Stratified surgery and the signature operator

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Based on joint work with Pierre Albin
(and also Eric Leichtnam, Rafe Mazzeo and Thomas Schick).

Mapping surgery to analysis (Higson-Roe)

I start by stating a fundamental theorem. Explanations in a moment.

Theorem

(N. Higson and J. Roe, 2004). Let V be a smooth, closed, oriented n-dimensional manifold and let $\Gamma := \pi_1(V)$. We consider a portion of the surgery sequence in topology:

$$L_{n+1}(\mathbb{Z}\Gamma) \dashrightarrow \mathcal{S}(V) \to \mathcal{N}(V) \to L_n(\mathbb{Z}\Gamma)\,.$$

There are natural maps α, β, γ and a commutative diagram

$$\begin{tikzcd}
L_{n+1}(\mathbb{Z}\Gamma) \arrow[r, dashed] \arrow[d, "\gamma"] & \mathcal{S}(V) \arrow[r] \arrow[d, "\alpha"] & \mathcal{N}(V) \arrow[r] \arrow[d, "\beta"] & L_n(\mathbb{Z}\Gamma) \arrow[d, "\gamma"] \\
K_{n+1}(C^*(\widetilde{V})^\Gamma) \arrow[r] & K_{n+1}(D^*(\widetilde{V})^\Gamma) \arrow[r] & K_n(V) \arrow[r] & K_n(C^*(\widetilde{V}))
\end{tikzcd}$$

The bottom sequence is the analytic surgery sequence associated to V and $\pi_1(V)$.

- Later Piazza-Schick gave a different description of the Higson-Roe theorem, employing Atiyah-Patodi-Singer index theory and using crucially the Hilsum-Skandalis perturbation associated to a homotopy equivalence.
- this more analytic treatment also gave the mapping of the Stolz surgery sequence for positive scalar curvature metrics to the same K-theory sequence.

The surgery sequence in topology

- the sequence actually extends to an infinite sequence to the left (but we only consider the displayed portion)

$$\cdots \to L_{n+1}(\mathbb{Z}\Gamma) \dashrightarrow \mathcal{S}(V) \to \mathcal{N}(V) \to L_n(\mathbb{Z}\Gamma)\,.$$

- one of the goals of this sequence for V a manifold is to understand the **structure set** $\mathcal{S}(V)$
- $\mathcal{S}(V)$ measures the non-rigidity of V (more later)
- $L_*(\mathbb{Z}\Gamma)$ are groups but $\mathcal{S}(V)$ is only a set. $\mathcal{N}(V)$ can be given the structure of a group but the map out of it is not a homomorphism.$\Rightarrow$ exactness must be suitably defined
- we now describe briefly the sequence

The structure set $\mathcal{S}(V)$ and the normal set $\mathcal{N}(V)$

- Elements in $\mathcal{S}(V)$ are equivalence classes $[X \xrightarrow{f} V]$ with X smooth oriented and closed and f an orientation preserving homotopy equivalence.
- $(X_1 \xrightarrow{f_1} V) \sim (X_2 \xrightarrow{f_2} V)$ if they are h-cobordant (there is a bordism X between X_1 and X_2 and a map $F\colon X \to V \times [0,1]$ such that $F|_{X_1} = f_1$ and $F|_{X_2} = f_2$ and F is a homotopy equivalence).
- $\mathcal{S}(V)$ is a pointed set with $[V \xrightarrow{\mathrm{Id}} V]$ as a base point
- V is rigid if $S(V) = \{[V \xrightarrow{\mathrm{Id}} V]\}$
- $\mathcal{N}(V)$ is the set of degree one normal maps $f : M \to V$ considered up to normal bordism (we shall forget about the adjective "normal" in this talk)
- there is a natural map $\mathcal{S}(V) \to \mathcal{N}(V)$

The L-groups. Exactness

- the L-groups $L_*(\mathbb{Z}\Gamma)$ are defined algebraically as equivalence classes of quadratic forms with coefficients in $\mathbb{Z}\Gamma$
- a fundamental theorem of Wall tells us that $L_*(\mathbb{Z}\Gamma)$ is isomorphic to a bordism group $L_*^1(B\Gamma)$ of manifolds with b.
- In fact, one can choose yet a more specific realization with "special cycles" $(L_*^2(B\Gamma))$; a special cycle is $(W, \partial W)$ with a degree one normal map $F : W \to V \times [0,1]$ such that $F|_{\partial W} : \partial W \to \partial(V \times [0,1])$ is a homotopy equivalence + $r : V \to B\Gamma$
- through this special realization $L_{n+1}(\mathbb{Z}\Gamma)$ acts on $\mathcal{S}(V)$ and exactness at $\mathcal{S}(V)$ means the following: $[X \xrightarrow{f} V]$ and $[Y \xrightarrow{g} V]$ are mapped to the same element in $\mathcal{N}(V)$ if and only if they belong to the same $L_{n+1}(\mathbb{Z}\Gamma)$-orbit.
- the map $\mathcal{N}(V) \to L_n(\mathbb{Z}\Gamma)$ is called the **surgery obstruction**
- exactness at $\mathcal{N}(V)$ means that $[X \xrightarrow{f} V] \in \mathcal{N}(V)$ is mapped to 0 in $L_n(\mathbb{Z}\Gamma)$ if and only if it is the image of an element in $\mathcal{S}(V)$ (i.e. can be surgered to an homotopy equivalence).

The Browder-Quinn surgery sequence for a smoothly stratified space

- Let now V be a smoothly stratified pseudomanifold.
- we bear in mind the Wall's realization of the L-groups
- we give " essentially " the same definitions but we require the maps to be *stratified* and *transverse* (will come back to definitions)
- we obtain the Browder-Quinn surgery sequence

$$\cdots \to L^{\mathrm{BQ}}_{n+1}(V) \dashrightarrow \mathcal{S}^{\mathrm{BQ}}(V) \to \mathcal{N}^{\mathrm{BQ}}(V) \to L^{\mathrm{BQ}}_{n}(V)$$

There are differences: for example $L^{\mathrm{BQ}}_{*}(V)$ depends now on the fundamental groups of all closed strata.
Warning: in the paper of Browder and Quinn there are precise statements but no proofs; a few key definitions are also missing.
Part of our work was to give a rigorous account.

Our program now:

- explain the Higson-Roe theorem (following Piazza-Schick)
- say why this is an interesting and useful theorem
- pass to stratified spaces and explain problems
- explain how to use analysis on stratified pseudomanifolds in order to achieve the same goal for the Browder-Quinn surgery sequence

$$L_{n+1}^{\mathrm{BQ}}(V) \dashrightarrow \mathcal{S}^{\mathrm{BQ}}(V) \to \mathcal{N}^{\mathrm{BQ}}(V) \to L_n^{\mathrm{BQ}}(V)$$

assuming V to be a Witt space or more generally a Cheeger space.

Higson-Roe analytic surgery sequence

- change of notation: M is a riemannian manifold with a free and cocompact isometric action of Γ. We write M/Γ for the quotient.
  Thus, with respect to the previous slides,
  $V = M/\Gamma$ and $\widetilde{V} = M$.
- we also have a Γ-equivariant complex vector bundle E
- $D_c^*(M)^\Gamma \subset \mathcal{B}(L^2(M,E))$ is the algebra of Γ-equivariant bounded operators on $L^2(M,E)$ that are of finite propagation and pseudolocal
- $D^*(M)^\Gamma$ is the norm closure of $D_c^*(M)^\Gamma$
- $C_c^*(M)^\Gamma \subset \mathcal{B}(L^2(M,E))$ is the algebra of Γ-equivariant bounded operators on $L^2(M,E)$ that are of finite propagation and locally compact
- $C^*(M)^\Gamma$ is the norm-closure of $C_c^*(M)$
- $C^*(M)^\Gamma$ is an ideal in $D^*(M)^\Gamma$

- we can consider the short exact sequence (of Higson-Roe);

$$0 \to C^*(M)^\Gamma \to D^*(M)^\Gamma \to D^*(M)^\Gamma / C^*(M)^\Gamma \to 0$$

- and thus

$$\cdots \to K_*(D^*(M)^\Gamma) \to K_*(D^*(M)^\Gamma / C^*(M)^\Gamma) \xrightarrow{\delta} K_{*+1}(C^*(M)^\Gamma) \to \cdot$$

- Paschke duality: $K_*(D^*(M)^\Gamma / C^*(M)^\Gamma) \simeq K_{*+1}(M/\Gamma)$
- one can also prove that $K_*(C^*(M)^\Gamma) \simeq K_*(C_r^*\Gamma)$
- these groups behave functorially (covariantly).

If $\widetilde{u} : M \to E\Gamma$ is a Γ-equiv. classifying map then we can use $\widetilde{u}_*$ to map the Higson-Roe sequence to the universal Higson-Roe sequence:

$$\cdots \to K_*(C_r^*\Gamma) \to K_*(D_\Gamma^*) \to K_{*+1}(B\Gamma) \xrightarrow{\delta} K_{*+1}(C_r^*\Gamma) \to \cdots$$

where $D_\Gamma^* := D^*(E\Gamma)^\Gamma$ (for simplicity $B\Gamma$ is a finite complex here). It turns out that δ is the assembly map.

Index and rho-classes

We assume that we now have a Γ-equivariant Dirac operator D. Let n be the dimension of M. We can define:

- the fundamental class $[D] \in K_n(M/\Gamma) = K_{n+1}(D^*(M)^\Gamma / C^*(M)^\Gamma)$
- the index class $\mathrm{Ind}(D) := \delta[D] \in K_n(C^*(M)^\Gamma)$
- If D is L^2-invertible we can use the same definition of $[D]$ but get the rho classes $\rho(D)$ in $K_{n+1}(D^*(M)^\Gamma)$ (no need to go to the quotient)
- For example if n is odd then

$$\rho(D) = [\frac{1}{2}(1 + \frac{D}{|D|})] = [\Pi_{\geq}(D)] \in K_0(D^*(M)^\Gamma)$$

- If we only know that $\mathrm{Ind}(D) = 0 \in K_n(C^*(M)^\Gamma)$ then $\exists$ a perturbation $C \in C^*(M)^\Gamma$ such that $D + C$ is L^2-invertible.
- can define $\rho(D + C) \in K_{n+1}(D^*(M)^\Gamma)$ as before; e.g. if n is odd $\rho(D + C) := [\Pi_{\geq}(D + C)] \in K_0(D^*(M)^\Gamma)$.
- notice that $\rho(D + C)$ does depend on C.

Atiyah-Patodi-Singer index theory: if W is an oriented manifold with free cocompact action and with boundary $\partial W = M$ then

- by bordism invariance we know that D_∂ has zero index
- $\exists\ C_\partial \in C^*(\partial W)^\Gamma$ such that $D_\partial + C_\partial$ is L^2-invertible
- one can prove that there exists an index class

$$\mathrm{Ind}(D, C_\partial) \in K_*(C^*(W)^\Gamma)$$

Mapping surgery to analysis

We can now explain the maps Ind, ρ, β in the following diagram

$$\begin{tikzcd}
L_{n+1}(\mathbb{Z}\Gamma) \arrow[r, dashed] \arrow[d, "\mathrm{Ind}"] & \mathcal{S}(V) \arrow[r] \arrow[d, "\rho"] & \mathcal{N}(V) \arrow[r] \arrow[d, "\beta"] & L_n(\mathbb{Z}\Gamma) \arrow[d, "\mathrm{Ind}"] \\
K_{n+1}(C^*(\widetilde{V})^\Gamma) \arrow[r] & K_{n+1}(D^*(\widetilde{V})^\Gamma) \arrow[r] & K_n(V) \arrow[r] & K_n(C^*(\widetilde{V}))
\end{tikzcd}$$

- $\mathrm{Ind}[F : W \to V \times [0,1], r : V \to B\Gamma]$: use the Hilsum-Skandalis perturbation of $F|_{\partial W}$ and take a suitable APS-index class for the signature operator. Well-definedness due to Charlotte Wahl.
- $\rho[X \xrightarrow{f} V]$: use the Hilsum-Skandalis perturbation of f and take the corresponding rho class for the signature operator
- $\beta[U \xrightarrow{f} V] := f_*[\eth^U_{sign}] - [\eth^V_{\mathrm{sign}}]$

Well-definedness of ρ and commutativity of diagram is all in the next Theorem.

Theorem

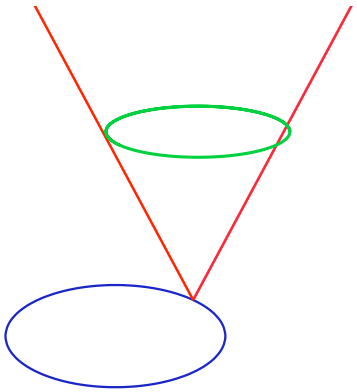
(P-Schick) Let C_∂ be a trivializing perturbation for D_∂. For the index class $\mathrm{Ind}(D, C_\partial) \in K_*(C^*(W)^\Gamma)$ *the following holds:*

$$\iota_*(\mathrm{Ind}(D, C_\partial)) = j_*(\rho(D_\partial + C_\partial)) \quad \textit{in} \quad K_0(D^*(W)^\Gamma).$$

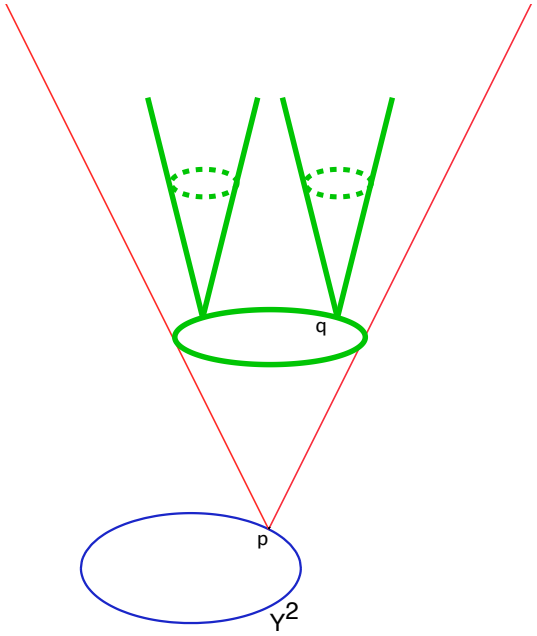
Here $j\colon D^*(\partial W)^\Gamma \to D^*(W)^\Gamma$ is induced by the inclusion $\partial W \hookrightarrow W$ and $\iota\colon C^*(W)^\Gamma \to D^*(W)^\Gamma$ the natural inclusion.

- Further contributions:
  - P-Schick for Stolz
  - Xie-Yu for Stolz using localization algebras
  - Zenobi (Higson-Roe à la P-Schick for V a topological manifold)
  - Zenobi (Higson-Roe via groupoids)
  - Weinberger-Xie-Yu (Higson-Roe for V a topological manifold)
- Many beautiful applications (Chang-Weinberger, P-Schick, Weinberger-Yu, Xie-Yu, Zeidler, Zenobi, Weinberger-Xie-Yu....)

Stratified pseudomanifolds

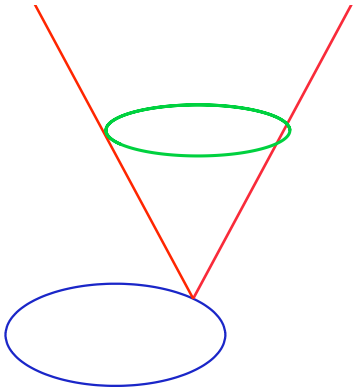


Above is an example of depth 1;below is an example of depth 2:



Basics

Let us concentrate on the depth one case.



So there is a decomposition of $\widehat{X}$ into two **strata**: $\widehat{X} = Y \cup X$
Y is the singular set (the bottom blue circle) and X is the regular part (the union of the red cones (without the vertices)).
The **link** of a point $p \in Y$ is a smooth closed manifold Z (the **green circle**).
A neighborhood of $p \in Y$ looks like $B \times C(Z)$, with B a ball in $\mathbb{R}^{\dim Y}$.In fact a tubular neighborhood T of Y is a bundle of cones $C(Z) \to T \xrightarrow{\pi} Y$, as in the figure.

Examples

- singular projective algebraic varieties
- quotients of non-free actions
- compactifications of locally symmetric spaces
- moduli spaces

Questions and problems:

- can we run the machine in the singular case ?
- problem 1: for stratified spaces Poincaré duality does **not** hold
- consequently, we do not have a signature
- we do analysis on the regular part X of $\widehat{X}$; we need to fix a metric g on X
- natural metrics are typically incomplete
- problem 2: the signature operator on $\Omega_c^*(X)$ has many extensions
  (so, even granting the Fredholm property, which one will be "connected to topology" ?!) .

Witt spaces

We now restrict the class of pseudomanifolds. We consider Witt spaces:

Definition

$\widehat{X}$ is a Witt space if any even-dimensional link L has $IH_{\overline{m}}^{\dim L/2}(L;\mathbb{Q}) = 0$.

If $\widehat{X}$ is Witt **then**.....everything works !

Cheeger spaces

We want to drop the Witt assumption and treat more general stratified spaces. We shall treat Cheeger spaces.

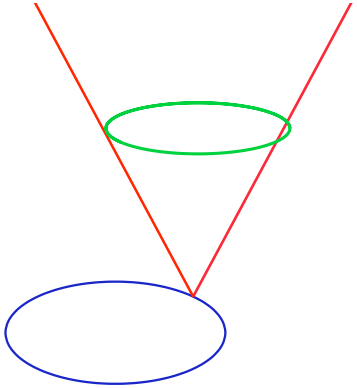
$$\{\text{Witt spaces}\} \subset \{\text{Cheeger spaces}\} \subset \{\text{Stratified spaces}\}$$

References:

- P. Albin, E. Leichtnam, R. Mazzeo, P.P.
  "Hodge theory on Cheeger spaces. "
  Crelle Journal (in press).
- P. Albin, E. Leichtnam, R. Mazzeo, P.P.
  "The Novikov conjecture on Cheeger spaces."
  JNCG (in press)
- P. Albin, M. Banagl, E. Leichtnam, R. Mazzeo, P.P.
  "Refined intersection homology on non-Witt spaces."
  Journal of Analysis and Topology. 2015

Iterated conic metrics.

Let us concentrate on the depth one case.



Recall that a neighborhood of p in the singular set Y (the blue circle) looks like $B \times C(Z)$, with B a ball in $\mathbb{R}^{\dim Y}$.
In fact a tubular neighborhood T of Y is a bundle of cones as in figure:

$$C(Z) \to T \xrightarrow{\pi} Y$$

If x is the variable along the cone then $x = 1$ defines a fibration

$$Z - H \longrightarrow Y$$

A conic metric on X is, by definition, an incomplete metric of the form $g := dx^2 + x^2 g_Z + \pi^* g_Y$.

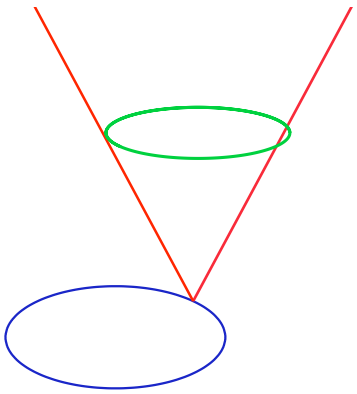
Closed extensions

- we want to use Hilbert-space techniques
- we want closed operators
- if $\widehat{X}$ is Witt, then $d_{\min} = d_{\max}$ and $\eth_{\text{sign}} : \Omega_c^+ \oplus \Omega_c^- \to \Omega_c^+ \oplus \Omega_c^- \to$ is essentially self-adjoint
- in the non-Witt case $d : \Omega_c^k \to \Omega_c^{k+1}$ has various closed extensions (between $d_{\min}$ and $d_{\max}$)
- similarly $\eth_{\text{sign}}$ is NOT essentially self-adjoint

Resolution

- we resolve the pseudomanifold $\widehat{X}$ to a manifold with corners $\widetilde{X}$ (Verona + Brasselet-Hector-Saralegi + ALMP). $\widetilde{X}$ has an additional structure: it has an an iterated fibration structure on the boundary (boundary hypersurfaces are fibrations + compatibility relations at the corners between these fibrations).

Example: if $\widehat{X}$ is a depth-one space



then $\widetilde{X}$ is a manifold with boundary and the boundary is our fibration $H \to Y$ (thus with base equal to the singular stratum (the bottom circle) and fiber the links (the green circles)).

Expansions

We first consider $\eth_{\mathrm{dR}} := d + d^*$. Recall that a tubular neighborhood T of the singular set Y looks like

$$C(Z) \to T \to Y$$

Consider the resolved manifold $\widetilde{X}$; a manifold with boundary with boundary equal to the fibration $Z \to H \to Y$.
If Z is even-dimensional and has cohomology in middle degree then we are NOT in the Witt case.

Fundamental Lemma *Any $u \in \mathcal{D}_{\max}(\eth_{\mathrm{dR}})$ has an asymptotic expansion at Y,*

$$u \sim x^{1/2}(\alpha_1(u) + dx \wedge \beta_1(u)) + \widetilde{u}$$

with the terms in this expansion distributional:

$$\alpha_1(u), \beta_1(u) \in H^{-1/2}(Y; \Lambda^* T^* Y \otimes \mathcal{H}^{f/2}(H/Y)), \quad \widetilde{u} \in xH^{-1}(X, \Lambda^* X)$$

Here $\mathcal{H}^{f/2}(H/Y)$ is the flat Hodge bundle over Y (with typical fiber $\mathcal{H}^{f/2}(Z_y)$) and $f = \dim Z$

Cheeger boundary condition

The distributional differential forms $\alpha(u)$, $\beta(u)$ serve as 'Cauchy data' at Y which we use to define *Cheeger ideal boundary conditions*. Here is what we do: for any subbundle

$$\begin{tikzcd} W \arrow[rr] \arrow[dr] & & \mathcal{H}^{f/2}(H/Y) \arrow[dl] \\ & Y & \end{tikzcd}$$

that is parallel with respect to the flat connection, we define

$$\mathcal{D}_W(\eth_{\mathrm{dR}}) = \{u \in \mathcal{D}_{\max}(\eth_{\mathrm{dR}}) :$$
$$\alpha_1(u) \in H^{-1/2}(Y; \Lambda^* T^* Y \otimes W), \quad \beta_1(u) \in H^{-1/2}(Y; \Lambda^* T^* Y \otimes (W)^{\perp})\}.$$

We call W a (Hodge) mezzoperversity adapted to g.

ANALYTIC RESULTS. Part 1.

- Every mezzoperversity induces a closed self-adjoint domain $\mathcal{D}_{\mathcal{W}}(\eth_{\mathrm{dR}})$;
- $(\eth_{\mathrm{dR}}, \mathcal{D}_{\mathcal{W}}(\eth_{\mathrm{dR}}))$ is Fredholm with discrete spectrum;
- We can define a domain for the exterior derivative as an unbounded operator on L^2 differential forms: $\mathcal{D}_{\mathcal{W}}(d)$;
- the corresponding de Rham cohomology groups, $H^*_{\mathcal{W}}(\widehat{X})$, are finite dimensional and metric independent;
- there is a Hodge decomposition theorem.

ANALYTIC RESULTS. Part 2

- given a mezzoperversity $\mathcal{W}$ there is a dual mezzoperversity $\mathcal{DW}$ defined in terms of the vertical Hodge-$\star$
- there is a natural non-degenerate pairing $H^{\ell}_{\mathcal{W}}(\widehat{X}) \times H^{n-\ell}_{\mathcal{DW}}(\widehat{X}) \to \mathbb{R}$
- if $\mathcal{W} = \mathcal{DW}$ then we say that $\mathcal{W}$ is self-dual (might not $\exists$)
- $\widehat{X}$ admitting a self-dual mezzoperversity is a **Cheeger space**;
- on a Cheeger space we have a non-degenerate pairing $H^{\ell}_{\mathcal{W}}(\widehat{X}) \times H^{n-\ell}_{\mathcal{W}}(\widehat{X}) \to \mathbb{R}$ and thus a signature $\sigma_{\mathcal{W}}(\widehat{X})$;
- a self-dual mezzoperversity defines a Fredholm signature operator $(\eth_{\mathrm{sign}}, \mathcal{D}_{\mathcal{W}}(\eth_{\mathrm{sign}}))$;
- the index is equal to the signature : $\sigma_{\mathcal{W}}(\widehat{X}) = \mathrm{ind}(\eth_{\mathrm{sign}}, \mathcal{D}_{\mathcal{W}}(\eth_{\mathrm{sign}})) \equiv \mathrm{ind}(\eth_{\mathrm{sign},\mathcal{W}})$
- there is a well defined K-homology class $[\eth_{\mathrm{sign},\mathcal{W}}]$ in $K_*(\widehat{X})$
- if $\pi_1(\widehat{X}) = \Gamma$ and $\widehat{X}_{\Gamma}$ is the universal cover of $\widehat{X}$ then we also have a higher index class

$$\mathrm{Ind}(\eth^{\Gamma}_{\mathrm{sign},\mathcal{W}}) \in K_*(C^*(\widehat{X}_{\Gamma})^{\Gamma})$$

Summary+ Crucial Questions

Given a Cheeger space $\widehat{X}$ with a fixed self-dual mezzoperversity $\mathcal{W}$ and a classifying map r we have defined

- $H^*_{\mathcal{W}}(\widehat{X})$
- $\sigma_{\mathcal{W}}(\widehat{X}) \in \mathbb{Z}$
- $[\eth_{\mathrm{sign},\mathcal{W}}]$ in $K_*(\widehat{X})$
- $\mathrm{Ind}(\eth^{\Gamma}_{\mathrm{sign},\mathcal{W}}) \in K_*(C^*(\widehat{X}_{\Gamma})^{\Gamma}) = K_*(C^*_r\Gamma)$

Question 1: what happens to these invariants if $F : \widehat{X} \to \widehat{M}$ is a stratified homotopy equivalence ??
Question 2: is there a Hilsum-Skandalis perturbation ??

Question 3: can we define the rho class of a **stratified** homotopy equivalence ??

Question 4: how does all this depend on the choice of $\mathcal{W}$??

Stratified maps

Let $F : \widehat{X} \to \widehat{M}$ be a smoothly stratified map between depth-1 stratified spaces. We denote by Y_X and Y_M the singular strata and by T_X and T_Y the corresponding tubular neighbourhoods.
Then

$$F|_{Y_X} : Y_X \to Y_M \quad F|_{T_X} : T_X \to T_Y$$

moreover $F|_{T_X}$ is a bundle map.
F is transverse if T_X is the pull-back of T_Y and $F|_{T_X}$ is a pull-back map.
Pull-back of forms is not L^2-bounded but we can consider the Hilsum-Skandalis replacement for the pull-back map. We work on the resolved manifold.

Proposition

Let $\mathcal{W}$ be a mezzoperversity for $\widehat{M}$. Then we can define the pull-back mezzoperversity $F^\sharp(\mathcal{W})$. If $\mathcal{W}$ is self-dual, so is $F^\sharp(\mathcal{W})$.

Stratified homotopy invariance

Theorem

If $F : \widehat{M}' \to \widehat{M}$ is a stratified homotopy equivalence and $\mathcal{W}$ is a mezzoperversity for $\widehat{M}$ then

$$H^*_{\mathcal{W}}(\widehat{M}) \simeq H^*_{F\sharp\mathcal{W}}(\widehat{M}')\,.$$

If $\mathcal{W}$ is self-dual

$$\sigma_{\mathcal{W}}(\widehat{M}) = \sigma_{F\sharp\mathcal{W}}(\widehat{M}')$$

$$\mathrm{Ind}(\eth^{\Gamma}_{\mathrm{sign},\mathcal{W}}) = \mathrm{Ind}(\eth^{\Gamma,\prime}_{\mathrm{sign},F\sharp\mathcal{W}}) \in K_*(C^*_r\Gamma)$$

proved via a Hilsum-Skandalis perturbation.

Bordism invariance

Theorem

Both $\sigma_{\mathcal{W}}(\widehat{M})$ *and* $\mathsf{Ind}(\eth^{\Gamma}_{\mathrm{sign},\mathcal{W}})$ *are Cheeger-bordism-invariant: if* $(\widehat{M},\mathcal{W})$ *is bordant to* $(\widehat{M}',\mathcal{W}')$ *through* $(\widehat{Z},\mathcal{W}_{\widehat{Z}})$ *then the signature and the index class are the same.*

Theorem

(from an idea of Markus Banagl)
Let $\mathcal{W}$ *and* $\mathcal{W}'$ *be two mezzoperversity for* $\widehat{M}$*. Then* $(\widehat{M},\mathcal{W})$ *is Cheeger-bordant to* $(\widehat{M},\mathcal{W}')$

Independence on $\mathcal{W}$. The L-class

Corollary

$\sigma_{\mathcal{W}}(\widehat{M})$ and $\mathrm{Ind}(\eth^{\mathcal{G}(r)}_{\mathrm{sign},\mathcal{W}})$ are stratified homotopy invariant and independent of $\mathcal{W}$!!

Consequently: for a Cheeger space $\widehat{M}$ we have a signature and a homology L-class $L_*(\widehat{M}) \in H_*(\widehat{M}, \mathbb{Q})$ defined à la Thom.
First defined by Banagl using topology. By our Hodge theorem we prove they are equal. Having $L_*(\widehat{M})$ we can define the higher signatures on a Cheeger space

$$\{< \alpha, r_*(L_*(\widehat{M})) >, \ \ \alpha \in H^*(B\Gamma, \mathbb{Q})\}$$

and formulate the Novikov Conjecture (stratified homotopy invariance)

Theorem

(Albin-Leichtnam-Mazzeo-P.)
If SNC holds for Γ then the Novikov conjecture holds for a Cheeger space $\widehat{M}$ with $\pi_1(\widehat{M}) = \Gamma$. In particular it holds for a Witt space.

Back to Browder-Quinn I

- we have seen that on a Cheeger space $\widehat{X}$ with mezzoperversity $\mathcal{W}$ there exists a K-homology class $[\tilde{\eth}_{\mathrm{sign},\mathcal{W}}] \in K_*(\widehat{X})$
- if $f : \widehat{M} \to \widehat{X}$ is a stratified homotopy equivalence then there is an associated Hilsum-Skandalis perturbation for the signature operator on $M \sqcup X$ with mezzoperversity $f^\sharp\mathcal{W} \sqcup \mathcal{W}$
- hence (modulo showing that the operators are in the right algebras) there is a well defined rho class $\rho(f, \mathcal{W}) \in K_*(D^*(\widehat{X}_\Gamma)^\Gamma)$
- Finally let B be a Cheeger space with boundary and $B \xrightarrow{F} \widehat{X} \times [0,1]$ a degree one transverse map. The mezzoperversity $\mathcal{W}$ on $\widehat{X}$, extends trivially to $\widehat{X} \times [0,1]$ (call it again $\mathcal{W}$) and pulling it back on B via $F^\sharp$ we obtain a mezzoperversity $F^\sharp\mathcal{W} \sqcup \mathcal{W}$ on $B \cup (\widehat{X} \times [0,1])$. If F_∂ is a stratified homotopy equivalence then there is a well defined APS-index class

$$\mathrm{Ind}_{\mathrm{APS},\mathcal{W}}(B \xrightarrow{F} \widehat{X} \times [0,1]) \in K_*(C^*(\widehat{X}_\Gamma)^\Gamma)$$

Back to Browder-Quinn II

Let $\widehat{X}$ be a Cheeger space. Choose a mezzoperversity $\mathcal{W}$.
Consider the Browder-Quinn surgery sequence
$L^{\mathrm{BQ}}_{n+1}(\widehat{X}) \dashrightarrow \mathcal{S}^{\mathrm{BQ}}(\widehat{X}) \to \mathcal{N}^{\mathrm{BQ}}(\widehat{X}) \to L^{\mathrm{BQ}}_{n}(\widehat{X})$
Define Ind : $L^{\mathrm{BQ}}_{*}(\widehat{X})$ on a refined cycle via $\mathrm{Ind}_{\mathrm{APS},\mathcal{W}}$
Define $\rho : \mathcal{S}^{\mathrm{BQ}}(\widehat{X}) \to K_*(D^*(\widehat{X}_\Gamma)^\Gamma)$ as $\rho[\widehat{Y} \xrightarrow{f} \widehat{X}] = \rho(f, \mathcal{W})$.
Define $\beta : \mathcal{N}^{\mathrm{BQ}}(\widehat{X}) \to K_*(\widehat{X})$ as
$\beta[\widehat{Y} \xrightarrow{f} \widehat{X}] = f_*[\eth_{\mathrm{sign},f^\sharp\mathcal{W}}] - [\eth_{\mathrm{sign},\mathcal{W}}]$.

Theorem

These maps are well defined and they are independent of $\mathcal{W}$.
Moreover, the following diagram is commutative

$$\begin{tikzcd}
L^{\mathrm{BQ}}_{n+1}(\widehat{X}) \arrow[r, dashed] \arrow[d, "\mathrm{Ind}"] & \mathcal{S}^{\mathrm{BQ}}(\widehat{X}) \arrow[r] \arrow[d, "\rho"] & \mathcal{N}^{\mathrm{BQ}}(\widehat{X}) \arrow[r] \arrow[d, "\beta"] & L^{\mathrm{BQ}}_{n} \\
K_{n+1}(C^*(\widehat{X}_\Gamma)^\Gamma) \arrow[r] & K_{n+1}(D^*(\widehat{X}_\Gamma)^\Gamma) \arrow[r] & K_n(\widehat{X}) \arrow[r] & K_n(C^*
\end{tikzcd}$$

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