

*On the Parthasarathy formula for quantized
irreducible flag manifolds*

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What is the **Parthasarathy formula**?

Given a **symmetric space** $M = G/K$ with Dirac operator D , it expresses D^2 in terms of the Casimir of G , namely

$$D^2 = \Omega_G + \frac{1}{8} \text{Scal}_M.$$

Allows to determine the spectrum via **representation theory** of G .

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Allows to determine the spectrum via **representation theory** of G .

We would like to have an analogue for quantum symmetric spaces.

Goal: spectral triples for quantized irreducible flag manifolds. Prove compact resolvent condition for Krähmer's Dirac operators.

A **generalized flag manifold** is a homogeneous space G/P , where P is a parabolic subgroup (contains a Borel subgroup).

They are classified by subsets $S \subseteq \Pi$ of simple roots. We have a decomposition $\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{l} \oplus \mathfrak{u}_+$ as \mathfrak{l} -modules, where

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{l})} \mathfrak{g}_\alpha, \quad \mathfrak{u}_\pm = \bigoplus_{\alpha \in \Delta(\mathfrak{u}_\pm)} \mathfrak{g}_{\pm\alpha}.$$

These are subalgebras of \mathfrak{g} . Some names: $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_+$ is the **standard parabolic subalgebra**, \mathfrak{l} is the **Levi factor** and \mathfrak{u}_+ is the **nilradical**.

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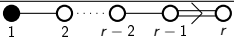
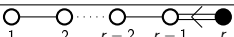
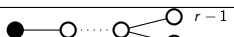
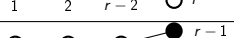
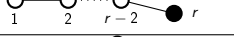
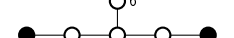
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We will consider parabolics of **cominuscule type**. One definition is that \mathfrak{u}_+ is abelian. In this case G/P is a symmetric space.

We also have the commutation relations $[\mathfrak{u}_+, \mathfrak{u}_-] \subset \mathfrak{l}$.

A parabolic subalgebra \mathfrak{p} is cominuscale iff $S = \Pi \setminus \{\alpha_t\}$ and α_t has **multiplicity 1** in the highest root of \mathfrak{g} .

Alg.	Dynkin diagram	Nomenclature
A_r		Grassmannian
B_r		Odd dimensional quadric
C_r		Lagrangian Grassmannian
D_r		Even dimensional quadric
D_r		Orthogonal Grassmannian
E_6		Cayley plane
E_7		Unnamed

These spaces are **compact Kähler manifolds**. Put an Hermitian metric on $\Omega^{(0,\bullet)}$. Then we have the **Dolbeault-Dirac** operators $D = \bar{\partial} + \bar{\partial}^*$.

Concretely pick dual bases $\{v_i\}_i \in \mathfrak{u}_+$ and $\{w_i\}_i \in \mathfrak{u}_-$. We identify the basis of \mathfrak{u}_+ with some root vectors $\{E_{\xi_i}\}_i \in \mathfrak{g}$. We define

$$\bar{\partial} = \sum_{\xi_i \in \Delta(\mathfrak{u}_+)} E_{\xi_i} \otimes \gamma_-(w_i) \in U(\mathfrak{g}) \otimes \text{Cl}.$$

It turns out that $\bar{\partial} = \bar{\partial}^*$. Hence we can write

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$$D = \bar{\partial} + \bar{\partial}^* \in U(\mathfrak{g}) \otimes \text{Cl}.$$

Now we want to compute D^2 . Since $\bar{\partial}^2 = 0$ we have

$$D^2 = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

Explicitly we can write

$$D^2 = \sum_{i,j} E_{\xi_i} F_{\xi_j} \otimes \gamma_-(w_i) \gamma_+(v_j) + \sum_{i,j} F_{\xi_i} E_{\xi_j} \otimes \gamma_+(v_i) \gamma_-(w_j).$$

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Can forget about terms in $U(\mathfrak{l}) \otimes \text{Cl}$, since these are **bounded operators**.
Write $A \sim B$ if $A - B = C$ with $C \in U(\mathfrak{l}) \otimes \text{Cl}$. Since $[\mathfrak{u}_+, \mathfrak{u}_-] \subset \mathfrak{l}$ we get

$$D^2 \sim \sum_{i,j} E_{\xi_i} F_{\xi_j} \otimes (\gamma_-(w_i) \gamma_+(v_j) + \gamma_+(v_j) \gamma_-(w_i)).$$

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Next we have the **Clifford algebra relations**

$$\gamma_-(w) \gamma_+(v) + \gamma_+(v) \gamma_-(w) = \langle w, v \rangle 1, \quad w \in \mathfrak{u}_-, \quad v \in \mathfrak{u}_+.$$

Since we are using dual bases we have $\langle w_i, v_j \rangle = \delta_{ij}$. Hence

$$D^2 \sim \sum_i E_{\xi_i} F_{\xi_i} \otimes 1.$$

Finally we have $C \sim \sum_i E_{\xi_i} F_{\xi_i}$, where C is the **quadratic Casimir** of \mathfrak{g} .
Then we have the (simplified) Parthasarathy formula

$$D^2 \sim C \otimes 1.$$

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First of all we need appropriate quantum versions of all the different structures appearing here (most of these are well known).

To obtain a simple formula for D^2 , following the classical case, we need:

- 1 commutation relations in the **quantized enveloping algebra** $U_q(\mathfrak{g})$,
- 2 commutation relations in the **quantum Clifford algebra** Cl_q .

$U(\mathfrak{g}) \rightsquigarrow U_q(\mathfrak{g})$ Hopf algebra deformation,
matrix coefficients of irreps $\rightsquigarrow \mathbb{C}_q[G]$.

When $q \in \mathbb{C}$ is not a root of unity the representation theory is essentially the same. Hence we have a $U_q(\mathfrak{l})$ -module \mathfrak{u}_+ .

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The category of $U_q(\mathfrak{g})$ -modules is **braided**, so we have isomorphisms

$$\widehat{R}_{V,W} : V \otimes W \rightarrow W \otimes V.$$

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[Berenstein, Zwicknagl (05)] define the **quantum exterior algebra** by

$$\Lambda_q(V) = T(V) / \langle \ker(\sigma_{V,V} - \text{id}) \rangle.$$

$\ker(\sigma_{V,V} - \text{id}) = \text{span of eigenspaces of } \widehat{R}_{V,V} \text{ with } \text{positive eigenvalues.}$

In general the dimension is different from the classical one (but not here).

Following Lusztig we can define **quantum root vectors**, which give a PBW-basis of $U_q(\mathfrak{g})$. What about the Clifford algebra?

[Krähmer, Tucker-Simmons (15)] define a certain quantum Clifford algebra $Cl_q \subseteq \text{End}(\Lambda_q(\mathfrak{u}_+))$. Then they define **Dolbeault-Dirac operators**

$$D = \tilde{\partial} + \tilde{\partial}^* \in U_q(\mathfrak{g}) \otimes Cl_q,$$

where $\tilde{\partial} \in U_q(\mathfrak{g}) \otimes Cl_q$ is defined by

$$\tilde{\partial} = \sum_i S^{-1}(E_{\xi_i}) \otimes \gamma_-(w_i).$$

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One of their main results is that $\tilde{\partial}^2 = 0$ when \mathfrak{u}_+ is cominuscule.

Then $D^2 = \tilde{\partial}\tilde{\partial}^* + \tilde{\partial}^*\tilde{\partial}$. What else can be said?

To proceed we need commutation relations in $U_q(\mathfrak{g})$.

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Theorem

Let $\xi, \xi' \in \Delta(\mathfrak{u}_+)$ in the cominuscule case. Then we have

$$E_\xi E_{\xi'}^* - q^{-(\xi, \xi')} E_{\xi'}^* E_\xi \sim \sum_{\eta, \eta'} c_{\xi, \xi'}^{\eta, \eta'} E_\eta^* E_{\eta'},$$

where $\eta, \eta' \in \Delta(\mathfrak{u}_+)$ and $\text{ht}(\eta') < \text{ht}(\xi)$.

Similar relations are obtained for $\mathcal{E}_i = S^{-1}(E_{\xi_i})$.

Word of caution: general commutation relations are much more complicated. Here we use the fact that $U_q(\mathfrak{l})$ acts adjointly on the **twisted Schubert cell** in the cominuscule case [Zwicky (09)].

Proposition

1 We have $D^2 \sim \sum_{i,j} \mathcal{E}_i \mathcal{E}_j^* \otimes T_{ij}$, where $T_{ij} \in \text{Cl}_q$ are given by

$$T_{ij} = \gamma_-(w_i) \gamma_-(w_j)^* + \sum_{k,l} b_{kl}^{ij} \gamma_-(w_k)^* \gamma_-(w_l), \quad b_{kl}^{ij} \in \mathbb{C}.$$

2 Suppose $D^2 \sim \sum_i C_i \otimes T_i$, where $C_i \in U_q(\mathfrak{g})$ are central elements. Then we must have $T_{ij} = 0$ for $i \neq j$ in the previous expression.

Hence a **necessary condition** to obtain a formula of Parthasarathy-type is to have certain quadratic commutation relations in Cl_q .

So we have to focus our attention on the Clifford algebra part. How is it defined? Classically we have the isomorphism

$$\Lambda(\mathfrak{u}_-) \otimes \Lambda(\mathfrak{u}_+) \cong \text{End}(\Lambda(\mathfrak{u}_+)).$$

The two components act by interior and exterior multiplication.

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Krähmer and Tucker-Simmons show that also in the quantum case

$$\Lambda_q(\mathfrak{u}_-) \otimes \Lambda_q(\mathfrak{u}_+) \cong \text{End}(\Lambda_q(\mathfrak{u}_+)).$$

This defines the **quantum Clifford algebra** Cl_q acting on $\Lambda_q(\mathfrak{u}_+)$. The action of $\Lambda_q(\mathfrak{u}_+)$ is obvious. On the other hand for $\Lambda_q(\mathfrak{u}_-)$ we have

$$\langle w, \gamma_-(y)x \rangle = \langle w \wedge y, x \rangle, \quad y, w \in \Lambda_q(\mathfrak{u}_-), \quad x \in \Lambda_q(\mathfrak{u}_+).$$

There is a unique up to a constant $U_q(\mathfrak{l})$ -invariant pairing $\langle \cdot, \cdot \rangle : \mathfrak{u}_- \otimes \mathfrak{u}_+ \rightarrow \mathbb{C}$. How to extend to exterior algebras?

Can naturally be extended to a pairing of tensor algebras by

$$\langle y_k \otimes \cdots \otimes y_1, x_1 \otimes \cdots \otimes x_k \rangle_T = \langle y_1, x_1 \rangle \cdots \langle y_k, x_k \rangle.$$

One can identify $\Lambda_q(\mathfrak{u}_\pm)$ with a subspace of $T(\mathfrak{u}_\pm)$ via some maps π_\pm [Chirvasitu, Tucker-Simmons (14)]. Then we obtain a pairing

$$\langle y, x \rangle_\Lambda = \langle \pi_-^{-1}(y), \pi_+^{-1}(x) \rangle_T, \quad y \in \Lambda_q(\mathfrak{u}_-), \quad x \in \Lambda_q(\mathfrak{u}_+).$$

In each degree we can rescale the pairing by a constant. Classically it would be a rescaling of $k!$ in degree k (from antisymmetrization).

Indeed classically a natural pairing is defined using the **determinant**

$$\langle y_k \wedge \cdots \wedge y_1, x_1 \wedge \cdots \wedge x_k \rangle = \det \begin{pmatrix} \langle y_1, x_1 \rangle & \cdots & \langle y_1, x_k \rangle \\ \vdots & \ddots & \vdots \\ \langle y_k, x_1 \rangle & \cdots & \langle y_k, x_k \rangle \end{pmatrix}.$$

However this is not possible if we want **$U_q(\mathfrak{l})$ -invariance**.

The situation is analogous for the definition of Hermitian inner products. Summarizing we have the following constants:

$$\text{pairings} \rightsquigarrow \{\lambda_k\}_{k=1}^N, \quad \text{inner products} \rightsquigarrow \{\lambda'_k\}_{k=1}^N.$$

For the relations appearing in D^2 these numbers only appear in the combination $c_k = |\lambda_k|^2 / \lambda'_k$, since we have quadratic expressions.

What about examples? The easiest class is that of **quantum projective spaces**. Let us write $D^2 = D_D^2 + D_O^2$ for diagonal and off-diagonal parts.

Lemma

We have $D_O^2 \sim 0$ if and only if $\frac{c_{k+1}}{c_k} = \frac{c_k}{c_{k-1}} q^{-2}$.

The naive choice $c_k = 1$ is not possible, unlike the classical case.

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Theorem

Suppose furthermore that $c_0 = c_1$. Then we have $D^2 \sim C_q \otimes 1$. Here C_q is a central element of $U_q(\mathfrak{g})$.

Since the spectrum of C_q can be computed we obtain that D has compact resolvent. This reproduces the results of [D'Andrea, Dabrowski (10)].

The relations are **more complicated** than in the classical case. For example for exterior and interior multiplication we have

$$\mathbf{e}_i \mathbf{i}_i + \mathbf{i}_i \mathbf{e}_i - q(q - q^{-1}) \sum_{j=1}^{i-1} \mathbf{e}_j \mathbf{i}_j = 1.$$

The extra terms vanish in the classical case $q \rightarrow 1$. Clearly their presence can be traced back to the **braiding**, which in this case gives

$$\hat{R}(\mathbf{e}_i \otimes \mathbf{e}_j) = \mathbf{e}_j \otimes \mathbf{e}_i + (q - q^{-1}) \mathbf{e}_i \otimes \mathbf{e}_j, \quad i < j.$$

The relations for the quantum root vectors are also more complicated. But they take exactly the right form to combine with the relations above to give the simple expression $D^2 \sim \mathcal{C}_q \otimes 1$.

Does this hold for all irreducible flag manifolds?

Look for a low-dimensional counter-example. We consider $C_2 = \mathfrak{sp}(4)$ and remove the long root α_2 . In this case we have

$$\Delta(\mathfrak{l}) = \{\pm\alpha_1\}, \quad \Delta(\mathfrak{u}_+) = \{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}.$$

Hence the semisimple part of \mathfrak{l} coincides with $\mathfrak{sl}(2)$ and \mathfrak{u}_+ can be identified with the adjoint module of $\mathfrak{sl}(2)$.

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Long computations show that in this case we **do not** have

$$\gamma_-(w_i)\gamma_-(w_j)^* = - \sum_{k,\ell} b_{ij}^{k\ell} \gamma_-(w_k)^* \gamma_-(w_\ell), \quad i \neq j.$$

Hence we do not get a formula of Parthasarathy-type. The problem regarding the compact resolvent of D remains open here.

So what goes wrong? Not clear!

For projective spaces $\mathfrak{l} = \mathfrak{sl}(N)$ (semisimple part) and \mathfrak{u}_+ is the fundamental representation. In this case we have the Hecke relation

$$(\hat{R} - q)(\hat{R} + q^{-1}) = 0.$$

On the other hand for the adjoint representation there is **no quadratic relation** for \hat{R} . In particular there is more than one positive eigenvalue.

Poor understanding of "quadratic" Casimirs in general. Higher order terms in the Dolbeault-Dirac operators?