Stratified surgery and the signature operator

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Based on joint work with Pierre Albin
(and also Eric Leichtnam, Rafe Mazzeo and Thomas Schick).
I start by stating a fundamental theorem. Explanations in a moment.

**Theorem**

(N. Higson and J. Roe, 2004). Let $V$ be a smooth, closed, oriented $n$-dimensional manifold and let $\Gamma := \pi_1(V)$. We consider a portion of the surgery sequence in topology:

$$L_{n+1}(\mathbb{Z}\Gamma) \longrightarrow S(V) \longrightarrow \mathcal{N}(V) \longrightarrow L_n(\mathbb{Z}\Gamma).$$

There are natural maps $\alpha, \beta, \gamma$ and a commutative diagram

$$
\begin{array}{cccccc}
L_{n+1}(\mathbb{Z}\Gamma) & \longrightarrow & S(V) & \longrightarrow & \mathcal{N}(V) & \longrightarrow & L_n(\mathbb{Z}\Gamma) \\
\downarrow\gamma & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma \\
K_{n+1}(C^*(\tilde{V})\Gamma) & \longrightarrow & K_{n+1}(D^*(\tilde{V})\Gamma) & \longrightarrow & K_n(V) & \longrightarrow & K_n(C^*(\tilde{V}))
\end{array}
$$

The bottom sequence is the **analytic surgery sequence** associated to $V$ and $\pi_1(V)$. 
Later Piazza-Schick gave a different description of the Higson-Roe theorem, employing Atiyah-Patodi-Singer index theory and using crucially the Hilsum-Skandalis perturbation associated to a homotopy equivalence.

This more analytic treatment also gave the mapping of the Stolz surgery sequence for positive scalar curvature metrics to the same K-theory sequence.
The surgery sequence in topology

- the sequence actually extends to an infinite sequence to the left (but we only consider the displayed portion)

\[ \cdots \to L_{n+1}(\mathbb{Z}\Gamma) \to S(V) \to \mathcal{N}(V) \to L_n(\mathbb{Z}\Gamma). \]

- one of the goals of this sequence for $V$ a manifold is to understand the **structure set** $S(V)$

- $S(V)$ measures the non-rigidity of $V$ (more later)

- $L_\ast(\mathbb{Z}\Gamma)$ are groups but $S(V)$ is only a set. $\mathcal{N}(V)$ can be given the structure of a group but the map out of it is not a homomorphism. $\Rightarrow$ exactness must be suitably defined

- we now describe briefly the sequence
The structure set $S(V)$ and the normal set $N(V)$

- Elements in $S(V)$ are equivalence classes $[X \xrightarrow{f} V]$ with $X$ smooth oriented and closed and $f$ an orientation preserving homotopy equivalence.

- $(X_1 \xrightarrow{f_1} V) \sim (X_2 \xrightarrow{f_2} V)$ if they are $h$-cobordant (there is a bordism $X$ between $X_1$ and $X_2$ and a map $F: X \to V \times [0, 1]$ such that $F|_{X_1} = f_1$ and $F|_{X_2} = f_2$ and $F$ is a homotopy equivalence).

- $S(V)$ is a pointed set with $[V \xrightarrow{Id} V]$ as a base point

- $V$ is rigid if $S(V) = \{[V \xrightarrow{Id} V]\}$

- $N(V)$ is the set of degree one normal maps $f: M \to V$ considered up to normal bordism (we shall forget about the adjective ”normal” in this talk)

- there is a natural map $S(V) \to N(V)$
The L-groups. Exactness

- The $L$-groups $L_\ast (\mathbb{Z} \Gamma )$ are defined algebraically as equivalence classes of quadratic forms with coefficients in $\mathbb{Z} \Gamma$
- A fundamental theorem of Wall tells us that $L_\ast (\mathbb{Z} \Gamma )$ is isomorphic to a bordism group $L_1^1 (B \Gamma )$ of manifolds with b.
- In fact, one can choose yet a more specific realization with "special cycles" ($L_2^2 (B \Gamma )$); a special cycle is $(W, \partial W)$ with a degree one normal map $F : W \to V \times [0, 1]$ such that $F|_{\partial W} : \partial W \to \partial (V \times [0, 1])$ is a homotopy equivalence + $r : V \to B \Gamma$
- Through this special realization $L_{n+1} (\mathbb{Z} \Gamma )$ acts on $S(V)$ and exactness at $S(V)$ means the following: $[X \xrightarrow{f} V]$ and $[Y \xrightarrow{g} V]$ are mapped to the same element in $\mathcal{N}(V)$ if and only if they belong to the same $L_{n+1} (\mathbb{Z} \Gamma )$-orbit.
- The map $\mathcal{N}(V) \to L_n (\mathbb{Z} \Gamma )$ is called the surgery obstruction.
- Exactness at $\mathcal{N}(V)$ means that $[X \xrightarrow{f} V] \in \mathcal{N}(V)$ is mapped to 0 in $L_n (\mathbb{Z} \Gamma )$ if and only if it is the image of an element in $S(V)$ (i.e. can be surgered to an homotopy equivalence).
The Browder-Quinn surgery sequence for a smoothly stratified space

- Let now $V$ be a smoothly stratified pseudomanifold.
- we bear in mind the Wall’s realization of the $L$-groups
- we give ”essentially” the same definitions but we require the maps to be *stratified* and *transverse* (will come back to definitions)
- we obtain the Browder-Quinn surgery sequence

$$
\cdots \to L_{n+1}^{BQ}(V) \to S^{BQ}(V) \to N^{BQ}(V) \to L_n^{BQ}(V)
$$

There are differences: for example $L_*^{BQ}(V)$ depends now on the fundamental groups of all closed strata.  

**Warning:** in the paper of Browder and Quinn there are precise statements but no proofs; a few key definitions are also missing.  

Part of our work was to give a rigorous account.
Our program now:

- explain the Higson-Roe theorem (following Piazza-Schick)
- say why this is an interesting and useful theorem
- pass to stratified spaces and explain problems
- explain how to use analysis on stratified pseudomanifolds in order to achieve the same goal for the Browder-Quinn surgery sequence

\[
L_{n+1}^{BQ}(V) \rightarrow S^{BQ}(V) \rightarrow N^{BQ}(V) \rightarrow L_n^{BQ}(V)
\]

assuming \( V \) to be a Witt space or more generally a Cheeger space.
Higson-Roe analytic surgery sequence

- **change of notation:** $M$ is a riemannian manifold with a free and cocompact isometric action of $\Gamma$. We write $M/\Gamma$ for the quotient.

  Thus, with respect to the previous slides,
  \[ V = M/\Gamma \text{ and } \tilde{V} = M. \]

- we also have a $\Gamma$-equivariant complex vector bundle $E$

- $D_c^*(M)^\Gamma \subset B(L^2(M, E))$ is the algebra of $\Gamma$-equivariant bounded operators on $L^2(M, E)$ that are of finite propagation and pseudolocal

- $D^*(M)^\Gamma$ is the norm closure of $D_c^*(M)^\Gamma$

- $C_c^*(M)^\Gamma \subset B(L^2(M, E))$ is the algebra of $\Gamma$-equivariant bounded operators on $L^2(M, E)$ that are of finite propagation and locally compact

- $C^*(M)^\Gamma$ is the norm-closure of $C_c^*(M)$

- $C^*(M)^\Gamma$ is an ideal in $D^*(M)^\Gamma$
we can consider the short exact sequence (of Higson-Roe);

\[ 0 \to C^*(M)^\Gamma \to D^*(M)^\Gamma \to D^*(M)^\Gamma / C^*(M)^\Gamma \to 0 \]

and thus

\[ \cdots \to K_*(D^*(M)^\Gamma) \to K_*(D^*(M)^\Gamma / C^*(M)^\Gamma) \xrightarrow{\delta} K_{*+1}(C^*(M)^\Gamma) \to \cdots \]

Paschke duality: \( K_*(D^*(M)^\Gamma / C^*(M)^\Gamma) \cong K_{*+1}(M/\Gamma) \)

one can also prove that \( K_*(C^*(M)^\Gamma) \cong K_*(C_r\Gamma) \)

these groups behave functorially (covariantly).

If \( \tilde{u}: M \to E\Gamma \) is a \( \Gamma \)-equiv. classifying map then we can use \( \tilde{u}_* \) to map the Higson-Roe sequence to the universal Higson-Roe sequence:

\[ \cdots \to K_*(C_r^\Gamma) \to K_*(D_f^\Gamma) \to K_{*+1}(B\Gamma) \xrightarrow{\delta} K_{*+1}(C_r^\Gamma) \to \cdots \]

where \( D_f^\Gamma := D^*(E\Gamma)^\Gamma \) (for simplicity \( B\Gamma \) is a finite complex here).

It turns out that \( \delta \) is the assembly map.
Index and rho-classes

We assume that we now have a $\Gamma$-equivariant Dirac operator $D$. Let $n$ be the dimension of $M$. We can define:

- the fundamental class
  
  $$[D] \in K_n(M/\Gamma) = K_{n+1}(D^*(M)\Gamma/C^*(M)\Gamma)$$

- the index class $\text{Ind}(D) := \delta[D] \in K_n(C^*(M)\Gamma)$

- If $D$ is $L^2$-invertible we can use the same definition of $[D]$ but get the rho classes $\rho(D)$ in $K_{n+1}(D^*(M)\Gamma)$ (no need to go to the quotient)

- For example if $n$ is odd then

  $$\rho(D) = \left[ \frac{1}{2} (1 + \frac{D}{|D|}) \right] = [\Pi_{\geq}(D)] \in K_0(D^*(M)\Gamma)$$
If we only know that $\text{Ind}(D) = 0 \in K_n(C^*(M)^\Gamma)$ then $\exists$ a perturbation $C \in C^*(M)^\Gamma$ such that $D + C$ is $L^2$-invertible.

can define $\rho(D + C) \in K_{n+1}(D^*(M)^\Gamma)$ as before; e.g. if $n$ is odd $\rho(D + C) := [\prod_{\geq}(D + C)] \in K_0(D^*(M)^\Gamma)$.

notice that $\rho(D + C)$ does depend on $C$.

Atiyah-Patodi-Singer index theory: if $W$ is an oriented manifold with free cocompact action and with boundary $\partial W = M$ then

by bordism invariance we know that $D_\partial$ has zero index

$\exists$ $C_\partial \in C^*(\partial W)^\Gamma$ such that $D_\partial + C_\partial$ is $L^2$-invertible

one can prove that there exists an index class

$$\text{Ind}(D, C_\partial) \in K_*(C^*(W)^\Gamma)$$
Mapping surgery to analysis

We can now explain the maps $\text{Ind}, \rho, \beta$ in the following diagram

\[
\begin{array}{cccccc}
L_{n+1}(\mathbb{Z} \Gamma) & \longrightarrow & S(V) & \longrightarrow & \mathcal{N}(V) & \longrightarrow & L_n(\mathbb{Z} \Gamma) \\
\downarrow \text{Ind} & & \downarrow \rho & & \downarrow \beta & & \downarrow \text{Ind} \\
K_{n+1}(C^*(\tilde{V})^\Gamma) & \longrightarrow & K_{n+1}(D^*(\tilde{V})^\Gamma) & \longrightarrow & K_n(V) & \longrightarrow & K_n(C^*(\tilde{V}))
\end{array}
\]

- $\text{Ind}[F : W \to V \times [0, 1], r : V \to B \Gamma]$: use the Hilsum-Skandalis perturbation of $F|_{\partial W}$ and take a suitable APS-index class for the signature operator. Well-definedness due to Charlotte Wahl.

- $\rho[X \xrightarrow{f} V]$: use the Hilsum-Skandalis perturbation of $f$ and take the corresponding rho class for the signature operator

- $\beta[U \xrightarrow{f} V] := f_*[\bar{\partial}^U_{\text{sign}}] - [\bar{\partial}^V_{\text{sign}}]$

Well-definedness of $\rho$ and commutativity of diagram is all in the next Theorem.
Theorem

(P-Schick) Let $C_\partial$ be a trivializing perturbation for $D_\partial$. For the index class $\text{Ind}(D, C_\partial) \in K_*(C^*(W)^\Gamma)$ the following holds:

$$\iota_*(\text{Ind}(D, C_\partial)) = j_*(\rho(D_\partial + C_\partial)) \quad \text{in} \quad K_0(D^*(W)^\Gamma).$$

Here $j: D^*(\partial W)^\Gamma \to D^*(W)^\Gamma$ is induced by the inclusion $\partial W \hookrightarrow W$ and $\iota: C^*(W)^\Gamma \to D^*(W)^\Gamma$ the natural inclusion.

Further contributions:

- P-Schick for Stolz
- Xie-Yu for Stolz using localization algebras
- Zenobi (Higson-Roe à la P-Schick for $V$ a topological manifold)
- Zenobi (Higson-Roe via groupoids)
- Weinberger-Xie-Yu (Higson-Roe for $V$ a topological manifold)

Stratified pseudomanifolds

Above is an example of depth 1; below is an example of depth 2:
Basics

Let us concentrate on the depth one case.

So there is a decomposition of $\hat{X}$ into two strata: $\hat{X} = Y \cup X$.

$Y$ is the singular set (the bottom blue circle) and $X$ is the regular part (the union of the red cones (without the vertices)).

The link of a point $p \in Y$ is a smooth closed manifold $Z$ (the green circle).

A neighborhood of $p \in Y$ looks like $B \times C(Z)$, with $B$ a ball in $\mathbb{R}^{\dim Y}$. In fact, a tubular neighborhood $T$ of $Y$ is a bundle of cones $C(Z) \to T \xrightarrow{\pi} Y$, as in the figure.
Examples

- singular projective algebraic varieties
- quotients of non-free actions
- compactifications of locally symmetric spaces
- moduli spaces
Questions and problems:

- can we run the machine in the singular case?
- problem 1: for stratified spaces Poincaré duality does not hold
- consequently, we do not have a signature
- we do analysis on the regular part $X$ of $\hat{X}$; we need to fix a metric $g$ on $X$
- natural metrics are typically incomplete
- problem 2: the signature operator on $\Omega^*_c(X)$ has many extensions
  (so, even granting the Fredholm property, which one will be "connected to topology" ?!)
Witt spaces

We now restrict the class of pseudomanifolds. We consider Witt spaces:

Definition
\( \hat{X} \) is a Witt space if any even-dimensional link \( L \) has
\[
\text{IH}_{m}^{\dim L/2}(L; \mathbb{Q}) = 0.
\]

If \( \hat{X} \) is Witt then.....everything works !
Cheeger spaces

We want to drop the Witt assumption and treat more general stratified spaces. We shall treat Cheeger spaces.

\[ \{\text{Witt spaces}\} \subset \{\text{Cheeger spaces}\} \subset \{\text{Stratified spaces}\} \]

References:

Iterated conic metrics.

Let us concentrate on the depth one case.

Recall that a neighborhood of \( p \) in the singular set \( Y \) (the blue circle) looks like \( B \times C(Z) \), with \( B \) a ball in \( \mathbb{R}^{\dim Y} \).

In fact a tubular neighborhood \( T \) of \( Y \) is a bundle of cones as in figure:

\[
C(Z) \to T \overset{\pi}{\longrightarrow} Y
\]

If \( x \) is the variable along the cone then \( x = 1 \) defines a fibration

\[
Z \to H \to Y
\]

A conic metric on \( X \) is, by definition, an incomplete metric of the form \( g := dx^2 + x^2 g_Z + \pi^* g_Y \).
Closed extensions

- we want to use Hilbert-space techniques
- we want closed operators
- if $\hat{X}$ is Witt, then $d_{\text{min}} = d_{\text{max}}$ and $\tilde{\partial}_{\text{sign}} : \Omega_c^+ \oplus \Omega_c^- \to \Omega_c^+ \oplus \Omega_c^- \to$ is essentially self-adjoint
- in the non-Witt case $d : \Omega_c^k \to \Omega_c^{k+1}$ has various closed extensions (between $d_{\text{min}}$ and $d_{\text{max}}$)
- similarly $\tilde{\partial}_{\text{sign}}$ is NOT essentially self-adjoint
Resolution

- we resolve the pseudomanifold $\hat{X}$ to a manifold with corners $\tilde{X}$ (Verona + Brasselet-Hector-Saralegi + ALMP). $\tilde{X}$ has an additional structure: it has an an iterated fibration structure on the boundary (boundary hypersurfaces are fibrations + compatibility relations at the corners between these fibrations).

Example: if $\hat{X}$ is a depth-one space

then $\tilde{X}$ is a manifold with boundary and the boundary is our fibration $H \to Y$ (thus with base equal to the singular stratum (the bottom circle) and fiber the links (the green circles)).
Expansions

We first consider $\partial_{dR} := d + d^*$. Recall that a tubular neighborhood $T$ of the singular set $Y$ looks like

$$C(Z) \to T \to Y$$

Consider the resolved manifold $\tilde{X}$; a manifold with boundary with boundary equal to the fibration $Z \to H \to Y$.

If $Z$ is even-dimensional and has cohomology in middle degree then we are NOT in the Witt case.

**Fundamental Lemma** Any $u \in D_{\text{max}}(\partial_{dR})$ has an asymptotic expansion at $Y$,

$$u \sim x^{1/2}(\alpha_1(u) + dx \wedge \beta_1(u)) + \tilde{u}$$

with the terms in this expansion distributional:

$$\alpha_1(u), \beta_1(u) \in H^{-1/2}(Y; \bigwedge^* T^* Y \otimes \mathcal{H}^{f/2}(H/Y)), \quad \tilde{u} \in xH^{-1}(X, \bigwedge^* X)$$

Here $\mathcal{H}^{f/2}(H/Y)$ is the flat Hodge bundle over $Y$ (with typical fiber $\mathcal{H}^{f/2}(Z_y)$) and $f = \dim Z$. 
Cheeger boundary condition

The distributional differential forms $\alpha(u), \beta(u)$ serve as ‘Cauchy data’ at $Y$ which we use to define Cheeger ideal boundary conditions. Here is what we do: for any subbundle

$$W \rightarrow \mathcal{H}^{f/2}(H/Y) \rightarrow Y$$

that is parallel with respect to the flat connection, we define

$$D_W(\tilde{\partial}_{dR}) = \{ u \in D_{\max}(\tilde{\partial}_{dR}) : \alpha_1(u) \in H^{-1/2}(Y; \Lambda^*T^*Y \otimes W), \beta_1(u) \in H^{-1/2}(Y; \Lambda^*T^*Y \otimes (W)^\perp) \}.$$ 

We call $W$ a (Hodge) mezzoperversity adapted to $g$. 
Every mezzoperversity induces a closed self-adjoint domain $\mathcal{D}_\mathcal{W}(\bar{\partial}_{dR})$;

$(\bar{\partial}_{dR}, \mathcal{D}_\mathcal{W}(\bar{\partial}_{dR}))$ is Fredholm with discrete spectrum;

We can define a domain for the exterior derivative as an unbounded operator on $L^2$ differential forms: $\mathcal{D}_\mathcal{W}(d)$;

the corresponding de Rham cohomology groups, $H^*_\mathcal{W}(\tilde{X})$, are finite dimensional and metric independent;

there is a Hodge decomposition theorem.
ANALYTIC RESULTS. Part 2

▶ given a mezzoperversity $\mathcal{W}$ there is a dual mezzoperversity $\mathcal{D}\mathcal{W}$ defined in terms of the vertical Hodge-$*$

▶ there is a natural non-degenerate pairing

$$H^\ell_\mathcal{W}(\hat{X}) \times H^{n-\ell}_{\mathcal{D}\mathcal{W}}(\hat{X}) \to \mathbb{R}$$

▶ if $\mathcal{W} = \mathcal{D}\mathcal{W}$ then we say that $\mathcal{W}$ is self-dual (might not $\exists$)

▶ $\hat{X}$ admitting a self-dual mezzoperversity is a **Cheeger space**;

▶ on a Cheeger space we have a non-degenerate pairing

$$H^\ell_\mathcal{W}(\hat{X}) \times H^{n-\ell}_\mathcal{W}(\hat{X}) \to \mathbb{R}$$

and thus a signature $\sigma_\mathcal{W}(\hat{X})$;

▶ a self-dual mezzoperversity defines a Fredholm signature operator $(\partial_{\text{sign}}, \mathcal{D}\mathcal{W}(\partial_{\text{sign}}))$;

▶ the index is equal to the signature :

$$\sigma_\mathcal{W}(\hat{X}) = \text{ind}(\partial_{\text{sign}}, \mathcal{D}\mathcal{W}(\partial_{\text{sign}})) \equiv \text{ind}(\partial_{\text{sign}}, \mathcal{W})$$

▶ there is a well defined $K$-homology class $[\partial_{\text{sign}}, \mathcal{W}]$ in $K_*(\hat{X})$

▶ if $\pi_1(\hat{X}) = \Gamma$ and $\hat{X}_\Gamma$ is the universal cover of $\hat{X}$ then we also have a higher index class

$$\text{Ind}(\partial_{\text{sign}}^\Gamma, \mathcal{W}) \in K_*(C^*(\hat{X}_\Gamma)^\Gamma)$$
Summary + Crucial Questions

Given a Cheeger space $\hat{X}$ with a fixed self-dual mezzoperversity $\mathcal{V}$ and a classifying map $r$ we have defined

- $H^\mathcal{V}_*(\hat{X})$
- $\sigma^\mathcal{V}(\hat{X}) \in \mathbb{Z}$
- $[\partial_{\text{sign}}^\mathcal{V}]$ in $K_*(\hat{X})$
- $\text{Ind}(\partial_{\text{sign}}^\mathcal{V}) \in K_*(C^*(\hat{X}_\Gamma)^\Gamma) = K_*(C^*_{\Gamma})$

**Question 1:** what happens to these invariants if $F : \hat{X} \to \hat{M}$ is a stratified homotopy equivalence ??

**Question 2:** is there a Hilsum-Skandalis perturbation ??

**Question 3:** can we define the rho class of a **stratified** homotopy equivalence ??

**Question 4:** how does all this depend on the choice of $\mathcal{V}$ ??
Stratified maps

Let $F : \hat{X} \to \hat{M}$ be a smoothly stratified map between depth-1 stratified spaces. We denote by $Y_X$ and $Y_M$ the singular strata and by $T_X$ and $T_Y$ the corresponding tubular neighbourhoods. Then

$$F|_{Y_X} : Y_X \to Y_M \quad F|_{T_X} : T_X \to T_Y$$

moreover $F|_{T_X}$ is a bundle map.

$F$ is transverse if $T_X$ is the pull-back of $T_Y$ and $F|_{T_X}$ is a pull-back map.

Pull-back of forms is not $L^2$-bounded but we can consider the Hilsum-Skandalis replacement for the pull-back map. We work on the resolved manifold.

Proposition

Let $\mathcal{W}$ be a mezzoperversity for $\hat{M}$. Then we can define the pull-back mezzoperversity $F^\#(\mathcal{W})$. If $\mathcal{W}$ is self-dual, so is $F^\#(\mathcal{W})$. 
Theorem

If \( F : \hat{M}' \to \hat{M} \) is a stratified homotopy equivalence and \( \mathcal{W} \) is a mezzoperversity for \( \hat{M} \) then

\[
H^*_{\mathcal{W}}(\hat{M}) \cong H^*_{F^\#\mathcal{W}}(\hat{M}') .
\]

If \( \mathcal{W} \) is self-dual

\[
\sigma_{\mathcal{W}}(\hat{M}) = \sigma_{F^\#\mathcal{W}}(\hat{M}')
\]

\[
\text{Ind}(\bar{\partial}_{\text{sign},\mathcal{W}}) = \text{Ind}(\bar{\partial}_{\text{sign},F^\#\mathcal{W}}) \in K_*(C^*_\Gamma)
\]

proved via a Hilsum-Skandalis perturbation.
Bordism invariance

Theorem
Both \( \sigma_W(\hat{M}) \) and \( \text{Ind}(\partial_{\text{sign}}^\Gamma, \mathcal{W}) \) are Cheeger-bordism-invariant: if \((\hat{M}, \mathcal{W})\) is bordant to \((\hat{M}', \mathcal{W}')\) through \((\hat{Z}, \mathcal{W}_{\hat{Z}})\) then the signature and the index class are the same.

Theorem
(from an idea of Markus Banagl)
Let \( \mathcal{W} \) and \( \mathcal{W}' \) be two mezzoperversity for \( \hat{M} \). Then \((\hat{M}, \mathcal{W})\) is Cheeger-bordant to \((\hat{M}, \mathcal{W}')\)
Independence on $\mathcal{W}$. The L-class

**Corollary**

$\sigma_{\mathcal{W}}(\hat{M})$ and $\text{Ind}(\bar{\partial}_{\text{sign},\mathcal{W}}^{G(r)})$ are stratified homotopy invariant and independent of $\mathcal{W}$ !!

**Consequently:** for a Cheeger space $\hat{M}$ we have a signature and a homology L-class $L_*(\hat{M}) \in H_*(\hat{M}, \mathbb{Q})$ defined à la Thom. First defined by Banagl using topology. By our Hodge theorem we prove they are equal. Having $L_*(\hat{M})$ we can define the higher signatures on a Cheeger space

$$\{\langle \alpha, r_*(L_*(\hat{M})) \rangle, \alpha \in H^*(B\Gamma, \mathbb{Q})\}$$

and formulate the Novikov Conjecture (stratified homotopy invariance)

**Theorem**

*(Albin-Leichtnam-Mazzeo-P.)*

If SNC holds for $\Gamma$ then the Novikov conjecture holds for a Cheeger space $\hat{M}$ with $\pi_1(\hat{M}) = \Gamma$. In particular it holds for a Witt space.
we have seen that on a Cheeger space $\hat{X}$ with mezzoperversity $\mathcal{W}$ there exists a K-homology class $[\partial_{\text{sign}}, \mathcal{W}] \in K_\ast(\hat{X})$

if $f : \hat{M} \to \hat{X}$ is a stratified homotopy equivalence then there is an associated Hilsum-Skandalis perturbation for the signature operator on $M \sqcup X$ with mezzoperversity $f^\# \mathcal{W} \sqcup \mathcal{W}$

hence (modulo showing that the operators are in the right algebras) there is a well defined rho class

$\rho(f, \mathcal{W}) \in K_\ast(D^\ast(\hat{X}_\Gamma)^\Gamma)$

Finally let $B$ be a Cheeger space with boundary and $B \xrightarrow{F} \hat{X} \times [0, 1]$ a degree one transverse map. The mezzoperversity $\mathcal{W}$ on $\hat{X}$, extends trivially to $\hat{X} \times [0, 1]$ (call it again $\mathcal{W}$) and pulling it back on $B$ via $F^\#$ we obtain a mezzoperversity $F^\# \mathcal{W} \sqcup \mathcal{W}$ on $B \cup (\hat{X} \times [0, 1])$. If $F_\partial$ is a stratified homotopy equivalence then there is a well defined APS-index class

$\text{Ind}_{\text{APS}, \mathcal{W}}(B \xrightarrow{F} \hat{X} \times [0, 1]) \in K_\ast(C^\ast(\hat{X}_\Gamma)^\Gamma)$
Let $\hat{X}$ be a Cheeger space. Choose a mezzoperversity $\mathcal{W}$.
Consider the Browder-Quinn surgery sequence
\[ L_{n+1}^{\text{BQ}}(\hat{X}) \to S^{\text{BQ}}(\hat{X}) \to N^{\text{BQ}}(\hat{X}) \to L_n^{\text{BQ}}(\hat{X}) \]
Define $\text{Ind} : L_*(\hat{X})$ on a refined cycle via $\text{Ind}_{\text{APS},\mathcal{W}}$
Define $\rho : S^{\text{BQ}}(\hat{X}) \to K_*(D^*(\hat{X}_\Gamma)^\Gamma)$ as $\rho[\hat{Y} \to \hat{X}] = \rho(f, \mathcal{W})$.
Define $\beta : N^{\text{BQ}}(\hat{X}) \to K_*(\hat{X})$ as
$\beta[\hat{Y} \to \hat{X}] = f_*[\partial_{\text{sign}, f^\#\mathcal{W}}] - [\partial_{\text{sign}, \mathcal{W}}]$. 

**Theorem**

These maps are well defined and they are independent of $\mathcal{W}$. Moreover, the following diagram is commutative.
THANK YOU