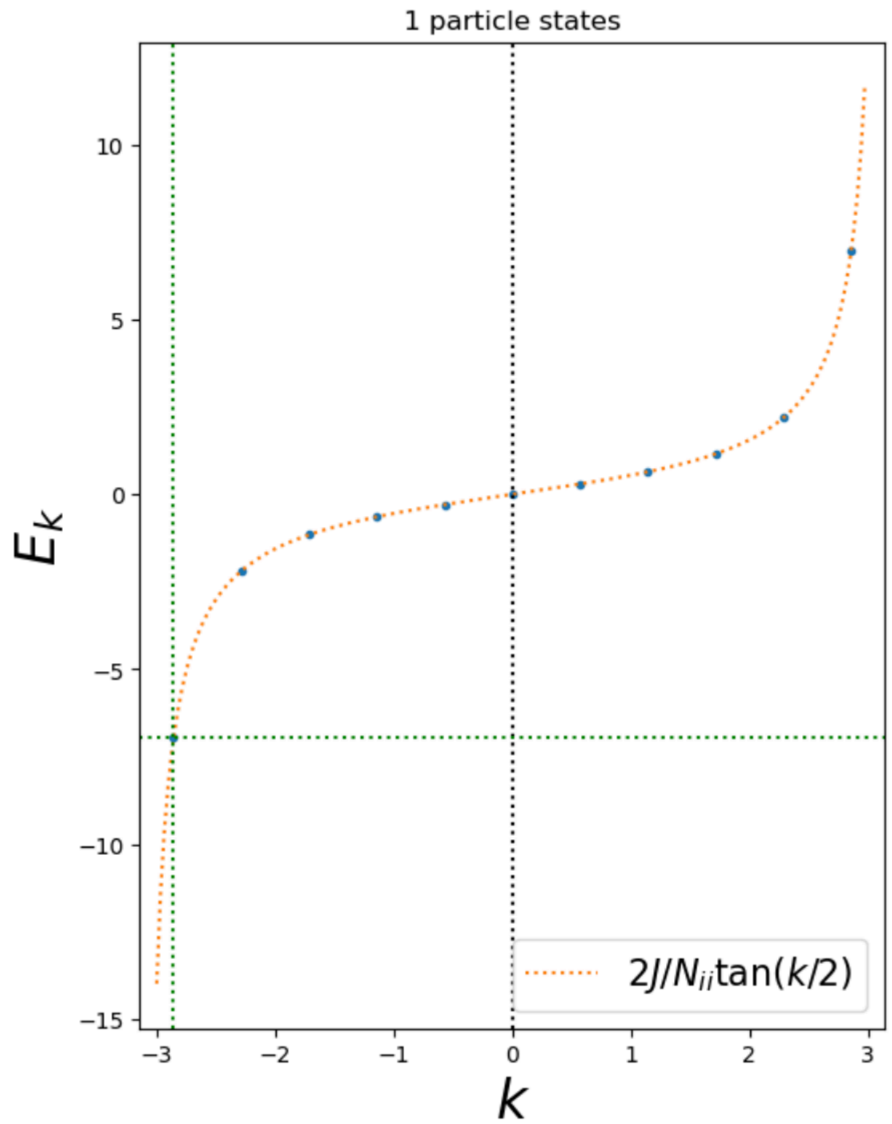
This is generalization of the conventional energy.

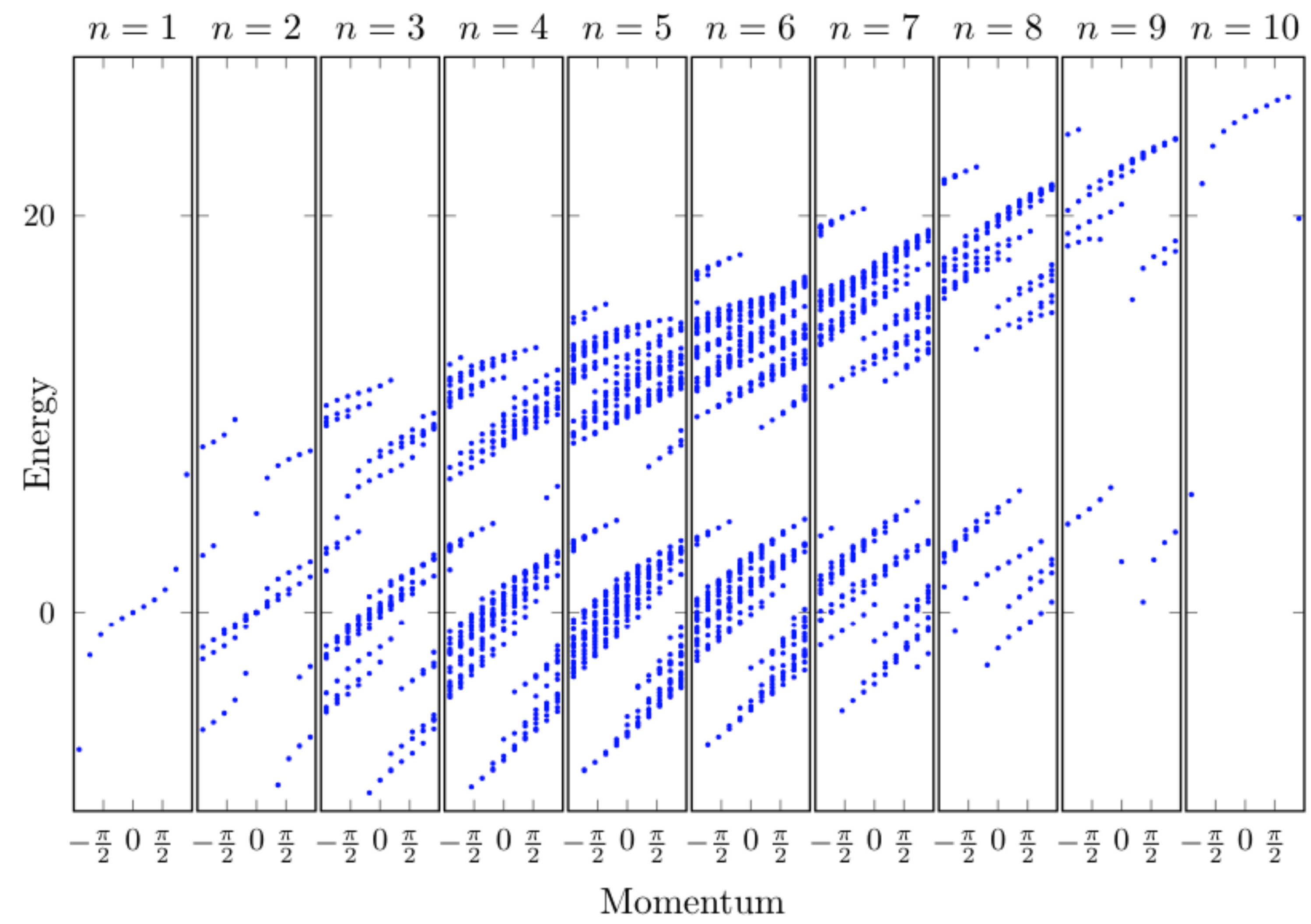
$$E = \frac{\langle\phi|\tilde{H}|\phi\rangle}{\langle\phi|\tilde{N}|\phi\rangle}$$

Norm is non-trivial

# Our result: exact diagonalization

One-particle sector of our hamiltonian indeed is Stacey fermion!




We can compute interacting case. The dispersion is chiral

$L = 11....$

# "Local" tensor network formalism: DMRG

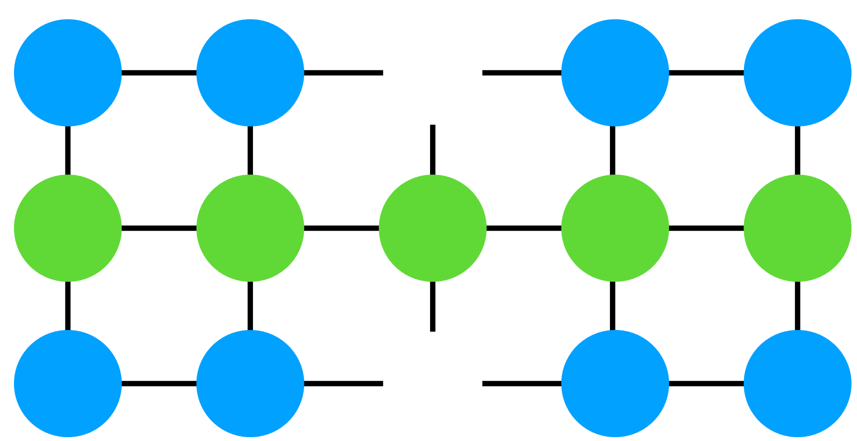
Tensor network can efficiently solve the generalized eigenvalue problem.

$$\lambda_{\min} = \min_{\phi} \frac{\langle \phi | \tilde{H} | \phi \rangle}{\langle \phi | \tilde{N} | \phi \rangle}$$

In MPS, solving it amounts to solving the local eigenvalue problem. This allows to simulate larger systems.

$$\langle \phi | \tilde{H} | \phi \rangle = A_i^\dagger H_{eff}(i) A_i$$

$$H_{eff}(i) =$$



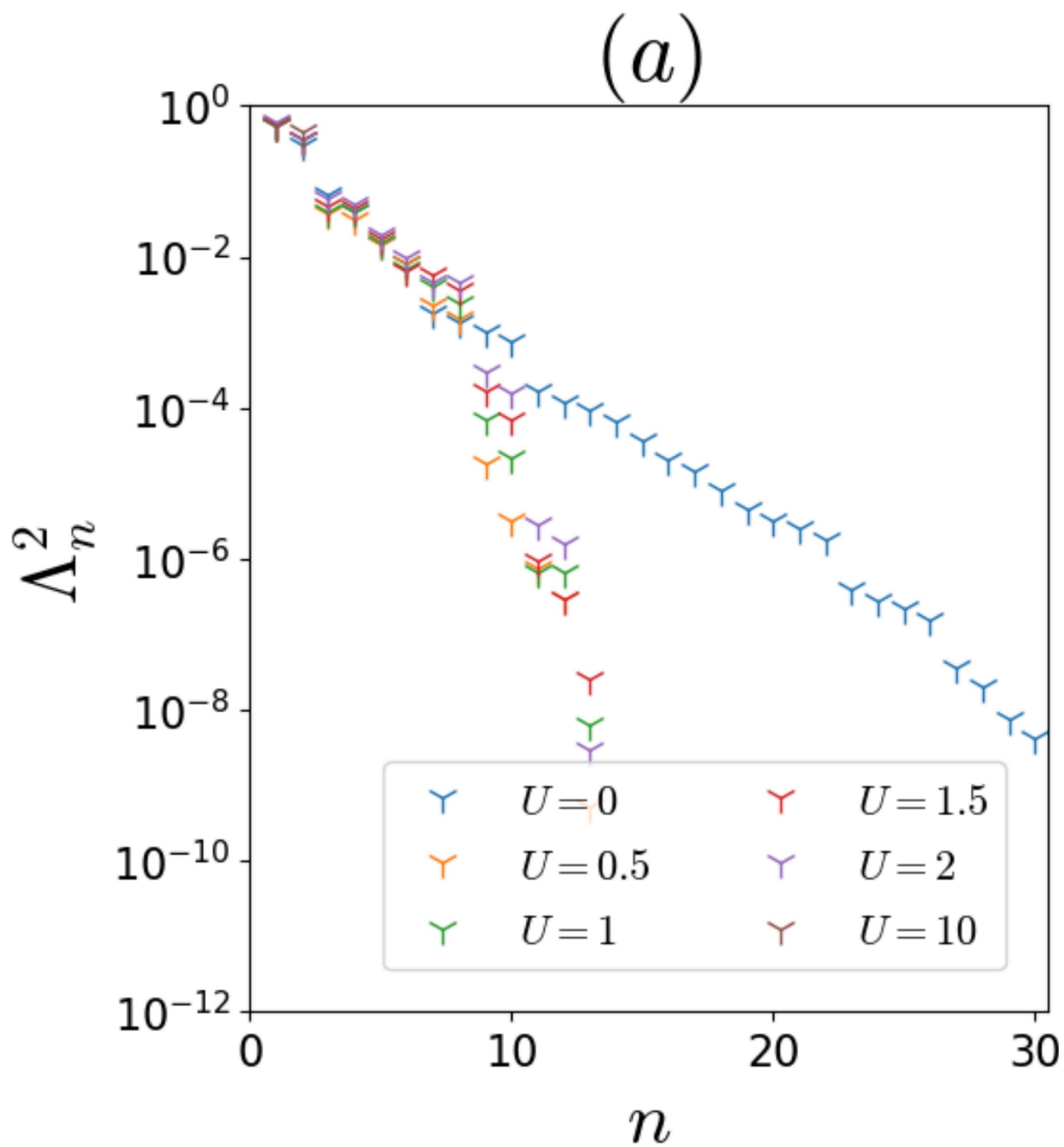
$$\tilde{H}_{eff}(i) A_i = \lambda_{\min} \tilde{N}_{eff}(i) A_i$$

$L > 100!$

# Schmidt values decay exponentially

The Schmidt values of the ground state decays exponentially.

This corroborates the efficient MPS simulation with finite $\chi$



# Summary of our formulation

We constructed the second-quantization of Stacey fermion.
This Hamiltonian becomes local in non-orthogonal basis.

## Orthogonal formulation

$$H_{\text{tangent}} = 2it_0 \sum_{n>m=1}^{N} (-1)^{n-m} \left( c_n^\dagger c_m - c_m^\dagger c_n \right)$$

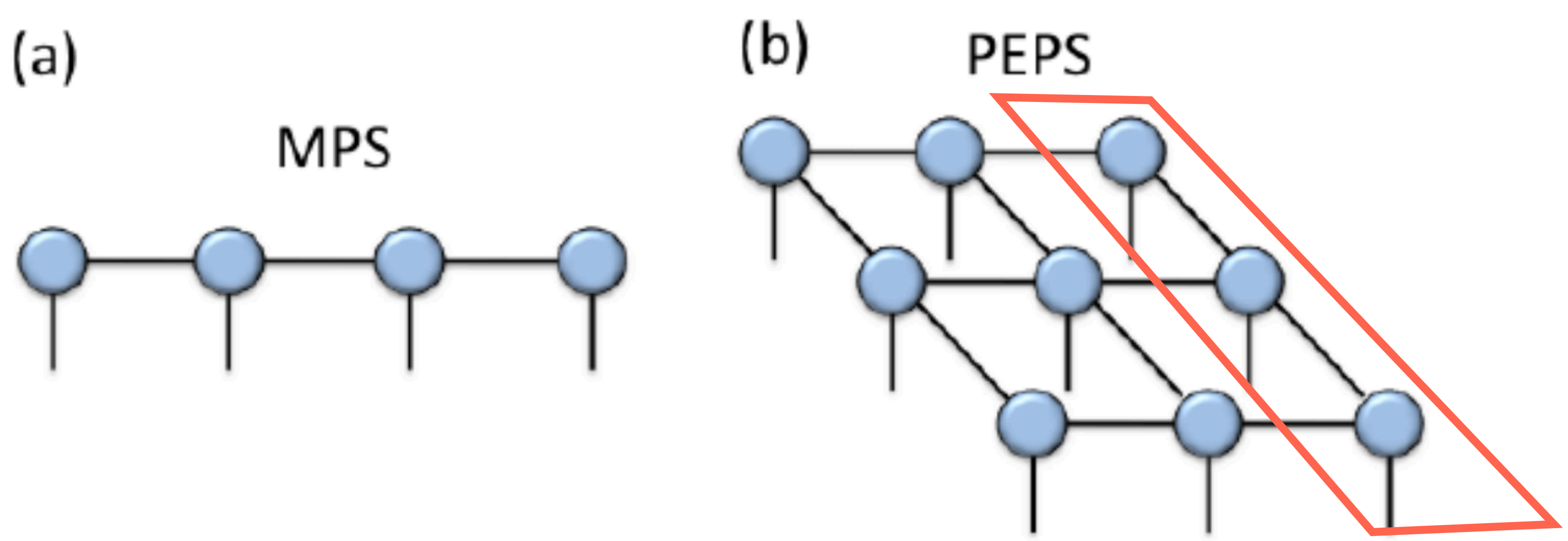
## Non-orthogonal formulation

$$= \frac{t_0}{2i} \sum_{n=1}^{N} \left( b_{n+1}^\dagger b_n - b_n^\dagger b_{n+1} \right).$$

# Physical picture — Why non-orthogonal?

Chiral fermions appear at the edge of integer quantum hall systems.

This two dimensional systems are described by **2D tensor network**— Projected Entanglement Product States(PEPS).

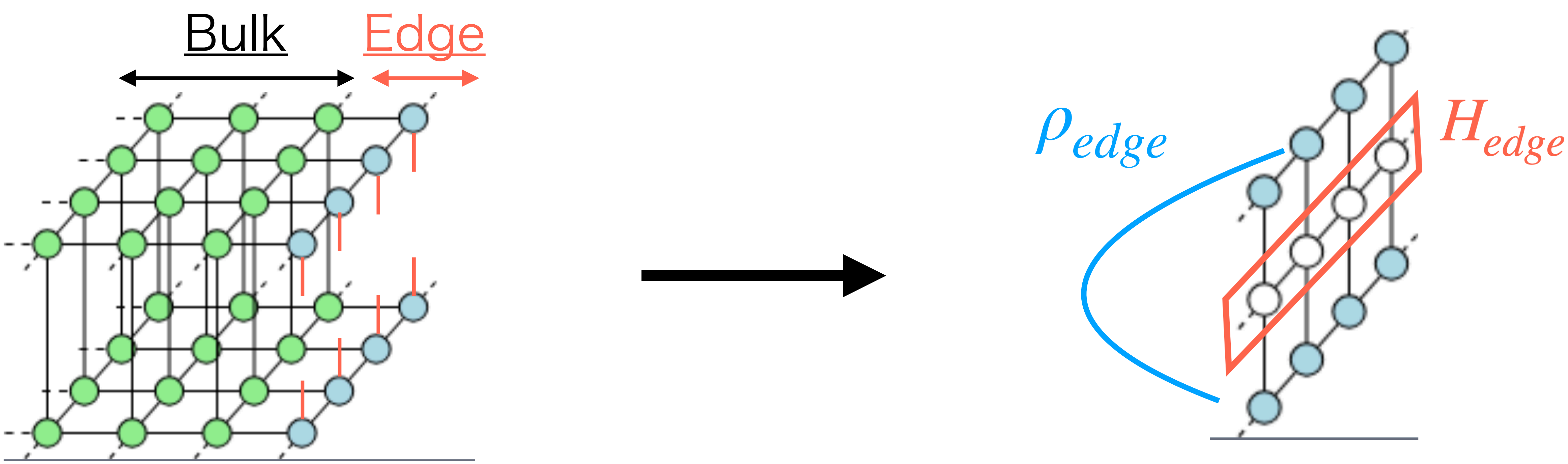



The chiral fermion should be represented by the edge MPS

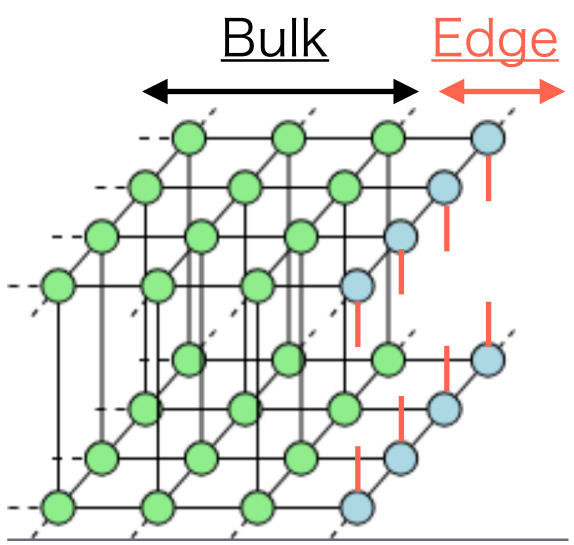
# Reduced density matrix -> mixed states

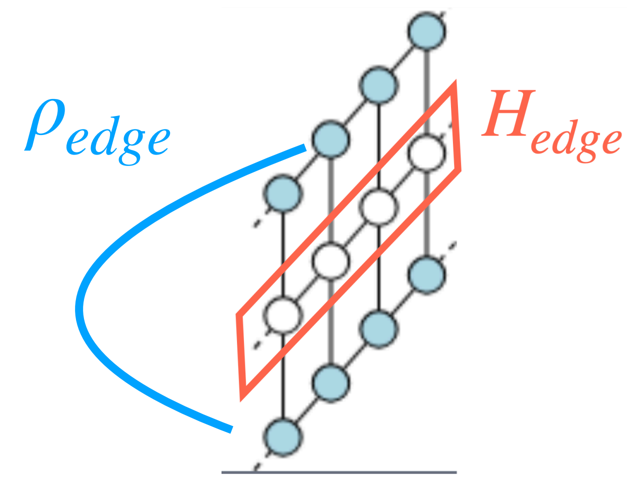
$$\langle \Psi | H_{edge} | \Psi \rangle = \mathrm{Tr}\,(\rho H_{edge}) \qquad \rho = | \Psi \rangle \langle \Psi |$$

$$= \mathrm{Tr}\,(\rho_{edge} H_{edge}) \qquad \rho_{edge} = \mathrm{Tr}_{bulk} | \Psi \rangle \langle \Psi |$$



# Reduced density matrix -> mixed states




$$\rho_{edge} = \mathrm{Tr}_{bulk} |\Psi\rangle\langle\Psi| \qquad = \mathrm{Tr}\,(\rho_{edge} H_{edge})$$

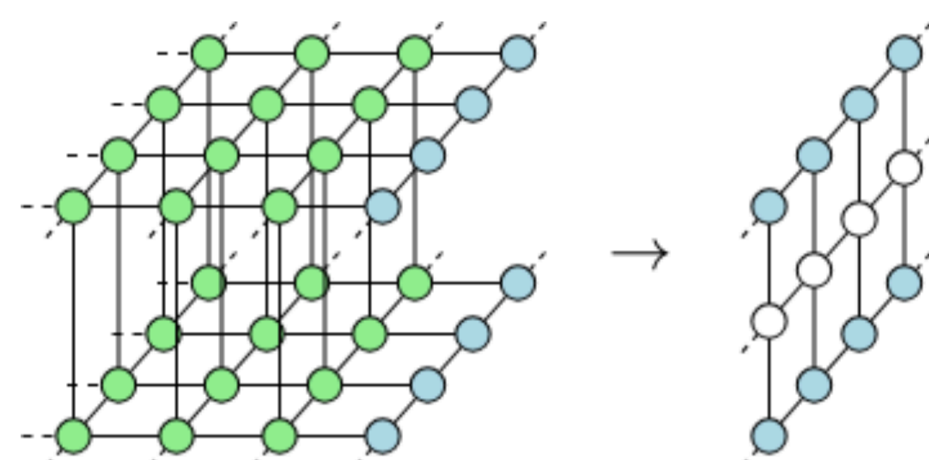
$$\rho_{edge} = \sum_n \Lambda_n |\psi_n^L\rangle\langle\psi_n^R| \qquad \Lambda_n^{-1} = \langle\psi_n^L|\hat{N}|\psi_n^R\rangle$$

$$\langle\Psi|H_{edge}|\Psi\rangle = \frac{\langle\psi_n^L|H_{edge}|\psi_n^R\rangle}{\langle\psi_n^L|\hat{N}|\psi_n^R\rangle}$$

Non-orthogonality of the chiral fermion is needed because the reduced density matrix is not a pure state.

# Summary

- Chiral theory cannot be discretized with preserving symmetry in a local manner. (Fermion doubling)
- Stacey fermion breaks the locality, but can be formulated "locally" in terms of **generalized eigenvalue problems**.
- We developed the second quantization of it to realize **chiral many-body states in one dimension** (extendable to higher dimension)
- MPS provides efficiently simulations.




To be continued…